

A. Luikov

HEAT

and

MASS



TRANSFER

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# HEAT and MASS TRANSFER



A. Luikov

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### **The Greek Alphabet**

<i>A α</i>	Alpha	<i>Ν ν</i>	Nu
<i>Β β</i>	Beta	<i>Ξ ξ</i>	Xi
<i>Γ γ</i>	Gamma	<i>Ο ο</i>	Omicron
<i>Δ δ</i>	Delta	<i>Π π</i>	Pi
<i>Ε ε</i>	Epsilon	<i>Ρ ρ</i>	Rho
<i>Ζ ζ</i>	Zeta	<i>Σ σ</i>	Sigma
<i>Η η</i>	Eta	<i>Τ τ</i>	Tau
<i>Θ θ</i>	Theta	<i>Υ υ</i>	Upsilon
<i>Ι ι</i>	Iota	<i>Φ φ</i>	Phi
<i>Κ κ</i>	Kappa	<i>Χ χ</i>	Chi
<i>Λ λ</i>	Lambda	<i>Ψ ψ</i>	Psi
<i>Μ μ</i>	Mu	<i>Ω ω</i>	Omega

*На английском языке*

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## EDITOR'S PREFACE

### TO THE RUSSIAN EDITION

The present book was written by the outstanding Soviet scientist, Academician of the Byelorussian Academy of Sciences, Professor Aleksei V. Luikov, not long before his decease in 1974.

This is the amended and supplemented second edition of the popular reference book published in 1972.

Heat-and mass-transfer theory deals with the transfer of energy (heat), momentum and mass and embraces some sections of molecular physics, aerohydrodynamics, reversible and irreversible thermodynamics, physico-chemistry of surface effects and chemical engineering.

Convective diffusion transfer processes are considered in terms of irreversible and nonlinear thermodynamics of continua. In the second edition considerable attention and space have been devoted to asymmetric hydrodynamics because of the increasing importance of rheological materials, for which the classical transfer equations are not applicable. Transfer equations based on nonlinear relations with memory, govern transfer phenomena in such materials more accurately.

The chapters "Heat Conduction" and "Convective Heat Transfer" have been amended and supplemented. In the solution of convective heat transfer problems the author substituted fourth-kind boundary conditions for boundary conditions of the third kind. In all cases heat transfer in fluids is analysed jointly with heat transfer in a solid wall.

The chapter "Transport Phenomena in Capillary-Porous Bodies" is supplemented with a theoretical analysis of mass transfer in such materials in the presence of phase conversions (liquid evaporation) which is of great practical importance for development of the calculation procedures for transpiration cooling and duration of drying processes.

The sixth chapter entitled "Analytical Heat and Mass Diffusion Theory" comprises an analysis of the differential heat and moisture transfer equations in capillary-porous colloid materials during limit transitions, which is applicable to drying processes and experimental methods of determining thermophysical properties.

The editor regards it his privilege to provide the reader the original, current and informative material of the manuscript, to which a few notes have been added. The valuable comments of Professors N. I. Gamayunov, A. I. Leontiev, A. P. Prudnikov, Assistant Professors V. A. Bubnov and A. A. Aleksashenko on certain sections of the manuscript are greatly appreciated.

*Professor V. V. Krasnikov*

## **AUTHOR'S PREFACE TO THE FIRST RUSSIAN EDITION**

The theory of energy (heat) and mass transfer is one of the domains of contemporary science; it is of great practical importance for increasing the process rates in heat-power engineering, power engineering, and chemical engineering in various branches of industry and agriculture. Heat and mass transfer has become especially important in modern technology, particularly atomic power engineering and space research, whose rapid development gave rise to the theory of energy and mass transfer. It comprises classical nonequilibrium thermodynamics, physical gas dynamics, equations from mathematical physics, physicochemical surface effects and dispersed systems.

For the past 10-15 years this branch of science has been considerably extended and advanced. Numerous applications of the theory may be found in power plant and industrial power engineering, technological processes, chemical engineering, the construction industry and agriculture. It should be noted that the level of scientific research and development varies in different branches of engineering. Whereas in the newest fields of scientific research the development of heat and mass transfer theory is at a rather high level, applications in other fields lag far behind.

In view of the above it has proved necessary to review the present state of heat and mass transfer theory and to outline its further development with an emphasis on the latest trends in the field.

This work is a new type of reference book, which is not a selection of formulas, tables and nomograms, but rather a systematic oriented review of the current heat and mass transfer theory, including a critical analysis; it contains an analytical description of various kinds of processes and an attempt to outline their further development.

The present book in no way replaces the published reference book on thermal engineering and heat transfer, since first, it uses a new methodological approach and second, analyses mainly simultaneous heat and mass transfer processes and mathematical transfer theory, which may equally be applied to transfer of heat and mass. Therefore, radiative heat transfer, pure heat transfer and other problems of heat transfer are not considered in the book. Much attention has been paid to analytical theory of heat and mass transfer, particularly to unsteady-state heat conduction problems (Sec. 2), where introduction of generalized functions has allowed the author to describe simultaneously one-dimensional temperature fields in classical geometries, to give a new, simpler description of propagation

of temperature waves, to generalize regular regimes of temperature heating and to make some other generalizations. As a further development of the analytical heat-conduction theory the author presents the most recent solutions of differential heat and mass transfer equations (Sec. 6), considers in detail hyperbolic heat- and mass-diffusion equations with regard to the finite rate of their propagation. This new approach to the description of heat and mass transfer effects is related with the work done in the USA and dealing with mass diffusion in porous materials.

One of the most important sections of the reference book is devoted to heat and mass transfer in capillary-porous materials (Sec. 5), since heat and mass transfer in such materials determines processing techniques in various industries. Moreover, the problem of heat and mass transfer in capillary-porous bodies has increased in importance because of their wide use in space research, particularly in devices for liquid and energy transfer (heat pipes, heat exchangers, etc.).

A new approach to convective heat transfer problems is the so-called similarity solutions, where, instead of traditional analysis, heat transfer of a solid in a fluid flow is considered as simultaneous heat and mass transfer in solids and fluids. In Ch. 4 are given the latest solutions of external and internal heat transfer problems. Similarity solutions are quite important, especially for unsteady-state convective heat and mass transfer. Description of new phenomena is also presented: free convection at heating from above (coinciding vectors of the heat flux and gravity force), thermoconvective waves and some other problems discussed in recent works on heat and mass transfer (Ch. 3).

Compilation of a reference book on a complicated problem of present-day science is quite a difficult task and the book cannot cover all the works in the field. The author will be grateful for any criticism and advice. He hopes that this first attempt at a review reference book may encourage others to write similar books in other fields of science and technology that will promote further technological progress in this country.

The author is thankful to Mrs. E. A. Bogacheva of the USSR Editorial Office, International Journal of Heat and Mass Transfer, for her assistance in preparation of the manuscript.

*Academician A. V. Luikov*

# CONVECTIVE DIFFUSIONAL TRANSFER

1



Any quantity such as mass, momentum, energy, etc. may be transferred in fluids both by molecular mode (random molecular motion) and convection (hydrodynamic macromolecular motion). Physically molecular transfer is caused by diffusion (molecular agitation) and this mode may, therefore, be called diffusional transfer. In this case heat transfer by heat conduction in homogeneous fluids is a self-diffusion transfer\*. Convective transfer is induced by visible (ordered) motion of the fluid itself due to external forces and pressure drop.

First we shall consider the basic analytical relations forming a basis for mathematical transformations and derivations of transfer equations.

## 1-1. BASIC ANALYTICAL RELATIONS

Analytical relations which have not received adequate attention in the literature will be used for the derivation of transfer equations. Therefore, a discussion of the main relations follows.

### (a) Time Derivative

The derivative of a scalar function  $\varphi(\tau)$  with respect to time is defined by the relation

$$\frac{d\varphi}{d\tau} = \lim_{\Delta\tau \rightarrow 0} \left[ \frac{\varphi(\tau + \Delta\tau) - \varphi(\tau)}{\Delta\tau} \right] \quad (1-1-1)$$

If  $\varphi$  is a function of time  $\tau$  alone, then the relation (1-1-1) completely defines the time derivative. When  $\varphi$  is a function of coordinates, this definition requires further refinement.

Let us consider a fluid composed of moving particles. Let  $x_p$  be the coordinate of particle  $p$  (see Fig. 1-1). Then the derivative  $\frac{dx_p}{d\tau}$  is the velocity

---

\* This should not be confused with diffusional heat conduction (the Dufour effect) in gaseous mixtures.

$v_1$  of particle  $p$  along  $x$

$$\frac{dx_p}{d\tau} = \lim_{\Delta\tau \rightarrow 0} \left[ \frac{x_p(\tau + \Delta\tau) - x_p(\tau)}{\Delta\tau} \right] = v_1 \quad (1-1-2)$$

This derivative is usually called a material or substantial derivative

$$\frac{dx_p}{d\tau} = \left( \frac{dx_p}{d\tau} \right)_{\vec{R}} = v_1 \quad (1-1-3)$$

where the subscript  $\vec{R}$  indicates a fixed coordinate system of the fluid

When determining the derivative of temperature with respect to time, two cases are met:

(i) Temperature at a given point varies independent of the motion (the temperature sensor moves together with the particle). This is a local derivative  $\partial T / \partial \tau$  of temperature

$$\left( \frac{dT}{d\tau} \right)_i = \frac{\partial T}{\partial \tau} \quad (1-1-4)$$

Then the particle temperature changes due to the motion of the particle itself. For fixed coordinates the total derivative of temperature is the substantial derivative

$$\begin{aligned} \left( \frac{dT}{d\tau} \right)_R &= \frac{dT}{d\tau} = \frac{\partial T}{\partial x_1} \left( \frac{dx_1}{d\tau} \right)_{\vec{R}} + \frac{\partial T}{\partial x_2} \left( \frac{dx_2}{d\tau} \right)_{\vec{R}} \\ &+ \frac{\partial T}{\partial x_3} \left( \frac{dx_3}{d\tau} \right)_{\vec{R}} + \left( \frac{dT}{d\tau} \right)_i = \frac{\partial T}{\partial \tau} + v_1 \frac{\partial T}{\partial x_1} + v_2 \frac{\partial T}{\partial x_2} + v_3 \frac{\partial T}{\partial x_3} \end{aligned} \quad (1-1-5)$$

where  $v_1, v_2, v_3$  are the velocity components of fluid in the direction of Cartesian coordinates. This relation may be written as follows:

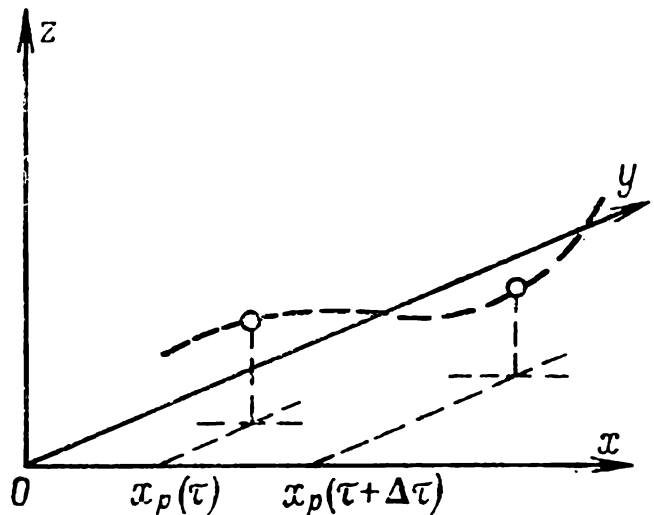


Fig. 1-1. Particle motion in space

$$\frac{dT}{d\tau} = \frac{\partial T}{\partial \tau} + \vec{v} \cdot \vec{\nabla} T \quad (1-1-6)$$

(ii) In the second case the coordinate system is the time function  $x(\tau)$ ,  $y(\tau)$ ,  $z(\tau)$ . Here the total derivative equals

$$\frac{DT}{D\tau} = \frac{\partial T}{\partial \tau} + \frac{\partial T}{\partial x} \left( \frac{dx}{d\tau} \right) + \frac{\partial T}{\partial y} \left( \frac{dy}{d\tau} \right) + \frac{\partial T}{\partial z} \left( \frac{dz}{d\tau} \right) \quad (1-1-7)$$

where  $dx/d\tau$ ,  $dy/d\tau$ ,  $dz/d\tau$  are the components of velocity vector  $\vec{w}$  that characterizes the movement of temperature sensor

$$\frac{DT}{D\tau} = \frac{\partial T}{\partial \tau} + \vec{w} \cdot \vec{\nabla} T \quad (1-1-8)$$

This is the most general expression of the time derivative [1-1]. Relation (1-1-6) is a particular case of (1-1-8) when  $\vec{w} = \vec{v}$ .

#### (b) Sliding Index Method

The sliding index method together with symbolic tensor analysis is very convenient for writing mathematical formulas as it allows shorter expression and better understanding of physical concepts. Independent variables are designated by different subscripts and not by different symbols. For example, Cartesian coordinates  $x$ ,  $y$ ,  $z$  are denoted by  $x_1$ ,  $x_2$ ,  $x_3$  ( $x_1 \equiv x$ ;  $x_2 \equiv y$ ;  $x_3 \equiv z$ ). For the terms of any sum a common term of the sum is found, from which individual terms are obtained for corresponding partial values of subscripts. This shall be explained with an example.

The squared length section  $ds^2$  in orthogonal coordinates  $(\xi_1, \xi_2, \xi_3)$  is expressed by the Gaussian formula

$$ds^2 = h_1^2 d\xi_1^2 + h_2^2 d\xi_2^2 + h_3^2 d\xi_3^2 = \sum_i h_i^2 d\xi_i^2 \quad (1-1-9)$$

where

$$h_i^2 = \sum_{k=1}^3 \left( \frac{\partial x_k}{\partial \xi_i} \right)^2, \quad i = 1, 2, 3 \quad (1-1-10)$$

Here, the summation sign is omitted and the terms of the sum should be represented by factors with repeated indices; from relations (1-1-9) and (1-1-10) we get:

$$ds^2 = h_i^2 d\xi_i^2 \equiv \left( \frac{\partial x_k}{\partial \xi_i} \right) \left( \frac{\partial x_k}{\partial \xi_i} \right) d\xi_i^2 \quad (1-1-11)$$

Repeated indices are very helpful for the interpretation of formulas, and they are referred to as sliding indices. For example, a Latin index followed by comma indicates the partial derivative with respect to the spatial coordinate  $x_{k,,}$ , and a superposed dot denotes the material derivative, e.g.

$$\frac{dT}{d\tau} = \frac{\partial T}{\partial \tau} + \vec{v} \cdot \vec{\nabla} T; \quad \dot{T} = \frac{\partial T}{\partial \tau} + v_i T_{,i}; \quad i = 1, 2 \quad (1-1-12)$$

$$\frac{d\vec{v}}{d\tau} = \frac{\partial \vec{v}}{\partial \tau} + (\vec{v} \cdot \vec{\nabla}) \vec{v}; \quad \dot{v}_i = \frac{\partial v_i}{\partial \tau} + v_{i,j} v_j; \quad i, j = 1, 2, 3 \quad (1-1-13)$$

If from the sum it is necessary to single out a term not involved in the summation, then one of the repeating indices is bracketed.

For example, if from relation (1-1-11) the component  $ds_k$  of the element  $\vec{ds}$  along the coordinate  $\xi_k$  is to be found, we may write

$$ds_k = h_{(k)} d\xi_k \quad (1-1-14)$$

The subscripts  $k$  in brackets indicate that no summation is to be made with respect to the index, although it is repeated in the formula

### (c) Transformation of Coordinates

Let us consider the transformation of the system of rectangular coordinates  $(x_i)$  into a system of orthogonal curvilinear coordinates

$$x_i = x_i(\xi_1, \xi_2, \xi_3), \quad i = 1, 2, 3$$

From relations (1-1-9) and (1-1-10), we obtain [1-2]

$$\text{grad } T = \vec{\nabla} T(\xi_1, \xi_2, \xi_3) = \sum_{i=1}^3 \vec{\xi}_i^1 \frac{1}{h_i} \frac{\partial T}{\partial \xi_i} \quad (1-1-15)$$

where  $\vec{\xi}_i^1$  is a unit vector along  $\xi_i$  of the system of coordinates  $(\xi_1, \xi_2, \xi_3)$ . The divergence of the heat flux density vector  $\vec{q}(\xi_1, \xi_2, \xi_3)$  is

$$\text{div } \vec{q} = \vec{\nabla} \vec{q}(\xi_1, \xi_2, \xi_3) = \frac{1}{h_1 h_2 h_3} \sum_{i=1}^3 \frac{\partial}{\partial \xi_i} \left( \frac{h_1 h_2 h_3}{h_i} q_i \right) \quad (1-1-16)$$

where  $q_i$  are the components of vector  $\vec{q}$  along  $\vec{\xi}_i^1$ . The Laplace operator  $\nabla^2 T$  equals

$$\nabla^2 T = \vec{\nabla} \vec{\nabla} T(\xi_1, \xi_2, \xi_3) = \frac{1}{h_1 h_2 h_3} \sum_{i=1}^3 \frac{\partial}{\partial \xi_i} \left( \frac{h_1 h_2 h_3}{h_i^2} \frac{\partial T}{\partial \xi_i} \right) \quad (1-1-17)$$

As an example, we shall consider the system of cylindrical coordinates  $(r, \theta, z)$ , where  $\theta$  is the polar angle (the angle between radius-vector  $r$  and axis  $x_1$ )

$$x_1 \equiv x = r \cos \theta; \quad x_2 \equiv y = r \sin \theta; \quad x_3 \equiv z = z$$



$$h_r^2 = \left(\frac{\partial x_1}{\partial r}\right)^2 + \left(\frac{\partial x_2}{\partial r}\right)^2 + \left(\frac{\partial x_3}{\partial r}\right)^2 = \cos^2 \theta + \sin^2 \theta + 0 = 1$$

$$h_\theta^2 = \left(\frac{\partial x_1}{\partial \theta}\right)^2 + \left(\frac{\partial x_2}{\partial \theta}\right)^2 + \left(\frac{\partial x_3}{\partial \theta}\right)^2 = (-r \sin \theta)^2 + (r \cos \theta)^2 + 0 = r^2$$

$$h_z^2 = \left(\frac{\partial x_1}{\partial x_3}\right)^2 + \left(\frac{\partial x_2}{\partial x_3}\right)^2 + \left(\frac{\partial x_3}{\partial x_3}\right)^2 = 0 + 0 + 1 = 1$$

Hence,  $h_r = 1$ ,  $h_\theta = r$ ,  $h_z = 1$

$$\nabla^2 T(r, \theta, x_3) = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial x_3^2} \quad (1-1-18)$$

For a system of spherical coordinates  $(r, \theta, \Phi)$ , where  $\theta$  is the angle between the projection of radius-vector  $r$  on plane  $x_0x_2$  and the axis  $x_1$ ,  $\Phi$  is the angle between vector  $r$  and axis  $x_3$

$$x_1 = r \sin \Phi \cos \theta; \quad x_2 = r \sin \Phi \sin \theta; \quad x_3 = r \cos \Phi$$

In a similar way we find

$$h_r = 1, \quad h_\theta = r \sin \Phi, \quad h_\Phi = r;$$

$$\nabla^2 T(r, \theta, \Phi) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \Phi} \frac{\partial}{\partial \Phi} \left( \sin \Phi \frac{\partial T}{\partial \Phi} \right) + \frac{1}{r^2 \sin^2 \Phi} \frac{\partial^2 T}{\partial \theta^2} \quad (1-1-19)$$

#### (d) Transfer Theorems

Let us consider a volume  $V_a(\tau)$  moving at a velocity  $\vec{w}$  (see Fig. 1-2). In the given volume the derivative of the scalar function  $\varphi$  will then be expressed as

$$\frac{d}{d\tau} \int_{V_a(\tau)} \varphi dV = \lim_{\Delta\tau \rightarrow 0} \left[ \frac{\int_{V_a(\tau+\Delta\tau)} \varphi(\tau+\Delta\tau) dV - \int_{V_a(\tau)} \varphi(\tau) dV}{\Delta\tau} \right] \quad (1-1-20)$$

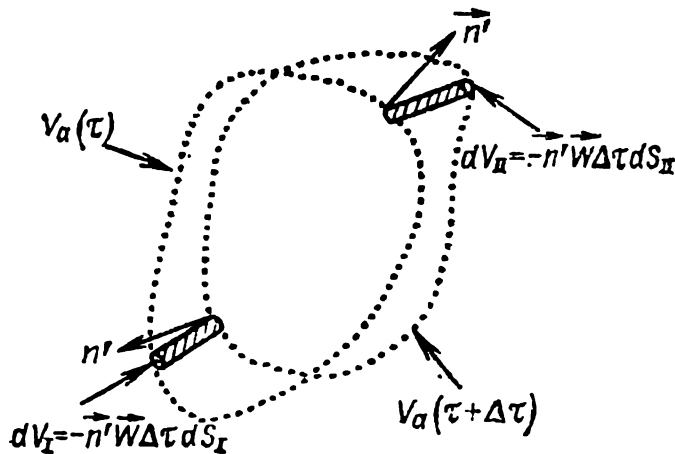


Fig. 1-2. Moving volume  $V_a(\tau)$  of the medium

Having expressed the elementary volumes  $dV$  in terms of elementary surfaces  $dS$ , the following expression can be obtained for the general transfer theorem:

$$\frac{d}{d\tau} \int_{V_a(\tau)} \varphi dV = \int_{V_a(\tau)} \frac{\partial \varphi}{\partial \tau} dV + \int_{S_a(\tau)} \varphi \bar{\mathbf{w}} \cdot \bar{\mathbf{n}}^1 dS \quad (1-1-21)$$

If  $\bar{\mathbf{w}} = 0$ , then  $V_a(\tau) = V$ , and

$$\frac{d}{d\tau} \int_V \varphi dV = \int_V \frac{\partial \varphi}{\partial \tau} dV \quad (1-1-22)$$

If  $\bar{\mathbf{w}}$  is equal to the fluid velocity  $\bar{\mathbf{v}}$  ( $\bar{\mathbf{w}} = \bar{\mathbf{v}}$ ), then from relation (1-1-22) the Reynolds transfer theorem is obtained

$$\frac{d}{d\tau} \int_{V_m(\tau)} \varphi dV = \int_{V_m(\tau)} \frac{\partial \varphi}{\partial \tau} dV + \int_{S_m(\tau)} \varphi \bar{\mathbf{v}} \cdot \bar{\mathbf{n}}^1 dS \quad (1-1-23)$$

where  $V_m(\tau)$  and  $S_m(\tau)$  are the volume and surface area of the flowing fluid.

If the transferred quantity is a vector, say,  $\bar{\mathbf{v}}$ , then the transfer theorem in a general case takes the form

$$\frac{d}{d\tau} \int_{V_a(\tau)} \bar{\mathbf{v}} dV = \int_{V_a(\tau)} \frac{\partial \bar{\mathbf{v}}}{\partial \tau} dV + \int_{S_a(\tau)} \bar{\mathbf{v}} (\bar{\mathbf{w}} \cdot \bar{\mathbf{n}}^1) dS \quad (1-1-24)$$

In [1-3] the generalized transfer theorem is given for any scalar quantity  $\varphi$  when it is differentiated along  $l$

$$\frac{d}{dl} \left( \int_{V(l)} \varphi dV \right) = \int_{V(l)} \frac{\partial \varphi}{\partial l} dV + \int_{S(l)} \varphi \left( \frac{d\bar{\mathbf{r}}}{dl} \cdot \bar{\mathbf{n}}^1 \right) dS \quad (1-1-25)$$

If there is a singular surface in a moving fluid, for example, a two-phase flow, then the equation of the transfer theorem will have a different form [1-4]. The singular surface with the area denoted by  $S_{sin}(\tau)$  divides the volume  $V$  into two regions, namely,  $V^+$  and  $V^-$ , and the surface  $S$ , into surfaces  $S^+$  and  $S^-$  (see Fig. 1-3).

If  $\bar{\mathbf{u}}$  is the velocity of the singular surface inside a fluid flow, then

$$\bar{\mathbf{u}}^+ = \begin{cases} \bar{\mathbf{v}} & \text{at the surface } S^+ \\ \bar{\mathbf{u}}_\xi \bar{\boldsymbol{\xi}}^1 & \text{at the surface } S_{sin} \end{cases}$$

$$\bar{\mathbf{u}}^- = \begin{cases} \bar{\mathbf{v}} & \text{at the surface } S^- \\ \bar{\mathbf{u}}_\xi \bar{\boldsymbol{\xi}}^1 & \text{at the surface } S_{sin} \end{cases}$$

where  $\vec{\xi}^1$  is a unit vector along the normal to the surface  $S_{sin}$ . Using the relation

$$\frac{d}{d\tau} \int_{V_m(\tau)} \varphi dV = \frac{d}{d\tau} \int_{V^+} \varphi dV + \frac{d}{d\tau} \int_{V^-} \varphi dV \quad (1-1-26)$$

for (1-1-23) we get

$$\frac{d}{d\tau} \int_{V_m^+(\tau)} \varphi dV = \int_{V_m^+(\tau)} \frac{\partial \varphi}{\partial \tau} dV + \int_{S_m^+(\tau)} \varphi \vec{v} \cdot \vec{n}^1 dS - \int_{S_{sin}(\tau)} \varphi^+ \vec{u}_\xi \cdot \vec{\xi}^1 dS \quad (1-1-27)$$

$$\frac{d}{d\tau} \int_{V_m^-(\tau)} \varphi dV = \int_{V_m^-(\tau)} \frac{\partial \varphi}{\partial \tau} dV + \int_{S_m^-(\tau)} \varphi \vec{v} \cdot \vec{n}^1 dS + \int_{S_{sin}(\tau)} \varphi^- \vec{u}_\xi \cdot \vec{\xi}^1 dS \quad (1-1-28)$$

From relation (1-1-26) it follows

$$\frac{d}{d\tau} \int_{V_m(\tau)} \varphi dV = \int_{V_m(\tau)} \frac{\partial \varphi}{\partial \tau} dV + \int_{S_m(\tau)} \varphi \vec{v} \cdot \vec{n}^1 dS - \int_{S_{sin}(\tau)} [\varphi] \vec{u}_\xi \cdot \vec{\xi}^1 dS \quad (1-1-29)$$

where

$$[\varphi] \equiv \varphi^+ - \varphi^- \quad (1-1-30)$$

For the derivation of transfer equations the following theorems are used:  
The Stokes theorem

$$\oint_L \vec{A} \cdot d\vec{L} = \int_S [\vec{\nabla} \times \vec{A}] \cdot d\vec{S} = \int_S \text{curl } \vec{A} \cdot \vec{n}^1 dS \quad (1-1-31)$$

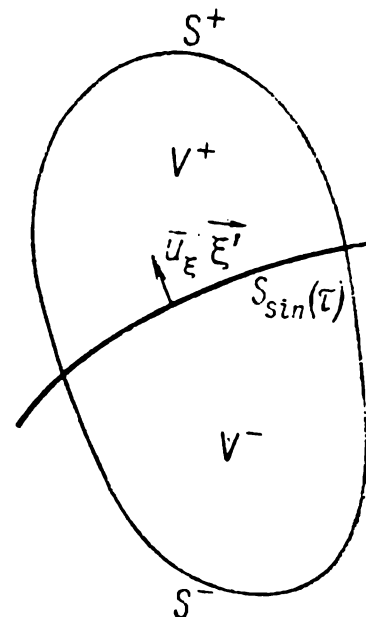


Fig. 1-3. Singular surface in a moving medium

where  $\vec{A}$  is a vector quantity, which varies along  $\vec{L}$ .

The Gauss-Ostrogradsky theorem

$$\int_V \operatorname{div} \vec{A} dV = \oint_S \vec{A} \cdot d\vec{S} = \oint_S \vec{A} \cdot \vec{n}^1 dS \quad (1-1-32)$$

The Green theorem follows from the Gaussian formula with the substitution

$$\vec{A} = a \vec{\nabla} b = a \operatorname{grad} b \quad (1-1-33)$$

Then

$$\operatorname{div} \vec{A} = \vec{\nabla} \cdot \vec{A} = a \nabla^2 b + (\vec{\nabla} a \cdot \vec{\nabla} b) = a \operatorname{div} \operatorname{grad} b + \operatorname{grad} a \cdot \operatorname{grad} b \quad (1-1-34)$$

Hence

$$\int_V [a \nabla^2 b + (\vec{\nabla} a \cdot \vec{\nabla} b)] dV = \oint_S a \vec{\nabla} b \cdot d\vec{S} \quad (1-1-35)$$

Interchange of  $a$  and  $b$  results in a relation similar to (1-1-35). From these two relations the second form of the Green theorem may be obtained

$$\int_V (a \nabla^2 b - b \nabla^2 a) dV = \oint_S (a \vec{\nabla} b - b \vec{\nabla} a) \cdot \vec{n}^1 dS \quad (1-1-36)$$

### (e) Differentiation

The generalized differentiation operation is denoted by the symbol  $\vec{\nabla}$ , referred to as the Hamiltonian operator and defined by the relation

$$\vec{\nabla} \equiv \vec{n}_1^1 \frac{\partial}{\partial x_1} + \vec{n}_2^1 \frac{\partial}{\partial x_2} + \vec{n}_3^1 \frac{\partial}{\partial x_3} = \operatorname{grad} \quad (1-1-37)$$

where  $\vec{n}_1^1$ ,  $\vec{n}_2^1$ ,  $\vec{n}_3^1$  are the mutually orthogonal unit length vectors directed along the Cartesian coordinates  $x_1$ ,  $x_2$ ,  $x_3$ , or (in short) the unit vectors.

The gradient of scalar quantity  $\varphi$  is a vector (a tensor of the first order)

$$\operatorname{grad} \varphi \equiv \vec{\nabla} \varphi = \vec{n}_1^1 \frac{\partial \varphi}{\partial x_1} + \vec{n}_2^1 \frac{\partial \varphi}{\partial x_2} + \vec{n}_3^1 \frac{\partial \varphi}{\partial x_3} + \sum_{i=1}^3 \vec{n}_i^1 \frac{\partial \varphi}{\partial x_i} \quad (1-1-38)$$

A scalar product of two vectors  $\vec{\nabla}$  and  $\vec{A}$  is a divergence of vector  $\vec{A}$

$$\operatorname{div} \vec{A} \equiv \vec{\nabla} \cdot \vec{A} = \sum_{i=1}^3 \frac{\partial A_i}{\partial x_i} \quad (1-1-39)$$

The vector product  $(\vec{\nabla} \times \vec{A})$  is called operation rot  $\vec{A}$  or curl  $\vec{A}$ . This is a vector quantity

$$\text{curl } \vec{A} \equiv [\vec{\nabla} \times \vec{A}] = \begin{vmatrix} n_1^1 & n_2^1 & n_3^1 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ A_1 & A_2 & A_3 \end{vmatrix} = \sum_j \sum_k \epsilon_{ijk} \frac{\partial A_k}{\partial x_j} \quad (1-1-40)$$

where  $\epsilon_{ijk}$  is a function, which equals

$$\begin{aligned} &0 \quad \text{if all the indices are equal} \\ &+1 \quad \text{if } ijk = 123, 231 \text{ or } 312 \\ &-1 \quad \text{if } ijk = 132, 213 \text{ or } 321 \end{aligned} \quad (1-1-41)$$

The divergence of a scalar quantity gradient is a scalar quantity

$$\text{div (grad } \varphi) \equiv \vec{\nabla} \cdot (\vec{\nabla} \varphi) = \sum_{i=1}^3 \frac{\partial^2 \varphi}{\partial x_i^2} = \nabla^2 \varphi \quad (1-1-42)$$

where  $\nabla^2$  is the Laplacian operator,

$$\text{div grad} \equiv \nabla^2 = \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2} = \frac{\partial^2}{\partial x_i \partial x_i} \equiv \Delta \quad (1-1-43)$$

The operator  $(\vec{A} \cdot \vec{\nabla})$  equals

$$(\vec{A} \cdot \vec{\nabla}) = \sum_{i=1}^3 A_i \frac{\partial}{\partial x_i} \quad (1-1-44)$$

$$(\vec{A} \cdot \vec{\nabla}) \vec{A} = \frac{1}{2} \vec{\nabla} (\vec{A} \cdot \vec{A}) - (\vec{A} \times [\vec{\nabla} \times \vec{A}]) \quad (1-1-45)$$

The following relations are useful

$$\vec{\nabla} \cdot a \vec{A} = (\vec{A} \cdot \vec{\nabla} a) + a (\vec{\nabla} \cdot \vec{A}) \quad (1-1-46)$$

$$(\vec{\nabla} \cdot \{\vec{A} \vec{B}\}) = (\vec{A} \cdot \vec{\nabla}) \vec{B} + \vec{B} (\vec{\nabla} \cdot \vec{A}) \quad (1-1-47)$$

$$(\vec{\nabla} \times a \vec{A}) = a (\vec{\nabla} \times \vec{A}) + (\vec{\nabla} a \times \vec{A}) \quad (1-1-48)$$

$$[\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A})] = 0 \quad (1-1-49)$$

$$[\vec{\nabla} \times (\vec{A} \times \vec{B})] = [\vec{B} \cdot (\vec{\nabla} \vec{A})] - [\vec{A} \cdot (\vec{\nabla} \vec{B})] + \vec{A} (\vec{\nabla} \cdot \vec{B}) - \vec{B} (\vec{\nabla} \cdot \vec{A}) \quad (1-1-50)$$

where  $(\vec{A} \vec{B})$  and  $(\vec{\nabla} \vec{A})$  are dyads having nine components. They should not be confused with the scalar products  $\vec{A} \cdot \vec{B}$  and  $(\vec{\nabla} \cdot \vec{A})$

$$\vec{A} \vec{B} = \begin{vmatrix} A_1 B_1 & A_1 B_2 & A_1 B_3 \\ A_2 B_1 & A_2 B_2 & A_2 B_3 \\ A_3 B_1 & A_3 B_2 & A_3 B_3 \end{vmatrix} \quad (1-1-51)$$

It may be shown that

$$\text{div} (\text{grad } \vec{v})^T = \text{grad} (\text{div } \vec{v})$$

or

$$\vec{\nabla} \cdot (\vec{\nabla} \vec{v})^T = \vec{\nabla} (\vec{\nabla} \cdot \vec{v}) \quad (1-1-52)$$

where  $(\text{grad } v)^T \equiv (\vec{\nabla} v)^T \equiv \vec{v} \vec{\nabla}$  is the transposed dyad.

## 1-2. DIFFERENTIAL TRANSFER EQUATIONS

### (a) Equation of Substance Transfer

For an arbitrary volume  $V$  (bounded by surface  $S$ ) with an active source or sink of quantity  $C$  of a unit heat-flux power  $\dot{I}_V$ , the transfer equation is of the form

$$\int_V \frac{\partial C}{\partial \tau} dV + \oint_S C(\vec{w}_s \cdot \vec{n}^1) dS = - \oint_S (\vec{j}_C \cdot \vec{n}^1) dS + \int_V \dot{I}_V dV \quad (1-2-1)$$

where  $\vec{w}_s$  is the velocity of the surface  $S$ ;  $\vec{n}^1$  is a unit vector along the normal  $n$  to the surface  $S$ ;  $\vec{j}_C$  is the diffusional (molecular) flow of quantity  $C$ .

Using the Gauss-Ostrogradsky theorem and assuming  $\vec{w}_s \equiv \vec{v}$ , we shall have

$$\int_V \left( \frac{\partial C}{\partial \tau} + \text{div } C\vec{v} + \text{div } \vec{j}_C - \dot{I}_V \right) dV = 0 \quad (1-2-2)$$

From which we obtain the differential transfer equation

$$\frac{\partial C}{\partial \tau} + \text{div} (C\vec{v}) = - \text{div } \vec{j}_C + \dot{I}_V \quad (1-2-3)$$

The left-hand side of equation (1-2-3) is composed of terms expressing local variation in the concentration of the transferred quantity ( $\partial C/\partial \tau$ ) and its convective transfer ( $\text{div } C\vec{v}$ ). The right-hand side describes the diffusional transfer ( $\text{div } \vec{j}_C$ ) and the effect of quantity sources or sinks ( $\dot{I}_V$ ).

### (b) Mass Transfer Equation

Differential mass, momentum and energy transfer equations may be obtained from equation (1-2-3).

If a compressible fluid contains  $k$  components, then the diffusional flow equals

$$\vec{j}_k = \rho_k(\vec{v}_k - \vec{v}) \quad (1-2-4)$$

where  $\vec{v}_k$  is the molecular velocity of the  $k$ -th component;  $\bar{v}$  is the mean mass velocity determined from the relation

$$\bar{v} = \frac{1}{\rho} \sum_k \rho_k \vec{v}_k \quad (1-2-5)$$

where  $\rho_k$  is the concentration or density of the  $k$ -th component, which is related to the density  $\rho$  of the mixture by

$$\rho = \sum_k \rho_k \quad (1-2-6)$$

Assuming  $\rho_k = C$ , from equation (1-2-3) we obtain

$$\frac{\partial \rho_k}{\partial \tau} + \text{div}(\rho_k \bar{v}) = -\text{div} \vec{j}_k + \dot{I}_{vk} \quad (1-2-7)$$

Summation over all the  $k$ -th components yields

$$\frac{\partial \rho}{\partial \tau} + \text{div} \rho \bar{v} = 0 \quad (1-2-8)$$

since

$$\sum_k \vec{j}_k = 0; \quad \sum_k \dot{I}_{vk} = 0 \quad (1-2-9)$$

Using the relation

$$\text{div} \rho \bar{v} = \bar{v} \cdot \text{grad} \rho + \rho \text{div} \bar{v} \quad (1-2-10)$$

the transfer equation (1-2-8) may be written as

$$\frac{\partial \rho}{\partial \tau} + \bar{v} \cdot \text{grad} \rho = -\rho \text{div} \bar{v} \quad (1-2-11)$$

Having denoted the total derivative by

$$\frac{d}{d\tau} = \frac{\partial}{\partial \tau} + \bar{v} \cdot \vec{\nabla}$$

the equation (1-2-11) may be rewritten in the form

$$\frac{d\rho}{d\tau} = -\rho \text{div} \bar{v} \quad (1-2-12)$$

If the fluid is incompressible  $\left(\rho = \text{const}; \frac{d\rho}{d\tau} = 0\right)$ , then

$$\text{div} \bar{v} = 0 \quad (1-2-13)$$

Using the relation

$$\frac{\partial b\rho}{\partial \tau} + \text{div} (b\rho \vec{v}) = \rho \frac{db}{d\tau}$$

the mass transfer equation (1-2-7) for the  $k$ -th component may be rewritten as

$$\rho \frac{d\rho_{k0}}{d\tau} = - \text{div} \vec{j}_k + \dot{I}_{vk} \quad (1-2-14)$$

where  $\rho_{k0}$  is the relative concentration ( $\rho_{k0} = \rho_k/\rho$ ).

The molecular mass flow of the  $k$ -th component is defined by the diffusion law. For a gradient of relative concentration  $\vec{\nabla} \rho_{k0}$ , pressure  $\vec{\nabla} p$ , and temperature  $\vec{\nabla} T$ , the diffusion mass flow  $\vec{j}_k$  of the  $k$ -th component equals [1-2]

$$\vec{j}_k = - D\rho(\vec{\nabla} \rho_{k0} + k_T \vec{\nabla} \log_e T + k_p \vec{\nabla} \log_e p) \quad (1-2-15)$$

where the first term in parentheses is the mass transfer due to concentration diffusion or ordinary diffusion, the second term characterizes thermal diffusion and the last term accounts for the pressure diffusion effect.

The dimensionless coefficients of thermal diffusion  $k_T$  and pressure diffusion  $k_p$  are expressed in terms of the derivatives of chemical potential with respect to temperature and pressure, respectively.

Substituting (1-2-15) into equation (1-2-14), we get the differential mass-transfer equation

$$\rho \frac{d\rho_{k0}}{d\tau} = \text{div}[D\rho(\vec{\nabla} \rho_{k0} + k_T \vec{\nabla} \log_e T + k_p \vec{\nabla} \log_e p)] + \dot{I}_{vk} \quad (1-2-16)$$

If the diffusion coefficient and density are independent of the coordinates, then from equation (1-2-16) we obtain

$$\frac{d\rho_{k0}}{d\tau} = D \left( \nabla^2 \rho_{k0} + \frac{k_T}{T} \nabla^2 T + \frac{k_p}{p} \nabla^2 p \right) + \frac{\dot{I}_{vk}}{\rho} \quad (1-2-17)$$

where  $k_T/T$  and  $k_p/p$  are also assumed to be independent of the coordinates.

### (c) Momentum Transfer Equation

The momentum (or impulse) transfer equation is derived from the equilibrium equation of the momentum per unit mass of fluid

$$\frac{d}{d\tau} \int_{V_m(\tau)} \rho \vec{v} dV = \int_{V_m(\tau)} \rho \vec{F} dV + \int_{S_m(\tau)} \vec{\sigma} d\vec{S} \quad (1-2-18)$$



where  $\vec{\sigma}$  is the density of surface forces (the surface forces are calculated per unit surface area). The first term in (1-2-18) expresses the total rate of momentum change. It is equal to the sum of external  $\vec{F}$  and surface  $\vec{\sigma}$  forces acting within the volume  $V_m(\tau)$ .

The vector of surface forces  $\vec{\sigma}$  is defined in terms of unit vector  $\vec{n}^1$  directed along normal  $\vec{n}$  to the surface  $S$  and the tensor of internal stresses  $\Pi$  by the relation

$$\vec{\sigma} = \vec{n}^1 \Pi \quad (1-2-19)$$

Using the Gauss-Ostrogradsky theorem, we can obtain the Cauchy equation of motion from equation (1-2-18)

$$\rho \frac{d\vec{v}}{d\tau} = \rho \vec{F} + \text{div } \Pi \quad (1-2-20)$$

The pressure tensor  $P$  is introduced as a negative stress tensor:

$$P = -\Pi \quad (1-2-21)$$

Then instead of (1-2-20) we have

$$\rho \frac{d\vec{v}}{d\tau} = -\text{div } P + \rho \vec{F} \quad (1-2-22)$$

or

$$\frac{\partial \rho \vec{v}}{\partial \tau} + \text{div } (P + \rho \vec{v} \vec{v}) = \rho \vec{F} \quad (1-2-23)$$

where

$$\rho \vec{F} = \sum_k \rho_k \vec{F}_k$$

Equation (1-2-23) may be directly obtained from equation (1-2-3) by assuming that momentum ( $C = \rho \vec{v}$ ) is the transferred quantity. Then the pressure tensor  $P$  is a diffusion (molecular) flow of the quantity, and motion is induced by external forces ( $\dot{I}_V = \rho \vec{F}$ ).

The pressure tensor  $P$  may be divided into two tensor components, namely, a mechanical pressure tensor  $P^e$  (equilibrium component) and a viscous pressure tensor  $P^v$  (nonequilibrium component)

$$P = P^e + P^v \quad (1-2-24)$$

If the liquid is isotropic, then at the equilibrium state  $P^v = 0$ , i.e.

$$P = P^e = p\delta = \begin{vmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{vmatrix} \quad (1-2-25)$$

where  $\delta$  is a unit tensor and  $p$  is the hydrodynamic pressure.

Relation (1-2-25) may be written as:

$$P_{ii} = P_{ii}^e = p, \quad i = 1, 2, 3 \quad (1-2-26)$$

In this particular case of inviscid fluid ( $P^v = 0$ ), we obtain the Euler equation of motion from (1-2-22)

$$\rho \vec{v} + \overline{\nabla} p = \rho \vec{F} \quad (1-2-27)$$

where

$$\vec{v} = \frac{\partial \vec{v}}{\partial \tau} + (\vec{v} \cdot \overline{\nabla}) \vec{v} \quad (1-2-28)$$

If relation (1-1-50) is used, then equation (1-2-28) will take the form of a Lamb-Gromeki equation

$$\vec{v} = \frac{\partial \vec{v}}{\partial \tau} + \frac{1}{2} \overline{\nabla} \vec{v}^2 + [\vec{v} \times \text{curl } \vec{v}] = \vec{F} - \frac{1}{\rho} \overline{\nabla} p \quad (1-2-29)$$

For homogeneous isotropic viscous fluids the pressure tensor equals

$$P = p\delta + P^v \quad (1-2-30)$$

where  $P^v$  is a function of the velocity gradient  $\overline{\nabla} \vec{v}$ ; for Newtonian fluids  $P^v$  is a linear function of  $\overline{\nabla} \vec{v}$ .

Tensor  $P^v$  may be expressed as the sum of two tensors

$$P^v = p^v \delta + \overset{\circ}{P}^v \quad (1-2-31)$$

where  $p^v$  is one third of the spur of tensor  $P^v$

$$p^v = \frac{1}{3} P^v : \delta = \frac{1}{3} \text{tr } P^v = \frac{1}{3} P_{ii}^v \quad (1-2-32)$$

$\overset{\circ}{P}^v$  is the viscous pressure tensor with a zero spur ( $\overset{\circ}{P}^v : \delta = \overset{\circ}{P}_{ii}^v = 0$ ). Tensor  $P^v$  may be asymmetric, i.e.

$$P^v = p^v \delta + \overset{\circ}{P}^{rs} + P^{ra} = \frac{1}{3} \text{tr } P^v \delta + P^{ra} + \overset{\circ}{P}^{rs} \quad (1-2-33)$$

because the spur of the antisymmetric part of the tensor  $P^{ra}$  is equal to zero.

Equation (1-2-33) may be written in matrix form

$$\begin{vmatrix} P_{11}^r P_{21}^v P_{31}^v \\ P_{12}^r P_{22}^v P_{32}^v \\ P_{13}^r P_{23}^v P_{33}^v \end{vmatrix} = \frac{1}{3} (P_{11}^r + P_{22}^r + P_{33}^r) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$+ \begin{vmatrix} \frac{1}{3} (2P_{11}^r - P_{22}^r - P_{33}^r) & \frac{1}{2} (P_{21}^r + P_{12}^r) & \frac{1}{2} (P_{31}^r + P_{13}^r) \\ \frac{1}{2} (P_{12}^v + P_{21}^v) & \frac{1}{3} (2P_{22}^r - P_{11}^r - P_{33}^r) & \frac{1}{2} (P_{32}^r + P_{23}^r) \\ \frac{1}{2} (P_{13}^v + P_{31}^v) & \frac{1}{2} (P_{23}^r + P_{32}^r) & \frac{1}{3} (2P_{33}^r - P_{11}^r - P_{22}^r) \end{vmatrix}$$

$$+ \begin{vmatrix} 0 & \frac{1}{2} (P_{21}^v - P_{12}^v) & \frac{1}{2} (P_{31}^v - P_{13}^v) \\ \frac{1}{2} (P_{12}^r - P_{21}^r) & 0 & \frac{1}{2} (P_{32}^v - P_{23}^v) \\ \frac{1}{2} (P_{13}^v - P_{31}^v) & \frac{1}{2} (P_{23}^r - P_{32}^r) & 0 \end{vmatrix} \quad (1-2-34)$$

since

$$\begin{aligned} P_{ij}^{vs} &= \frac{1}{2} (P_{ij} + P_{ji}); & P_{ij}^{ra} &= \frac{1}{2} (P_{ij} - P_{ji}); & \overset{\circ}{P}_{ij}^{rs} &= \frac{1}{2} (P_{ij} + P_{ji}) \\ &+ \frac{1}{3} \delta_{ij} P_{kk}, & i, j, k &= 1, 2, 3, \end{aligned} \quad (1-2-35)$$

By keeping in view the relation (1-2-30), the pressure tensor  $P$  in a general case equals

$$P = (p + p^v) \delta + \overset{\circ}{P}^{vs} + P^{ra} \quad (1-2-36)$$

where  $p^v$  describes a viscous pressure tensor when the fluid is compressed from all sides. It is characterized by  $\text{div } \vec{v}$

$$p^v = -\eta_v \text{div } \vec{v} \quad (1-2-37)$$

where  $\eta_v$  is the coefficient of bulk viscosity.

The viscous pressure due to shear deformation\* is represented by

$$\overset{\circ}{\mathbf{P}}^{\nu s} = -2\eta(\overline{\nabla} \cdot \vec{v})^s = -2\eta(\overline{\nabla} \vec{v})^s + \frac{2}{3}\eta \operatorname{div} \vec{v} \delta \quad (1-2-38)$$

since

$$(\overline{\nabla} \vec{v})^s = (\overline{\nabla} \vec{v})^s + \frac{1}{3} \operatorname{div} \vec{v} \delta \quad (1-2-39)$$

where  $\eta$  is the coefficient of shear viscosity.

The antisymmetrical component of the viscous pressure tensor is equal to the axial vector, which depends upon  $\operatorname{rot} v$  and  $\Omega$ , i.e. it is a linear function of the velocity gradient

$$\overset{\circ}{\mathbf{P}}^{\nu a} = -\eta_r(\overline{\nabla} \times \vec{v} - 2\Omega) \quad (1-2-40)$$

where  $\Omega$  is the intrinsic angular velocity of particle rotation,  $\eta_r$  is the rotation viscosity.

In [1-7] it is established that the antisymmetric component of the viscous pressure tensor is caused by the internal micromomenta of surface forces.

Substitution of the following pressure tensor expression

$$\mathbf{P} = p\delta - \left(\eta_\nu - \frac{2}{3}\eta\right) \operatorname{div} \vec{v} \delta - 2\eta(\overline{\nabla} \vec{v})^s - \eta_r(\overline{\nabla} \times \vec{v} - 2\vec{\Omega}) \quad (1-2-41)$$

into (1-2-22), yields the differential equation of motion

$$\begin{aligned} \rho \frac{d\vec{v}}{d\tau} = & \rho \vec{F} - \overline{\nabla} p + \eta \nabla^2 \vec{v} + \left(\frac{1}{3}\eta + \eta_\nu\right) \overline{\nabla} \operatorname{div} \vec{v} \\ & + \eta_r \operatorname{curl} (2\vec{\Omega} - \operatorname{curl} \vec{v}) \end{aligned} \quad (1-2-42)$$

The following relations have been used in the above derivation:

$$\operatorname{div} p\delta \equiv \overline{\nabla} (p\delta) = \sum_{i,j} \vec{n}_i^1 \frac{\partial}{\partial x_j} (p\delta_{ij}) = \sum_i \vec{n}_i^1 \frac{\partial p}{\partial x_i} = \overline{\nabla} p; i, j=1,2,3 \quad (1-2-43)$$

where

$$\delta_{ij} = 0; \quad i \neq j; \quad \delta_{ij} = 1; \quad i = j; \quad 2(\overline{\nabla} \vec{v})^s = [\overline{\nabla} \vec{v} + (\overline{\nabla} \vec{v})^T] \quad (1-2-44)$$

$$\operatorname{div} (\overline{\nabla} \vec{v})^T = \overline{\nabla} (\operatorname{div} \vec{v}) \quad (1-2-45)$$

\* According to the law of internal friction, friction stress is a linear function of the velocity gradient.

It is easy to verify the validity of the last equality. The component  $Q_{ijk}$  of the third-order tensor equals:

$$Q_{ijk} = \overline{\nabla}_i \overline{\nabla}_j \bar{v}_k - \overline{\nabla}_j \overline{\nabla}_i \bar{v}_k = \frac{\partial^2 v_k}{\partial x_i \partial x_j} - \frac{\partial^2 v_k}{\partial x_j \partial x_i} = 0 \quad (1-2-46)$$

Hence,  $Q_{ijk} = 0$  for any curvilinear coordinate system. The result of repeated covariant differentiation is therefore independent of the differentiation order for Euclidian space. Equation (1-2-42) was originally derived by de Groot [1-5].

If the viscous stress tensor is symmetrical, i.e.  $P^v = (P^v)^T$ , then equation (1-2-42) differs from the general Navier-Stokes equation

$$\rho \frac{d\bar{v}}{d\tau} = \rho \bar{F} - \overline{\nabla} p + \eta \overline{\nabla}^2 \bar{v} + \left( \frac{1}{3} \eta + \eta_v \right) \text{grad div } \bar{v} \quad (1-2-47)$$

by the term  $\eta_v \text{curl} (2\bar{\Omega} - \text{curl } \bar{v})$ , which describes momentum transfer by rotational diffusion.

For incompressible fluids ( $\dot{\rho} = 0$ ), the last term of equation (1-2-47) is equal to zero ( $\text{div } \bar{v} = 0$ ), then equation (1-2-27) is generated from equation (1-2-47).

#### (d) Moment-of-Momentum Transfer Equation

The system of differential equations of motion of continuum is not complete. Other types of universal equations, independent of the properties of moving fluid, may be obtained. If in the classical equation of mechanics

$$m \frac{d\bar{v}}{d\tau} = \bar{F} \quad (1-2-48)$$

both sides are multiplied by position-vector  $\bar{r}$ , we find a moment-of-momentum equation for a material point

$$\frac{d\bar{K}}{d\tau} = \bar{r} \times \bar{F} \quad (1-2-49)$$

where

$$\bar{K} = [\bar{r} \times m\bar{v}] \quad (1-2-50)$$

Therefore, equation (1-2-49) is a corollary of the second law of Newtonian mechanics.

The moment of momentum of volume  $V$  of a continuum is

$$\bar{K} = \int_V [\bar{r} \times \bar{v}] \rho dV \quad (1-2-51)$$

For certain model fluids, in particular when they are affected by external fields, the density of intrinsic or internal angular moments of momentum  $\vec{K}^i$  should be taken into account. Then, for the general case we have

$$\vec{K} = \int_V [\vec{r} \times \vec{v}] \rho dV + \int_V \vec{K}^i \rho dV \quad (1-2-52)$$

All atoms and molecules possess intrinsic angular moments of momentum  $\vec{K}^i$ , but because of random motion the sum of these moments equals zero. But, under certain conditions, i.e. when an electromagnetic field is applied and the fluid has rheological properties, the sum of internal moments will be different from zero. In classical mechanics of continuum the internal angular moments of momentum  $\vec{K}^i$  are not taken into consideration.

If there are internal angular moments of momentum  $\vec{K}^i$ , it is necessary to allow for the existence of distributed surface and body couples acting on a particle of the continuum.

Further  $\vec{F}^i$  and  $\vec{\sigma}^i$  will be used to denote the moments of body forces calculated per unit mass and of surface forces acting per unit area. The moment-of-momentum equation for a finite volume of continuum will then take the form

$$\begin{aligned} \frac{d}{d\tau} \left[ \int_V (\vec{r} \times \vec{v}) \rho dV + \int_V \vec{K}^i \rho dV \right] &= \int_V [(\vec{r} \times \vec{F}) + \vec{F}^i] \rho dV \\ &+ \int_V [(\vec{r} \times \vec{\sigma}) + \vec{\sigma}^i] d\vec{S} \end{aligned} \quad (1-2-53)$$

The total derivative with respect to time of the moment of momentum of continuum volume  $V$ , taking into account the intrinsic moments, is equal to the sum of moments of external mass and surface forces acting on this volume, and to the sum of internal moments of distributed mass and surface forces. By using the internal stress tensor  $\Pi$  with reference to (1-2-19) instead of surface forces, and on replacing the stress tensor  $\Pi$  by pressure tensor  $P$  ( $P = -\Pi$ ), we obtain equation (1-2-50) in the absence of external ( $\vec{F} = 0$ ) and internal forces and moments ( $\vec{F}^i = \vec{K}^i = 0$ ) in the following form:

$$\dot{\vec{K}}^e + [\vec{r} \times \text{div } P] = 0 \quad (1-2-54)$$

where  $\vec{K}^e$  is the external angular momentum or external moment of momentum; the dot above the quantity denotes a time derivative ( $\dot{\vec{K}}^e = d\vec{K}^e/d\tau$ ).

Using the relation

$$[\vec{r} \times \text{div } \mathbf{P}] = \text{div } (\vec{r} \times \mathbf{P}) + \mathbf{P}^T - \mathbf{P} \quad (1-2-55)$$

and also

$$\mathbf{P} - \mathbf{P}^T = 2\vec{P}^{va} \quad (1-2-56)$$

where  $\vec{P}^{va}$  is an axial vector equivalent to the antisymmetrical part of the pressure tensor  $\mathbf{P}$ . Then from (1-2-54) we get

$$\rho \vec{K}^e + \text{div } [\vec{r} \times \mathbf{P}] = 2\vec{P}^{va} \quad (1-2-57)$$

Equation (1-2-57) may be rewritten in the form

$$\frac{\partial \rho \vec{K}^e}{\partial \tau} + \text{div } [\vec{r} \times \mathbf{P} + \rho \vec{K}^e \vec{v}] = 2\vec{P}^{va} \quad (1-2-58)$$

Equation (1-2-58) is similar to the transfer equation for any quantity (1-2-3). The axial vector  $\vec{P}^{va}$  acts as a source of angular momenta, the quantity  $(\vec{r} \times \mathbf{P})$  is a molecular component of angular momentum transfer, and  $\rho \vec{K}^e \vec{v}$  defines the convective component of transfer.

In case of a symmetric viscous stress tensor ( $\mathbf{P} = \mathbf{P}^T$ ), the axial vector  $\vec{P}^{va}$  equals zero.

If the internal angular moments of momentum  $K^i$  are accounted for, it may be assumed that

$$\vec{K} = \vec{K}^e + \vec{K}^i \quad (1-2-59)$$

where  $\vec{K}^i = I\Omega$ ;  $\Omega$  is the "intrinsic" angular velocity\* of fluid particle rotation;  $I$  is the moment of inertia of fluid particles. If the internal angular momentum tensor is denoted by  $\mathcal{P}$ , then for a conservative system [1-6]

$$\rho \vec{K} + \text{div } ([\vec{r} \times \mathbf{P}] + \mathcal{P}) = 0 \quad (1-2-60)$$

where  $\mathcal{P}$  is the third-order polar tensor.

If use is made of (1-2-57), then from (1-2-60) it follows that

$$\rho \vec{K}^i + \text{div } \mathcal{P} = -2\vec{P}^{va} \quad (1-2-61)$$

Thus, the asymmetry of the stress tensor is caused by micromoments. If there are body-distributed moments due to internal forces  $\vec{F}^i$ , then they are also responsible for the asymmetry of stress tensor  $\Pi$  ( $\Pi = -\mathbf{P}$ ).

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\* Angular velocity is a quantity, averaged over a physically small volume, which characterizes internal rotation in a volume, different from the motion of the volume as a whole.

Given an external force  $\vec{F}$ , equation (1-2-60) will take the form:

$$\rho \vec{K} + \text{div} ([\vec{r} \times \mathbf{P}] + \mathcal{P}) = \rho (\vec{r} \times \vec{F}) \quad (1-2-62)$$

*(e) Energy Transfer Equation*

First we shall consider the kinetic energy transfer equation. To do so equation (1-2-22) will be used in the form

$$\rho \vec{v} = -\text{div} \mathbf{P} + \rho \vec{F} \quad (1-2-63)$$

Multiplying both sides of equality (1-2-63) by  $\vec{v}$  and using the relation

$$\vec{v} \text{ div } \mathbf{P} = \text{div} (\mathbf{P} \cdot \vec{v}) + (\mathbf{P}^T : \vec{\nabla} \vec{v}) \quad (1-2-64)$$

we get a kinetic energy transfer equation in the vectorial form

$$\rho \dot{e}_k = -\text{div} (\mathbf{P} \cdot \vec{v}) + (\mathbf{P}^T : \vec{\nabla} \vec{v}) + \rho \vec{F} \cdot \vec{v} \quad (1-2-65)$$

where  $e_k = \frac{1}{2} v^2$  is the kinetic energy of a unit mass of fluid.

This equation may be written as

$$\frac{\partial \rho e_k}{\partial \tau} + \text{div} (\mathbf{P} \cdot \vec{v} + \rho e_k \vec{v}) = \text{tr} (\mathbf{P} \vec{\nabla} \vec{v}) + \rho (\vec{F} \cdot \vec{v}) \quad (1-2-66)$$

Thus, diffusion transfer of kinetic energy is characterized by  $(\mathbf{P} \cdot \vec{v})$ , and the work done by external forces  $(\vec{F} \cdot \vec{v})$  and internal forces  $\text{tr} (\mathbf{P} \vec{\nabla} \vec{v})$  acts as the energy source. We shall discuss the latter in more detail.

The velocity gradient  $\vec{\nabla} \vec{v}$  is a dyad (a second-order tensor); it may be expressed as

$$\vec{\nabla} \vec{v} = \frac{1}{3} \text{tr} (\vec{\nabla} \vec{v}) \delta + (\vec{\nabla} \vec{v})^s + (\vec{\nabla} \vec{v})^a \quad (1-2-67)$$

where

$$\text{tr} (\vec{\nabla} \vec{v}) = (\vec{\nabla} \vec{v} : \delta) = \text{div } \vec{v} \quad (1-2-68)$$

is the trace of tensor  $\vec{\nabla} \vec{v}$ . The subsequent terms are

$$(\vec{\nabla} \vec{v})^s = \frac{1}{2} [\vec{\nabla} \vec{v} + (\vec{\nabla} \vec{v})^T] - \frac{1}{3} \text{div } \vec{v} \delta \quad (1-2-69)$$

$$(\vec{\nabla} \vec{v})^s = \frac{1}{2} [\vec{\nabla} \vec{v} + (\vec{\nabla} \vec{v})^T] \quad (1-2-70)$$

$$(\vec{\nabla} \vec{v})^a = \frac{1}{2} [\vec{\nabla} \vec{v} - (\vec{\nabla} \vec{v})^T] \quad (1-2-71)$$



Furthermore,  $\mathbf{P} = p\delta + \mathbf{P}^v$ , and  $\mathbf{P}^v$  is determined from (1-2-36), therefore

$$(\mathbf{P}^T : \vec{\nabla} \vec{v}) = (p + p^v) \operatorname{div} \vec{v} + [\overset{\circ}{\mathbf{P}}^{vs} : (\vec{\nabla} \vec{v})^s] - [\mathbf{P}^{va} : (\vec{\nabla} \vec{v})^a] \quad (1-2-72)$$

In the derivation the relations  $\mathbf{P}^{vs} = (\mathbf{P}^{vs})^T$ ;  $\mathbf{P}^{va} = -(\mathbf{P}^{va})^T$  were used. In hydrodynamics the rotational vector  $\vec{\Omega}$  is defined by

$$\frac{1}{2} [\vec{\nabla} \times \vec{v}] = (\vec{\nabla} \vec{v})^a = \vec{\Omega} \quad (1-2-73)$$

Then we may write

$$(\mathbf{P}^{va} : \vec{\nabla} \vec{v}^a) = -\vec{P}^{va} (\vec{\nabla} \times \vec{v}) = -2\vec{\Omega} \cdot \vec{P}^{va} \quad (1-2-74)$$

Thus, kinetic energy transfer is induced by the rotation of fluid, its compressibility and the variation in temperature due to the dissipation of mechanical energy. When a fluid rotates its kinetic energy is composed of energies of translational  $\frac{1}{2} v^2$  and rotational  $\frac{1}{2} J\Omega^2$  motions.

$$e_k = \frac{1}{2} v^2 + \frac{1}{2} J\Omega^2 \quad (1-2-75)$$

where  $J$  is the moment of inertia of a unit mass.

Equation (1-2-65) takes the form

$$\begin{aligned} \rho[(\vec{v} \cdot \vec{v}) + J(\vec{\Omega} \cdot \vec{\Omega})] + \operatorname{div} (\mathbf{P} \cdot \vec{v}) &= \rho \vec{F} \cdot \vec{v} + (p + p^v) \operatorname{div} \vec{v} \\ &+ [\overset{\circ}{\mathbf{P}}^{vs} : (\vec{\nabla} \vec{v})^s] + [\vec{P}^{va} (\vec{\nabla} \times \vec{v} - 2\vec{\Omega})] \end{aligned} \quad (1-2-76)$$

for which the following equation has been used

$$\Omega : \mathbf{P}^{va} = -2\Omega \cdot \mathbf{P}^{va} \quad (1-2-77)$$

Given a field of conservative external forces, a scalar potential  $\varphi_k$  with  $k$  components is introduced in accordance with

$$\vec{F}_k = -\vec{\nabla} \varphi_k; \quad \frac{\partial \varphi_k}{\partial \tau} = 0 \quad (1-2-78)$$

Then the fluid potential equals

$$\varphi = \frac{1}{\rho} \sum_{k=1}^n \rho_k \varphi_k = \sum_{k=1}^n \rho_{k0} \varphi_k \quad (1-2-79)$$

Consequently,

$$\vec{F} = \frac{1}{\rho} \sum_{k=1}^n \rho_k \vec{F}_k = -\vec{\nabla} \varphi \quad (1-2-80)$$

Then the total mechanical energy of a unit mass is

$$e_m = \frac{1}{2} v^2 + \frac{1}{2} J\Omega^2 + \varphi \quad (1-2-81)$$

and the mechanical energy transfer equation will take the form

$$\rho \dot{e}_m + \text{div } j_{e_m} = \dot{I}_{e_m} \quad (1-2-82)$$

where  $\vec{j}_{e_m}$  is the diffusional flux of mechanical energy

$$\vec{j}_{e_m} = (\mathbf{P} \cdot \vec{v}) + \sum_{k=1}^n \varphi_k \vec{j}_k \quad (1-2-83)$$

and  $\dot{I}_{e_m}$  is the source of mechanical energy.

$$\begin{aligned} \dot{I}_{e_m} = & (p + p^v) \text{div } \vec{v} + [\vec{P}^{vs} : (\vec{\nabla} \vec{v})^o]^s + \vec{P}^{va} (\vec{\nabla} \times \vec{v} - 2\Omega) \\ & + \sum_{k=1}^n \sum_{\gamma=1}^{\Gamma} \varphi_k v_{k\gamma} j_\gamma - \sum_{k=1}^n \vec{j}_k \vec{F}_k \end{aligned} \quad (1-2-84)$$

where  $v_{k\gamma}$  is the stoichiometric coefficient of the chemical reaction  $\gamma$  ( $\gamma = 1, 2, 3, \dots, \Gamma$ );  $j_\gamma$  is the mass flux for the chemical reaction  $\gamma$ .

All terms in the expression for the source  $\dot{I}_{e_m}$  represent internal sources, except for the last one, which represents the work done by external forces.

The mechanical energy transfer equation may be written as

$$\frac{\partial \rho e_m}{\partial \tau} + \text{div} (\vec{j}_{e_m} + \rho e_m \vec{v}) = \dot{I}_{e_m} \quad (1-2-85)$$

We will conclude this section by considering the transfer of internal energy  $u$ .

The unit mass energy  $e$  is equal to

$$e = e_m + u \quad (1-2-86)$$

From the law of conservation of energy

$$\dot{e} + \text{div } \vec{j}_e = 0 \quad (1-2-87)$$

where  $\vec{j}_e$  is the diffusional energy flux.

Using relation

$$\rho(\dot{e} - \dot{e}_m) + \text{div}(\vec{j}_e - \vec{j}_{e_m}) = -\dot{I}_{e_m} \quad (1-2-88)$$

we get

$$\rho \dot{u} + \text{div } \vec{j}_u = \dot{I}_u \quad (1-2-89)$$

where

$$\rho \dot{u} = \frac{\partial \rho u}{\partial \tau} + \text{div } \rho u \vec{v} \quad (1-2-90)$$

$$\begin{aligned} \dot{I}_u = -\dot{I}_{em} = & -(p + p^v) \text{div } \vec{v} - [\vec{P}^{vs} : (\vec{\nabla} \vec{v})] - \vec{P}^{va} \cdot (\vec{\nabla} \times \vec{v} - 2\Omega) \\ & + \sum_{k=1}^n \vec{j}_k \cdot \vec{F}_k \end{aligned} \quad (1-2-91)$$

If the specific enthalpy is denoted by  $h$ , then in the absence of chemical reactions and for a symmetrical internal stress tensor the differential equation of enthalpy transfer will take the form

$$\rho \frac{dh}{d\tau} + \text{div } \vec{j}_h = \frac{dp}{d\tau} - (P^v : \vec{\nabla} \vec{v}) + \sum_k \vec{j}_k \cdot \vec{F}_k \quad (1-2-92)$$

where  $j_h$  is the enthalpy flux.

Using transfer equation (1-2-16) and assuming the specific enthalpy flux  $j_h$  to be equal to the heat flux  $\vec{j}_q$  ( $\vec{j}_h = \vec{j}_q$ ), from equation (1-2-87) we get the Fourier-Kirchhoff differential equation [1-2]

$$\begin{aligned} c_p \rho \frac{dT}{d\tau} = & \text{div } (\lambda \vec{\nabla} T + Q^* D \rho \vec{\nabla} \rho_{k0}) + \sum_k h_k \dot{I}_{V_k} + \frac{dp}{d\tau} + \eta \Phi_v \\ & + \sum_k c_{pk} \vec{j}_k \cdot \vec{\nabla} T + \sum_k \vec{j}_k \cdot \vec{F}_k \end{aligned} \quad (1-2-93)$$

where  $c_p$  is the specific heat at constant pressure

$$c_p = \frac{dh}{dT} = \sum_k c_{pk} \rho_{k0} = \sum_k \frac{dh_k}{dT} \rho_{k0} \quad (1-2-94)$$

where  $c_{pk}$  is the specific heat of the  $k$ -th component at constant pressure,  $h_k$  is the specific enthalpy of the  $k$ -th component,  $\lambda$  is the thermal conductivity coefficient, and

$$\Phi_v = -\frac{1}{\eta} (P^v : \vec{\nabla} \vec{v}) = \frac{1}{2} \sum_i \sum_j \left[ \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - \frac{2}{3} \text{div } \vec{v} \delta_{ij} \right]^2 \quad (1-2-95)$$

where  $\Phi_v$  is the Rayleigh dissipative function.

In the derivation of equation (1-2-88) the pressure diffusion effect was neglected and  $\eta_v$  was taken equal to zero (bulk viscosity equals zero).

As follows from Truesdell's studies [1-8], the range of validity of Navier-Stokes equations for a rarefied gas is defined by the relation  $\text{Tr} < 1$ , where  $\text{Tr}$  is the Truesdell number

$$\text{Tr} = 2\eta\varepsilon/p \quad (1-2-96)$$

If the Maxwellian molecular distribution law is taken into consideration, then the Truesdell number is proportional to the product of the Knudsen number ( $Kn$ ) and the Mach number ( $M$ )

$$Tr = \frac{3}{2} Kn M \quad (1-2-97)$$

Thus, for a rarefied gas flowing at high velocity ( $M \gg 1$ ) the Truesdell number may exceed unity ( $Tr > 1$ ). In this case the Navier-Stokes equation will not be applicable for the description of compressible fluid motion.

However in the case of an incompressible, but highly viscous fluid the Truesdell number for large shear rates may also exceed unity. Therefore, in this case the Navier-Stokes equation will not hold true.

Another particularity of the motion equation is that it fails to explain the vortex initiation process. The solutions of the Navier-Stokes equations can explain the disappearance of vortices rather than their formation.

It can be shown that the motion of a discrete structure fluid is governed by the generalized Navier-Stokes equation [1-8]. The discreteness of structure of a rarefied gas is determined by the fact that within the limits of a physically small volume the transfer rates of molecules are different. In other words, within the limits of a small volume, over which the microscopic quantities were averaged, the velocity of apparent motion varies. For this reason it is necessary to revise the definition of the average velocity of motion. A similar physical picture is seen in a fluid having vortex structure (the fluid consists of discrete hollow vortices). In this case the fluid velocity distribution is described by a discontinuous function.

To derive the motion equation, it is necessary to employ the averaging rule or a method of finding derivatives of discontinuous functions.

### 1-3. THERMODYNAMICS OF TRANSFER PROCESSES

Transfer of momentum, energy and mass is accompanied by an increase in the entropy of the fluid. The entropy production in convective diffusional transfer processes may characterize thermodynamic motive forces  $X$  and the corresponding fluxes  $J$ . According to the basic relation of irreversible thermodynamics, an entropy source is proportional to the sum of the products of fluxes and thermodynamic forces [1-5]

$$T\dot{I}_s = T \frac{d_i s}{d\tau} = \sum_i J_i X_i \quad (1-3-1)$$

where fluxes  $J_i$  and forces  $X_i$  may be tensors of any order.

The quantity  $d_i s/d\tau$  is the power of the entropy source  $\dot{I}_s$  ( $\dot{I}_s = d_i s/d\tau$ ).

To determine  $\dot{I}_s$ , it is necessary to know the entropy transfer equation which is derived from the internal energy transfer equation by using the Gibbs equation

$$T\dot{s} = \dot{u} + p\dot{v} - \sum_k \mu_k \dot{\rho}_{k0} \quad (1-3-2)$$

where  $\dot{v}$  is the derivative of specific volume ( $v = 1/\rho$ ), and  $\mu_k$  is the chemical potential of the  $k$ -th component.

This equation will take the form

$$T\rho\dot{s} + \text{div } \vec{j}_q - \sum_k \mu_k \text{div } \vec{j}_k = \sum_k \sum_\gamma \mu_k v_{k\gamma} J_\gamma + \sum_k \vec{j}_k \cdot \vec{F}_k - \text{tr}(\mathbf{P} \vec{\nabla} \vec{v}) + 2\vec{\Omega} \cdot \vec{P}^{va} \quad (1-3-3)$$

Using relations

$$\text{div } \vec{j}_q = T \text{div } (\vec{j}_q/T) - T \vec{j}_q \cdot \vec{\nabla} \left( \frac{1}{T} \right) \quad (1-3-4)$$

$$\mu_k \text{div } \vec{j}_k = T \text{div } \frac{\vec{\mu}_k \vec{j}_k}{T} - T \vec{j}_k \cdot \vec{\nabla} \frac{\mu_k}{T} \quad (1-3-5)$$

we can rewrite (1-3-3) in the following form

$$\rho\dot{s} + \text{div } \vec{j}_s = \frac{1}{T} \dot{I}_s \quad (1-3-6)$$

where  $\vec{j}_s$  is the diffusional entropy flux

$$\vec{j}_s = \frac{1}{T} (\vec{j}_q - \sum_k \mu_k \vec{j}_k) \quad (1-3-7)$$

$$\dot{I}_s = \sum_\gamma J_\gamma A_\gamma + \vec{j}_q \cdot \vec{X}_q + \sum_k \vec{j}_k \cdot \vec{X}_k + p^v \vec{X}_v + \vec{P}^{vs} \cdot \vec{X}_v^s + \vec{P}^{va} \cdot \vec{X}_v^a \quad (1-3-8)$$

here  $A_\gamma$  is the chemical affinity of the  $\gamma$  reaction

$$A_\gamma = - \sum_k \mu_k v_{k\gamma}; \quad \gamma = 1, 2, 3 \dots \Gamma \quad (1-3-9)$$

From relation (1-3-1) we find that the thermodynamic force of reaction  $\gamma$  is the chemical affinity  $A_\gamma$  (a scalar quantity), and the mass flux in the chemical reaction is the reaction rate (a scalar quantity)

$$X_\gamma \equiv A_\gamma; \quad J_\gamma \equiv j_\gamma \quad (1-3-10)$$

The polar vector (a first-order tensor)  $\vec{X}_q$  is the thermodynamic force of heat conduction, the flux of which equals  $\vec{j}_q$

$$\vec{X}_q = -\frac{1}{T} \vec{\nabla} T = -\vec{\nabla} \log_e T; \quad \vec{J}_q \equiv \vec{j}_q \quad (1-3-11)$$

The vector  $\vec{X}_k$

$$\vec{X}_k = \vec{F}_k - T \vec{\nabla} \left( \frac{\mu_k}{T} \right); \quad \vec{J}_k \equiv \vec{j}_k \quad (1-3-12)$$

is a thermodynamic force of diffusion.

The scalar quantity  $X_v$  is a thermodynamic force of momentum transfer when fluid is compressed from all sides with flux  $p^v$

$$X_v \equiv -\operatorname{div} \vec{v}; \quad J_v \equiv p^v \quad (1-3-13)$$

The viscous tensor force  $\overset{\circ}{X}_v^s$  corresponding to the symmetrical part of the viscous pressure tensor with a zero spur  $\overset{\circ}{P}_v^s$  results in momentum transfer during shear deformation

$$\overset{\circ}{X}_v^s = -(\vec{\nabla} \vec{v})^s; \quad J_v \equiv \overset{\circ}{P}^{vs} \quad (1-3-14)$$

The axial vector  $\vec{X}_v^a$  is a thermodynamic force of momentum transfer with rotational viscosity

$$\vec{X}_v^a = -(\vec{\nabla} \times \vec{v} - 2\vec{\Omega}); \quad \vec{J}_v^a = \vec{P}^{va} \quad (1-3-15)$$

According to the basic laws of irreversible thermodynamics fluxes are defined by Onsager's reciprocal relation

$$\vec{J}_i = \sum_k L_{ik} X_k \quad (1-3-16)$$

where  $L_{ik}$  are kinetic coefficients.

For an isotropic fluid, according to the Curie principle, the combination  $\sum_k L_{ik} \vec{X}_k$  is possible only in that case when thermodynamic forces  $X_k$  are tensors of the same order or the difference between the orders is even [1-5].

Therefore, the forces of the second order can combine with themselves or with scalar forces (a scalar is a tensor of order zero), and vector forces combine with themselves and with the third order tensor forces, since a vector is a first order tensor.

Hence, for an isotropic fluid

$$J_{\gamma j} = \sum_{\gamma} L_{j\gamma} A_{\gamma} + L_j^{cv} X_v; \quad j = 1, 2, 3 \dots \Gamma \quad (1-3-17)$$

$$p^v = \sum_{\gamma} L_{\gamma}^{vc} A_{\gamma} + L^{(ss)} X_v \quad (1-3-18)$$

$$\vec{j}_q = L_{qq} \vec{X}_q + \sum_k L_{qk} \vec{X}'_k \quad (1-3-19)$$

$$\vec{j}_i = L_{iq} \vec{X}_q + \sum_k L_{ik} \vec{X}'_k; \quad i = 1, 2, 3 \dots k-1 \quad (1-3-20)$$

$$\vec{p}^{va} = L^{(aa)} \vec{X}_v^{(a)} \quad (1-3-21)$$

$$\vec{p}^{vs} = L^{(tt)} \vec{X}_v^s \quad (1-3-22)$$

The bracketed indices of kinetic coefficients  $(ss)$ ,  $(aa)$ ,  $(tt)$  indicate the nature of flux and force, namely, scalar-scalar, axial vector-axial vector, tensor-tensor.

By comparing relations (1-3-17) through (1-3-22) with the known relations of diffusion, heat conduction and viscosity, we get

$$L_{qq} = \lambda T; \quad L^{(ss)} = \eta_V; \quad L^{(tt)} = 2\eta; \quad L^{(aa)} = \eta_r \quad (1-3-23)$$

where  $\eta_r$  are the coefficients of rotational viscosity.

For an anisotropic fluid a nonlinear system of equations will take the form [1-6]

$$J_j = \sum_{r=1}^R L_j^{(ss)c} A_r + L_j^{(ss)v} X_v + L_j^{(sv)q} X_q + \sum_{k=1}^{K-1} L_{jk}^{(sv)d} X'_k + L_j^{(sa)v} X_v^a + \quad (1-3-24)$$

$$+ L_j^{(st)v} X_v^s \quad (j = 1, 2, \dots k)$$

$$p^v = \sum_{r=1}^R L_r^{(ss)v} A_r + L^{(ss)v} X_v + L^{(sv)q} X_q + \sum_{k=1}^{K-1} L_k^{(sv)d} X'_k + L^{(sa)v} X_v^a + L^{(st)v} X_v^s \quad (1-3-25)$$

$$J_q = \sum_{r=1}^R L_r^{(vs)q} A_r + L^{(vs)q} X_v + L^{(vv)q} X_q + \sum_{k=1}^{K-1} L_k^{(vv)d} X'_k + L^{(va)q} X_v^a + L^{(vt)q} X_v^s \quad (1-3-26)$$

$$J_j = \sum_{r=1}^R L_{jr}^{(vs)d} A_r + L_j^{(vs)d} X_v + L_j^{(vv)d} X_q + \sum_{k=1}^{K-1} L_{jk}^{(vv)d} X'_k + L_j^{(va)d} X_v^a + L_j^{(vt)d} X_v^s \quad (j = 1, 2, \dots k-1) \quad (1-3-27)$$

$$P^{va} = \sum_{r=1}^R L_r^{vc} A_r + L^{vv} X_v + L^{vq} \cdot X_q + \sum_{k=1}^{K-1} L_k^{vd} \cdot X'_k + L^{vv} \cdot X_v^a + L^{vv} : X_v^s \quad (1-3-28)$$

$$P^{vs} = \sum_{r=1}^R L_r^{vc} A_r + L^{vv} X_v + L^{vq} \cdot X_q + \sum_{k=1}^{K-1} L_k^{vd} \cdot X'_k + L^{vv} \cdot X_v^a + L^{vv} : X_v^s \quad (1-3-29)$$

The bracketed indices denote the following:  $(sa)$  — scalar-axial vector;  $(ta)$  — symmetrical tensor with zero trace-axial vector, etc. The kinetic coefficients together with an explanation of their tensor quantities are tabulated in Table 1-1.

Table 1-1

## Kinetic Coefficients [1-6]

Tensorial order	Polar tensor	Axial tensor
0 (scalar)	$L_{jj}^{cc}, L_j^{CV}, L_j^{VC}, L^{VV}$	
1 (vector)	$L_j^{Cq}, L_j^{Vq}, L_{jk}^{Cd}, L_k^{Vd}, L_j^{qC},$ $L_{iy}^{dC}, L_i^{qV}, L_i^{dV}$	$L_j^{CV}, L^{VV}, L_j^{VC}, L^{VV}$
2 (tensor)	$L^{qq}, L_k^{qd}, L_i^{dq}, L_{ik}^{dd}, L_j^{CV},$	$L^{qV}, L_i^{dV}, L^{Vq}, L_k^{Vd}$
3 (tensor)	$L^{VV}, L_j^{VC}, L^{VV}, L^{VV}$	
	$L^{qV}, L_i^{dV}, L^{Vq}, L_k^{Vd}$	$L^{VV}, L^{VV}$
4 (tensor)	$L^{VV}$	

From relations (1-3-24) to (1-3-29) it is inferred that in a multicomponent fluid all processes of momentum, mass and energy transfer, as well as chemical and phase transformations are interrelated and affect one another. This suggests that transfer equations significantly depend on the properties of the fluid.

The nonlinear thermodynamics of continua, evolved by Truesdell and his followers [1-8]—[1-10], is a further development of irreversible thermodynamics. It considers transfer processes in any media obeying nonlinear laws.



#### 1-4. MULTICOMPONENT MIXTURES

The differential equations of mass, momentum and energy transfer for a multicomponent system remain the same as those for one-component materials, namely

$$\frac{\partial \rho_k}{\partial \tau} = -\operatorname{div}(\vec{j}_k + \rho_k \vec{v}) + \dot{I}_{vk}; \quad k = 1, 2, 3 \dots n \quad (1-4-1)$$

$$\frac{\partial \rho \vec{v}}{\partial \tau} = -\operatorname{div}(\mathcal{P} + \rho \vec{v} \vec{v}) + \sum_k \rho_k \vec{F}_k \quad (1-4-2)$$

$$\frac{\partial \left[ \rho \left( U + \frac{1}{2} v^2 \right) \right]}{\partial \tau} = -\operatorname{div} \left[ \vec{j}_U + \left( \rho U + \rho \frac{1}{2} v^2 \right) \vec{v} + (\vec{v} \cdot \mathcal{P}) \right] \quad (1-4-3)$$

$$+ \sum_k \vec{F}_k (\rho_k \vec{v} + \vec{j}_k)$$

We use the notations

$$\vec{j}_e = \vec{j}_U + \rho_1 \left( U + \frac{1}{2} v^2 \right) \vec{v} + (\vec{v} \cdot \mathcal{P}) \quad (1-4-4)$$

$$\pi = \mathcal{P} + \rho \vec{v} \vec{v} \quad (1-4-5)$$

$$\vec{m}_k = \rho_k \vec{v} + \vec{j}_k = \rho \rho_{k0} \vec{v} + \vec{j}_k \quad (1-4-6)$$

where  $\vec{j}_e$  is the total energy flux of the fluid, which consists of the diffusion flux of internal energy  $\vec{j}_U$ , convective flux of kinetic energy of macromotion and internal energy of apparent motion, and convective transfer pressure energy and internal friction forces. Tensor  $\pi$  consists of pressure tensor  $\mathcal{P}$  and dyad  $\rho \vec{v} \vec{v}$ ; the mass flux  $\vec{m}_k$  is the sum of diffusion  $\vec{j}_k$  and convective  $\rho_k \vec{v}$  mass fluxes.

Equations (1-4-1) to (1-4-3) may then be written as

$$\frac{\partial \rho_k}{\partial \tau} = -\operatorname{div} \vec{m}_k + \dot{I}_{vk}; \quad k = 1, 2, \dots \quad (1-4-7)$$

$$\frac{\partial \rho \vec{v}}{\partial \tau} = -\operatorname{div} \pi + \sum_k^n \vec{F}_k \rho_k \quad (1-4-8)$$

$$\frac{\partial}{\partial \tau} \left( \rho U + \frac{1}{2} \rho v^2 \right) = -\operatorname{div} \vec{j}_e + \sum_k \vec{F}_k \cdot \vec{m}_k \quad (1-4-9)$$

If the external forces are conservative and independent of time, they may be substituted by suitable potentials, and the work done by external forces — by potential energy

$$\vec{F}_k = - \text{grad } \psi_k = - \vec{\nabla} \psi_k; \quad \partial \psi_k / \partial \tau = 0 \quad (1-4-10)$$

If the potential energy of a unit mass is denoted by

$$\psi = \frac{1}{\rho} \sum_k \rho_k \psi_k \quad (1-4-11)$$

then the differential equation of potential energy transfer may be written as

$$\frac{\partial \rho \psi}{\partial \tau} = - \text{div} (\rho \psi \vec{v} + \sum_k \vec{j}_k \psi_k) - \vec{v} \cdot \sum_k \rho_k \vec{F}_k - \sum_k \vec{j}_k \cdot \vec{F}_k \quad (1-4-12)$$

The partial time derivative of volume density of potential energy is equal to the sum of divergence of energy flux and the potential energy sinks. The potential energy flux has convective  $\rho \psi \vec{v}$  and conductive  $\sum_k \vec{j}_k \psi_k$  components. The potential energy sink is caused by the work done by external forces during fluid flow  $\vec{v} \cdot \sum_k \rho_k \vec{F}_k$  and the work done by external forces during diffusion  $\sum_k \vec{j}_k \cdot \vec{F}_k$ . In this case the total specific energy is

$$e = \frac{1}{2} v^2 + \psi + U \quad (1-4-13)$$

and the total energy flux is

$$\vec{J}_e = \rho e \vec{v} + \vec{j}_U + \sum_k \vec{j}_k \psi_k + (\vec{v} \cdot \mathcal{P}) \quad (1-4-14)$$

The total energy flux  $\vec{J}_e$  consists of:

1. the convective energy flux caused by fluid motion,  $\rho e \vec{v}$ ;
2. the diffusion (molecular) energy flux  $\vec{j}_U$ , due to random molecular motion;
3. the molecular potential energy flux  $\sum_k \vec{j}_k \psi_k$  due to the diffusion of matter in the external forces field;
4. the convective energy flux of molecular pressure and the forces of internal friction  $\vec{v} \cdot \mathcal{P}$ .

The differential equation of energy transfer can now be expressed in the form

$$\frac{\partial \rho e}{\partial \tau} = - \text{div } \vec{J}_e \quad (1-4-15)$$

Various forms of expressing the energy transfer equation are presented in Table 1-2. They follow from the physical essence of energy. One state-

Table 1-2

## The Energy Transfer Equation for a Multicomponent System

$$E = U + K + \psi \left( \text{for } \frac{\partial \psi}{\partial \tau} = 0 \right):$$

$$\rho \frac{dE}{d\tau} = -(\vec{\nabla} \cdot j_u) - (\vec{\nabla} \cdot [\vartheta \cdot v]) + \sum_{i=1}^n (j_i \cdot g_i)$$

$$U + K = U + \frac{1}{2} v^2:$$

$$\rho \frac{d}{d\tau} (U + K) = -(\vec{\nabla} \cdot j_u) - (\vec{\nabla} \cdot [\vartheta \cdot v]) + \sum_{i=1}^n \dot{m}_i \cdot g_i$$

$$K = \frac{1}{2} v^2:$$

$$\rho \frac{dK}{d\tau} = - (v \cdot [\vec{\nabla} \cdot \vartheta]) + \sum_{i=1}^n \rho_i (v \cdot g_i)$$

U:

$$\rho \frac{dU}{d\tau} = -(\vec{\nabla} \cdot j_u) - (\vartheta : \vec{\nabla} v) + \sum_{i=1}^n (j_i \cdot g_i)$$

H:

$$\rho \frac{dH}{d\tau} = -(\vec{\nabla} \cdot j_u) + \frac{dp}{d\tau} - (\sigma : \vec{\nabla} v) + \sum_{i=1}^n (j_i \cdot g_i)$$

 $c_p$ :

$$\begin{aligned} \rho c_p \frac{dT}{d\tau} = & -(\vec{\nabla} \cdot j_u) - (\sigma : \vec{\nabla} v) + \sum_{i=1}^n (j_i \cdot g_i) + \left( \frac{\partial \log_e V}{\partial \log_e T} \right)_{p, C_{i0}} \frac{dp}{d\tau} \\ & + \sum_{i=1}^n H_i [(\vec{\nabla} \cdot J_i) - I_i] \end{aligned}$$

 $c_v$ :

$$\begin{aligned} \rho c_v \frac{dT}{d\tau} = & -(\vec{\nabla} \cdot j_u) - (\vartheta : \vec{\nabla} v) + \sum_{i=1}^n (j_i \cdot g_i) + \left[ p - T \left( \frac{\partial p}{\partial T} \right)_{p, C_{i0}} \right] (\vec{\nabla} \cdot v) \\ & + \sum_{i=1}^n \left[ U_i + \left( p - T \frac{\partial p}{\partial T} \right) V_i \right] [(\vec{\nabla} \cdot J_i) - I_i] \end{aligned}$$

 $H_i$ :

$$\frac{\partial}{\partial \tau} \left( \sum_{i=1}^n c_i H_i \right) + \left( \vec{\nabla} \cdot \sum_{i=1}^n \mu_i H_i \right) = (\vec{\nabla} \cdot \lambda \vec{\nabla} T) + \frac{dp}{d\tau} - (\sigma : \vec{\nabla} v) + \sum_{i=1}^n (j_i \cdot g_i)$$

ment about the principle of conservation of energy is "energy can neither be created nor destroyed". Therefore if a transfer equation is written for the total energy  $e$ , the transfer equation will not contain sources or sinks. The energy transfer equation then reads as follows: a local time change in the volume energy concentration is equal to the divergence of the energy flux density. Equation (1-4-15) is the Umov equation, and the total energy transfer vector  $\vec{J}_e$  is called the Umov vector.

The differential mass flow  $\dot{j}_{dk}$  of the  $k$ -th component is defined by the relation

$$\vec{j}_{dk}^a = \rho_k(\vec{v}_k - \vec{v}^a); \quad k = 1, 2, \dots, n \quad (1-4-16)$$

where  $\vec{j}_{dk}^a$  is the diffusion flux within a frame of reference  $a$  of fluid velocity  $\vec{v}^a$ , i.e. the superscript  $a$  denotes the frame of reference. All the frames of reference should satisfy the relation

$$\vec{v}^a = \sum_{k=1}^n a_k \vec{v}_k \quad (1-4-17)$$

where  $a_k$  are the normalized weights satisfying the equality

$$\sum_{k=1}^n a_k = 1 \quad (1-4-18)$$

The most common weight factors, frames of reference and diffusion fluxes are summarized in Table 1-3.

Table 1-3

Weight Factors  $a_k$  in Various Frames of Reference

Weights $a_k$	Velocity $\vec{v}^a$ of the frame of reference	Diffusion flux $j_k$
Mass concentration $\rho_{k0}$	Mass velocity $\vec{v} = \sum_k \rho_{k0} \vec{v}_k$	$\vec{j}_k = \rho_k(\vec{v}_k - \vec{v})$
Mole concentration $C_{k0}$	Mean mole velocity $\vec{v}^M = \sum_k C_{k0} \vec{v}_k$	$\vec{j}_k^M = \rho_k(\vec{v}_k - \vec{v}^M)$
$\rho_k \vec{v}_k$	Mean bulk velocity $\vec{v}^0 = \sum_k \rho_k v_k \vec{v}_k$	$\vec{j}_k^0 = \rho_k(\vec{v}_k - \vec{v}^0)$
$\tilde{n}_{ki}$	Mean velocity of the $i$ -th component $\vec{v}_i = \sum_k \tilde{n}_{ki} \vec{v}_k$	$\vec{j}_k^i = \rho_k(\vec{v}_k - \vec{v}_i)$

Note:  $\rho_{k0}$  is the mass fraction,  $\rho_{k0} = \rho_k/\rho$ ;  $C_{k0} = C_k/C$  is the mole fraction.

The mass fluid velocity

$$\bar{v} = \frac{\sum_k \rho_k \bar{v}_k}{\rho} = \sum_k \rho_{k0} \bar{v}_k; \quad \rho = \sum_k \rho_k \quad (1-4-19)$$

where the relative concentration  $\rho_{k0}$  is the weight.

The mole fluid velocity

$$\bar{v}^M = \frac{\sum_k C_k \bar{v}_k}{C} = \sum_k C_{k0} \bar{v}_k; \quad C = \sum_k C_k \quad (1-4-20)$$

where the relative mole concentration is the weight.

The basic relations for a binary mixture, consisting of species  $A$  and  $B$ , are presented in Table 1-4. Mass and mole fluxes are tabulated in Table 1-5 [1-11].

From the basic relation  $\sum_k \vec{j}_k = 0$  for a binary mixture, we have

$$\vec{j}_A = -\vec{j}_B \quad (1-4-21)$$

Table 1-4

#### Basic Relations for a Binary Mixture ( $A + B$ )

$C_{A0} + C_{B0} = 1$	$\rho_{A0} + \rho_{B0} = 1$
$C_{A0}M_A + C_{B0}M_B = M$	$\rho_{A0}/M_A + \rho_{B0}/M_B = \frac{1}{M}$
$C_{A0} = \frac{\rho_{A0}/M_A}{\rho_{A0}/M_A + \rho_{B0}/M_B}$	$\rho_{A0} = \frac{C_{A0}M_A}{C_{A0}M_A + C_{B0}M_B}$
$dC_{A0} = \frac{d\rho_{A0}}{M_A M_B \left( \frac{\rho_{A0}}{M_A} + \frac{\rho_{B0}}{M_B} \right)^2}$	$d\rho_{A0} = \frac{M_A M_B dC_{A0}}{(C_{A0}M_A + C_{B0}M_B)^2}$

- Note: 1.  $\rho = \rho_A + \rho_B$  is the mixture density, g/cm<sup>3</sup>  
 2.  $\rho_A = C_A M_A$  is the concentration of component  $A$ , g/cm<sup>3</sup>  
 3.  $\rho_{A0} = \rho_A / \rho$  is the relative concentration  
 4.  $C = C_A + C_B$  is the mole density of the mixture, gmol/cm<sup>3</sup>  
 5.  $C_A = \rho_A / M$  is the mole concentration of component  $A$ , gmol/cm<sup>3</sup>  
 6.  $C_{A0} = C_A / C$  is the mole fraction  
 7.  $M = \rho / C$  is the molecular mass of the mixture

Different forms of the concentration diffusion law for a binary mixture are given in Table 1-6. In general, the diffusion flux of concentration diffusion for a multicomponent mixture is

$$\vec{j}^a = -D^{a\omega} \nabla \omega \quad (1-4-22)$$

Table 1-5

## Mass and Mole Fluxes in a Binary Mixture

Quantity	With respect to		
	fixed axes	$v$	$v^*$
Fluid velocity, cm/s	$v_A$	$v_A - v$	$v_A - v^*$
Mass flux density of component $A$ , g/(cm $\cdot$ s)	$\dot{m}_A = \rho_A v_A$	$j_A = \rho_A(v_A - v)$	$j_A^* = \rho_A(v_A - v^*)$
Mole flux density of component $A$ , gmol/(cm $^2\cdot$ s)	$\dot{\mu}_A = C_A v_A$	$J_A = C_A(v_A - v)$	$J_A^* = C_A(v_A - v^*)$
Total mass flux, g/(cm $^2\cdot$ s)	$\dot{m}_A + \dot{m}_B = \rho v$	$j_A + j_B = 0$	$j_A^* + j_B^* = \rho(v - v^*)$
Total mole flux, gmol/(cm $^2\cdot$ s)	$\dot{\mu}_A + \dot{\mu}_B = C v^*$	$J_A + J_B = C(v^* - v)$	$J_A^* + J_B^* = 0$
Flux in terms of $\dot{m}_A$ and $\dot{m}_B$	$\dot{\mu}_A = \dot{m}_A/M_A$	$j_A = \dot{m}_A - \rho_{A0}(\dot{m}_A + \dot{m}_B)$	$j_A = \dot{m}_A - C_{A0}\left(\dot{m}_A + \frac{M_A}{M_B}\dot{m}_B\right)$
Flux in terms of $\dot{\mu}_A$ and $\dot{\mu}_B$	$\dot{m}_A = \dot{\mu}_A M_A$	$J_A = \dot{\mu}_A - \rho_{A0}\left(\dot{\mu}_A + \frac{M_B}{M_A}\dot{\mu}_B\right)$	$J_A^* = \dot{\mu}_A - C_{A0}(\dot{\mu}_A + \dot{\mu}_B)$
Flux in terms of $j_A$ and $v$	$\dot{m}_A = j_A + \rho_A v$	$J_A = j_A/M_A$	$j_A^* = \frac{M_A}{M_B} j_A$
Flux in terms of $J_A$ and $v^*$	$\dot{\mu}_A = J_A^* + C_A v^*$	$J_A = \frac{M_B}{M} J_A^*$	$j_A^* = J_A^* M_A$

Table 1-6

## Equivalent Forms of the Fick Diffusion Equation for a Binary Mixture

Flux	Gradient	Diffusion equation forms
$\dot{m}_A$	$\vec{\nabla} \rho_{A0}$	$\dot{m}_A - \rho_{A0}(\dot{m}_A + \dot{m}_B) = -\rho D_{AB} \vec{\nabla} \rho_{A0}$
$\dot{\mu}_A$	$\vec{\nabla} C_{A0}$	$\dot{\mu}_A - C_{A0}(\dot{\mu}_A + \dot{\mu}_B) = -CD_{AB} \vec{\nabla} C_{A0}$
$j_A$	$\vec{\nabla} \rho_{A0}$	$j_A = -\rho D_{AB} \vec{\nabla} \rho_{A0}$
$J_A^*$	$\vec{\nabla} C_{A0}$	$J_A^* = -CD_{AB} \vec{\nabla} C_{A0}$
$j_A$	$\vec{\nabla} C_{A0}$	$j_A = -\frac{C^2}{\rho} M_A M_B D_{AB} \vec{\nabla} C_{A0}$
$J_A^*$	$\vec{\nabla} \rho_{A0}$	$J_A^* = -\left(\frac{\rho^2}{CM_A M_B}\right) D_{AB} \vec{\nabla} \rho_{A0}$
$C(v_A - v_B)$	$\vec{\nabla} C_{A0}$	$C(v_A - v_B) = -\frac{CD_{AB}}{C_{A0}C_{B0}} \vec{\nabla} C_{A0}$

where  $\vec{\nabla} \omega$  is an  $(n-1)$ -dimensional vector having components  $\vec{\nabla} \omega_1, \vec{\nabla} \omega_2, \dots, \vec{\nabla} \omega_{n-1}$ , and  $\omega_k$  is a quantity, which characterizes mixture components, for example  $\rho_k, C_k, n_k$ , etc.

The quantity  $D^{a\omega}$  is a tensor among  $(n-1)^2$  diffusion coefficients corresponding to the frame of reference velocity  $\vec{v}^a$  and composition parameters  $\omega$ . For the reference frame velocity  $\vec{v}^b$ , we obtain

$$\vec{j}^b = -D^{b\omega} \vec{\nabla} \omega \quad (1-4-23)$$

The tensor  $D^{b\omega}$  is related to  $D^{a\omega}$  by

$$D^{a\omega} = B^{ab} D^{b\omega} \quad (1-4-24)$$

where  $B^{ab}$  is a tensor independent of  $\omega$  and defined by

$$B_{kj}^{ab} = \delta_{kj} + \left(a_n \frac{b_j}{b_n} - a_j\right) \frac{\rho_{k0}}{\rho} \quad (1-4-25)$$

where  $k, j = 1, 2, \dots, n-1$ .

Therefore, for the determination of diffusion coefficients  $D^{a\omega}$  a frame of reference and parameter  $\omega$  of mixture composition are selected, then relation (1-4-22) is the basic formula. If the mixture composition is characterized by a different parameter  $\psi$ , then they are related by

$$\vec{\nabla} \omega = \frac{\partial \omega}{\partial \psi} \vec{\nabla} \psi \quad (1-4-26)$$

where  $\frac{\partial \omega}{\partial \psi}$  is the  $(n-1)$ -dimensional matrix with elements  $\frac{\partial \omega_i}{\partial \psi_j}$ .

Thus, in a general case

$$j^b = - B^{ba} D^{aw} \frac{\partial \omega}{\partial \psi} \vec{\nabla} \psi \quad (1-4-27)$$

The procedure of calculating diffusional fluxes in a multicomponent mixture is described in detail in [1-5].

Molecular (diffusion) transfer is caused by several thermodynamic forces. Diffusion processes are generally classified as follows: concentration diffusion (thermodynamic force is the concentration gradient), thermo-diffusion ( $\vec{\nabla} T$ ), pressure diffusion ( $\vec{\nabla} p$ ) and forced diffusion (due to external forces  $\vec{F}_k$ ).

If we use  $G$  and  $V$  to denote a relative free mole enthalpy and a mole volume, respectively, we have

$$j_k^{(c)} = - \frac{C^2}{\rho RT} \sum_{i=1}^n M_k M_i D_{ki} \left[ C_{i0} \sum_{\substack{j=1 \\ s \neq i, j}}^n \left( \frac{\partial G_i}{\partial C_{j0}} \right)_{T, p, C_{s0}} \vec{\nabla} \omega_{j0} \right] \quad (1-4-28)$$

$$j_k^{(p)} = - \frac{C^2}{\rho RT} \sum_{i=1}^n M_k M_i D_{ki} \left[ C_{i0} M_i \left( \frac{V_i}{M_i} - \frac{1}{\rho} \right) \vec{\nabla} p \right] \quad (1-4-29)$$

$$j_k^{(F)} = - \frac{C^2}{\rho RT} \sum_{i=1}^n M_k M_i D_{ki} \left[ C_{i0} M_i \left( F_i - \sum_{j=1}^n \frac{\rho_j}{\rho} \vec{F}_j \right) \right] \quad (1-4-30)$$

$$j_k^{(T)} = - D_k^T \vec{\nabla} \log_e T \quad (1-4-31)$$

where  $D_{ki}$  is the diffusion coefficient of the gaseous mixture;  $D_k^T$  is the thermal diffusion coefficient. They satisfy the relations

$$D_{kk} = 0; \quad \sum_{k=1}^n D_k^T = 0 \quad (1-4-32)$$

$$\sum_{k=1}^n (M_k M_i D_{ki} - M_k M_j D_{kj}) = 0 \quad (1-4-33)$$

For  $n > 2$   $D_{ij} \neq D_{ji}$ .

In a particular case, for a binary mixture we have

$$\vec{j}_A = - \vec{j}_B = - \frac{C^2}{\rho RT} M_A^2 M_B D_{AB} C_{A0} \quad (1-4-34)$$

$$\times \left[ \left( \frac{\partial G_A}{\partial C_{A0} M_A} \right)_{p, T} \vec{\nabla} C_{A0} - \frac{\rho_B}{\rho} (\vec{F}_A - \vec{F}_B) + \left( \frac{V_A}{M_A} - \frac{1}{\rho} \right) \vec{\nabla} p \right] - D_A^T \vec{\nabla} \log_e T$$

By using the equality

$$(dG_A)_{p, T} = RT d(\log_e a)_T.$$



in which  $a$  is the chemical affinity, and by introducing the thermodiffusive relation

$$K_T = \alpha C_{A0} C_{B0} = \sigma C_{A0} C_{B0} T \quad (1-4-35)$$

where  $\alpha$  is the thermodiffusion constant and  $\sigma$  is the Soret coefficient, we may rewrite equation (1-4-34) in the form

$$\vec{j}_A = -\vec{j}_B = -\left(\frac{C^2}{\rho}\right) M_A M_B D_{AB} \left[ \left( \frac{\partial \log_e a_A}{\partial \log_e C_{A0}} \right)_{p,T} \vec{\nabla} C_{A0} \right. \quad (1-4-36)$$

$$\left. - \frac{M_A \rho_B C_{A0}}{\rho R T} (\vec{F}_A - \vec{F}_B) + \frac{M_A C_{A0}}{R T} \times \left( \frac{V_A}{M_A} - \frac{1}{\rho} \right) \vec{\nabla} p + K_T \vec{\nabla} \log_e T \right]$$

To conclude this section, we shall consider molecular (diffusion) energy flux  $j_U$ . It is composed of: (a) heat flux due to heat conduction, defined by the Fourier law, (b) heat flux due to diffusion heat conduction (Dufour effect)  $j_d$ , and (c) energy transfer due to diffusion.

$$\vec{j}_U = -\lambda \vec{\nabla} T + \vec{j}_d + \sum_k h_k \vec{j}_k \quad (1-4-37)$$

In general, the heat-conduction coefficient  $\lambda$  is a tensor. In the absence of a magnetic field, the heat-conduction tensor is symmetrical

$$\lambda = \tilde{\lambda} \quad (1-4-38)$$

The diffusion heat conduction  $\vec{j}_d$  for a multicomponent system has not been studied adequately; moreover, its value is insignificant ( $\vec{j}_d = 0$ ). The total energy flux

$$\begin{aligned} \vec{j}_e &= \left( U + \psi + \frac{1}{2} v^2 \right) \rho \vec{v} + (\mathcal{P} \cdot \vec{v}) \\ &+ \sum_k \vec{j}_k \psi_k + \vec{j}_d + \sum_k h_k \vec{j}_k - \lambda \vec{\nabla} T \end{aligned} \quad (1-4-39)$$

If there is no external force field ( $\psi = 0$ ), then by neglecting the transfer of kinetic energy of apparent motion, the diffusion heat conduction and the work done by internal friction forces, we get

$$\begin{aligned} \vec{j}_e &\approx -\lambda \vec{\nabla} T + \sum_k h_k \vec{j}_k + \rho U \vec{v} + p \vec{v} \approx -\lambda \vec{\nabla} T + \sum_k h_k \vec{j}_k + \rho h \vec{v} \\ &= -\lambda \vec{\nabla} T + \sum_k (h_k \vec{i}_k + \rho_k h_k \vec{v}) = -\lambda \vec{\nabla} T + \sum_k (h_k \vec{j}_k + \rho_k h_k \vec{v}) \\ &= -\lambda \vec{\nabla} T + \sum_k h_k \dot{m}_k \end{aligned} \quad (1-4-40)$$

where

$$h = \sum_k \rho_k h_k$$

Expression (1-4-40) for the energy flux is found in a number of published works.

### 1-5. DERIVATION OF TRANSFER EQUATIONS FROM THE KINETIC THEORY OF GASES

Equations expressing transfer of mass, momentum and energy may be obtained from solutions of the Boltzmann integro-differential equation

$$\frac{\partial f_k}{\partial \tau} + \vec{V}_k \frac{\partial f_k}{\partial \vec{r}} + \vec{F}_k \frac{\partial f_k}{\partial \vec{V}_k} = \sum_i B(f_k, f_i) \quad (1-5-1)$$

where  $f_k(\vec{r}, \vec{V}_k, \tau) d\vec{r} d\vec{V}_k$  is the number of molecules of the  $k$ -th species present at time  $\tau$  in the range  $(\vec{r}, \vec{r} + d\vec{r})$  and having velocities  $(\vec{V}_k, \vec{V}_k + d\vec{V}_k)$  in a multicomponent mixture of nonreacting gases.

The second term on the left-hand side of the equation to be solved represents variation in the distribution function  $f_k$  due to molecular motion, the third term is the acceleration due to external force  $\vec{F}_k$  (per unit mass) and, finally, the term on the right-hand side defines the rate of change of  $f_k$  due to binary collisions with particles of the same species  $k$  and other species  $i$ . Here, it is assumed that the molecules have no internal degrees of freedom and the instantaneous interaction forces are central.

The mass density is

$$\rho_k = n_k m_k = m_k \int f_k d\vec{V}_k \quad (1-5-2)$$

where  $m_k$  and  $n_k$  are molecular mass and density of the  $k$ -th species, respectively. The total density of the mixture

$$\rho = \sum_k \rho_k = \sum_k m_k \int f_k d\vec{V}_k \quad (1-5-3)$$

Having differentiated equation (1-5-2) with respect to time with due consideration for equation (1-5-1), and using the well-known property of collision integrals, we get the mass conservation equation of the  $k$ -th species

$$\begin{aligned} \frac{\partial \rho_k}{\partial \tau} = \int m_k \frac{\partial f_k}{\partial \tau} d\vec{V}_k = - \int \left[ m_k \vec{V}_k \frac{\partial f_k}{\partial \vec{r}_k} + \right. \\ \left. + m_k \vec{F}_k \frac{\partial f_k}{\partial \vec{V}_k} - \sum_i m_k B(f_k, f_i) \right] d\vec{V}_k \end{aligned} \quad (1-5-4)$$

Since  $f_k$  quickly tends to zero at large values of  $\vec{V}_k$ , the second term on the right-handside of equation (1-5-4) equals zero. Similarly, the last term also equals zero by virtue of the known properties of collision integrals.

Therefore

$$\frac{\partial \rho_k}{\partial \tau} = - \operatorname{div} \rho_k \vec{v}_k \quad (1-5-5)$$

where  $\vec{v}_k = \frac{m_k}{\rho_k} \int \vec{V}_k f_k d\vec{V}_k$  is the mean velocity of the species.

Introduction of mass velocity defined by

$$\rho \vec{v} = \sum_k \rho_k \vec{v}_k = \sum_k m_k \int \vec{V}_k f_k d\vec{V}_k \quad (1-5-6)$$

yields the following form of the mass conservation equation

$$\frac{\partial \rho_k}{\partial \tau} = - \operatorname{div} \rho_k \vec{v} - \operatorname{div} \vec{j}_k \quad (1-5-7)$$

where

$$\vec{j}_k = m_k \int (\vec{V}_k - \vec{v}) f_k d\vec{V}_k \quad (1-5-8)$$

is the diffusion mass flux.

The equation of motion or the equation of momentum transfer is obtained after differentiation of equation (1-5-6) with respect to time

$$\frac{\partial \rho \vec{v}}{\partial \tau} = - \operatorname{div} (\rho \vec{v} \vec{v} + \mathcal{P}) + \sum_k \rho_k \vec{F}_k \quad (1-5-9)$$

where the pressure tensor

$$\mathcal{P} = \sum_k m_k \int (\vec{V}_k - \vec{v}) (\vec{V}_k - \vec{v}) f_k d\vec{V}_k \quad (1-5-10)$$

The density  $\rho U$  of internal energy is defined by the relation

$$\rho U = \frac{1}{2} \sum_k m_k \int (\vec{V}_k - \vec{v})^2 f_k d\vec{V}_k \quad (1-5-11)$$

For temperature  $T$  (kinetic temperature) the following relation holds

$$\frac{3}{2} nKT = \rho U = \frac{1}{2} \sum_k m_k \int (\vec{V}_k - \vec{v})^2 f_k d\vec{V}_k \quad (1-5-12)$$

where  $n$  is the mixture particle density ( $n = \sum_k n_k$ ).

Having differentiated (1-5-11) with respect to time and by using the Boltzmann equation, we obtain

$$\frac{\partial \rho U}{\partial \tau} = -\operatorname{div}(\rho U \bar{v} + \vec{j}_q) - (\mathcal{P} : \vec{\nabla} \bar{v}) - \sum_k \vec{j}_k \vec{F}_k \quad (1-5-13)$$

where the heat flux (the energy flux) equals

$$\vec{j}_q = \frac{1}{2} \sum_k m_k \int (\vec{V}_k - \bar{v})^2 (\vec{V}_k - \bar{v}) f_k d\vec{V}_k \quad (1-5-14)$$

Thus, the classical transfer equations describe the transfer mechanism in systems governed by the Boltzmann equation.

However, equations (1-5-9) and (1-5-13) are not complete, in as much as the pressure tensor  $\mathcal{P}$  and the heat flux  $j_q$ , appearing in these equations, are not defined; for their determination it is necessary to know the distribution function  $f_k$ , which may be found from the Boltzmann equation. Therefore, completeness of the above mentioned equations is closely related with the accuracy of the Boltzmann equation.

It is well known that the solution of the Boltzmann equation in the first approximation leads equation (1-5-9) to the Navier-Stokes form. The second approximation, found by Barnett as per the Chapman-Enskog method, introduced new terms in the system of motion equations that, to a certain extent, take into account variations in velocity and temperature gradients over the mean free path length of molecules. A third approximation of the Boltzmann equation also exists. This is known as the super-Barnett solution.

The above approximations remained indubitable for several decades. But, between 1949 and 1954 experiments were carried out to determine the thickness of a shock wave and to measure the velocity of ultra-acoustic waves in rarefied helium. These experiments uncovered the following unexpected facts.

1. The Navier-Stokes equations hold true in wider ranges than the subsequent approximations.

2. Both the theories prove to be inadequate to describe the processes in highly rarefied fluids.

Analysing the above disagreement between the molecular-kinetic theory and the experimental findings, A. S. Predvoditelev [1-12] wrote: "If we assume that the mathematical procedures used for the solution of the Boltzmann equation are quite correct, then we shall have to attribute the above disagreements to the incompleteness of this equation".

In fact, in the Boltzmann equation two operations are performed over the function expressing distribution of molecules according to their thermal velocities. The first operation is used to calculate per unit time the variation in the function with respect to time, coordinates and velocities. If masses of all molecules are assumed equal, this change may be written as

$$\frac{df}{d\tau} = \frac{\partial f}{\partial \tau} + V_x \frac{\partial f}{\partial x} + V_y \frac{\partial f}{\partial y} + V_z \frac{\partial f}{\partial z} + F_x \frac{\partial f}{\partial V_x} + F_y \frac{\partial f}{\partial V_y} + F_z \frac{\partial f}{\partial V_z} \quad (1-5-15)$$

The second operation is employed to calculate the change in the same function per unit time due to molecular collision.

If  $V_{1x}$ ,  $V_{1y}$ ,  $V_{1z}$  are used to denote the components of the molecule velocity of the first species, and  $V_{2x}$ ,  $V_{2y}$ ,  $V_{2z}$  represent velocity components of the second species of molecules prior to their collision, then the measure of probability of decreasing intermolecular distances may be taken as  $f_1 f_2$ . The subscripts indicate the thermal velocity components for the distribution function.

After collision the velocity components of the first and the second species molecules will attain the values

$$(V'_{1x}, V'_{1y}, V'_{1z}) \text{ and } (V'_{2x}, V'_{2y}, V'_{2z}) \quad (1-5-16)$$

and the product  $f'_1 f'_2$  will serve as the probability measure of increasing intermolecular distances.

Thus, each collision of one species of molecules with another species may be characterized by the difference

$$f'_1 f'_2 - f_1 f_2 = \delta f_1 f_2 \quad (1-5-17)$$

Summation of these differences over the whole phase volume gives the change in the function  $f$  per unit time due to molecular collisions. Thus, we have

$$\frac{df}{d\tau} = \int \dots \int \delta f_1 f_2 g b \, db \, d\omega_1 \, d\omega_2 \, d\varepsilon \quad (1-5-18)$$

Here  $g b \, db \, d\omega_1 \, d\omega_2 \, d\varepsilon$  is the phase volume element, in which  $g$  is the relative velocity of colliding molecules;  $b$  is the shortest distance between rectilinear trajectories of molecules;  $\varepsilon$  is the angle between the molecule path plane of the first species and a fixed plane passing through  $g$ ;  $d\omega_1$  is the product  $dV_{1x} dV_{1y} dV_{1z}$ . Here, a question may arise as to whether these two operations always yield same results? The first operation, by its nature may give continuous values of phase space coordinates. The second one may give rise to discontinuous values as the variation  $\delta f_1 f_2$  may not be infinitesimal. This means that, in principle, we cannot regard the above mentioned operations equivalent.

The critical remarks compel us to return to the Maxwell theory.

The transfer equation for any quantity  $Q$  according to Maxwell has the form [1-9, 1-13]

$$\frac{\partial}{\partial \tau} (n\bar{Q}) + \frac{\partial}{\partial x_i} (nw_i \bar{Q}) = \Delta Q + nF_i \left( \frac{\partial \bar{Q}}{\partial w_i} \right) \quad (1-5-19)$$

where  $x_i$  are Cartesian coordinates ( $x_1 \equiv x$ ,  $x_2 \equiv y$ ,  $x_3 \equiv z$ ) and  $\bar{Q}$  is the averaged value of  $Q(w_i)$

$$\bar{Q} = \frac{1}{n} \int_{-\infty}^{+\infty} \int \int Q(nf) \, dw_1 \, dw_2 \, dw_3 \quad (1-5-20)$$

where  $f$  is the molecular velocity distribution function and  $w_i$  is the velocity of molecules.

If the velocity of apparent (hydrodynamic) motion of a fluid is designated by  $v_i$ , then the velocity of any molecule  $w_i$  relative to a fixed coordinate system may be represented as the sum of apparent and thermal motions velocities  $v_i$  and  $u_i$  respectively

$$w_i = v_i + u_i \quad (1-5-21)$$

Thus, a gas is represented as the motion of a model continuous medium in which the mass points are in a state of random motion. In this case, obviously, the equalities

$$\bar{w}_i = v_i, \quad \bar{u}_i = 0 \quad (1-5-22)$$

hold true, i.e. the mean velocity of thermal motion equals zero. If all the terms of equation (1-5-19) are multiplied by the molecular mass, then under conditions that  $\rho$  ( $\rho = mn$ ) is the density, for  $Q = 1$  we get the continuity equation

$$\frac{\partial \rho}{\partial \tau} + \frac{\partial}{\partial x_i} (\rho v_i) = 0 \quad (1-5-23)$$

From equations (1-5-19) and (1-5-23), we get

$$n \frac{d\bar{Q}}{d\tau} + \frac{\partial}{\partial x_i} (n \bar{u}_i \bar{Q}) = n F_i \frac{\partial \bar{Q}}{\partial u_i} + \Delta Q \quad (1-5-24)$$

where  $i = 1, 2, 3$ .

The quantity  $\Delta Q$  is determined from the collision integral

$$\Delta Q = \int_0^{2\pi} \int_0^\infty \dots \int (Q' - Q'') n^2 g b d\varepsilon f' f'' d\omega_1 d\omega_2 \quad (1-5-25)$$

where  $d\omega_1 = dV_{11} dV_{21} dV_{31}$ ;  $g = \sum_i (V_{i1} - V_{i2})^2$ ;  $b$  is the distance between impinging molecules in the direction of relative velocity;  $\varepsilon$  is the dihedral angle between the plane passing through the line connecting the approached molecules and the relative velocity vector and the plane passing through axis  $x_1$  ( $x_1 \equiv x$ ) and the relative velocity vector.

Consequently, the line showing the direction of relative velocity will be a secant for the above planes. The superscripts ' and '' correspond to the values before and after collisions.

To calculate the collision integral (1-5-25), it is necessary to know the distribution functions  $f'$  and  $f''$ . However, if the mean of all the values of the transferred quantity  $Q$  is introduced by the relation

$$n^2 \bar{Q} = \int_0^\infty \dots \int Q n^2 f' f'' d\omega_1 d\omega_2 \quad (1-5-26)$$

then there is no need in finding these functions. In this lies the advantage of the Maxwell method compared to the method based on the Boltzmann equation.

The quantity  $b db$  characterizes the effective section of molecule collisions.

If the interaction potential is inversely proportional to the fifth power of the distance between molecules, then we obtain the equality:

$$b db = \sqrt{2 km} \frac{\alpha}{g} d\alpha. \quad (1-5-27)$$

Then [1-13]

$$\begin{aligned} \Delta Q(\bar{V}_{11}^2) = & \int \frac{\bar{k}}{2m^3} \left\{ \rho^2 A_1[(v_1'^2) - (v_1'^2)] + \rho^2 A_2(\bar{c}^2 - 3\bar{u}_1^2) \right. \\ & \left. + \frac{1}{2} \rho^2 A_2[(v_2' - v_2'')^2 + (v_3' - v_3'')^2 - 2(v_1' - v_1'')^2] \right\} \end{aligned} \quad (1-5-28)$$

where

$$\bar{c}^2 = \sum_i \bar{u}_i^2 \quad (i = 1, 2, 3)$$

$$A_1 = 4\pi \int_0^\infty \alpha \cos^2 \theta d\alpha; \quad A_2 = \pi \int_0^\infty \alpha \sin^2 2\theta d\alpha \quad (1-5-29)$$

If

$$Q = mw_i = m(v_i + u_i) \quad (1-5-30)$$

then from equation (1-5-24) we obtain

$$\rho \frac{dv_i}{d\tau} + \frac{\partial(\rho u_i u_j)}{\partial x_i} = \rho F_i; \quad i = 1, 2, 3; \quad j = 1, 2, 3 \quad (1-5-31)$$

From equation (1-5-31) it is possible to obtain transfer equations for ideal and viscous fluids.

If the apparent motion does not affect the thermal one, then

$$\overline{u_i u_j} = 0 \quad \text{at} \quad \bar{u}_1^2 = \bar{u}_2^2 = \bar{u}_3^2 \quad (1-5-32)$$

The pressure in such a medium is uniform in all directions and equals

$$p = \frac{1}{3} \rho \sum_i \bar{u}_i^2 = \rho \bar{u}_1^2 = \rho \bar{u}_2^2 = \rho \bar{u}_3^2; \quad i = 1, 2, 3 \quad (1-5-33)$$

Using relation (1-5-33), from equation (1-5-31) we arrive at the Euler equation for an ideal fluid

$$\rho \frac{d\vec{v}}{d\tau} + \vec{\nabla} p = \rho \vec{F} \quad (1-5-34)$$

The quantity  $\Delta Q$  may be calculated from equation (1-5-24). Assuming  $Q = w_i$ , multiplying equation (1-5-24) by  $(\partial \bar{Q})/(\partial v_i)$  and subtracting the equation obtained from transfer equation (1-5-33), we get

$$n \left( \frac{d\bar{Q}}{d\tau} - \frac{\partial \bar{Q}}{\partial v_i} \frac{dv_i}{d\tau} \right) = - \frac{\partial}{\partial x_i} (\overline{nu_i Q}) + \frac{\partial \bar{Q}}{\partial v_i} \frac{\partial}{\partial x_i} (\overline{nu_i u_j}) + \Delta Q;$$

$$i = 1, 2, 3; \quad j = 1, 2, 3 \quad (1-5-35)$$

Assume that there exists the relation

$$(\overline{V_{i1}})^2 = (\overline{V_{i2}})^2 = (\overline{w_i^2}) \quad (1-5-36)$$

then from equation (1-5-35) it follows:

$$\Delta Q(\overline{V_{i1}^2}) = \Delta Q(\overline{V_{i2}^2}) = \Delta Q(\overline{w_i^2}) \quad (1-5-37)$$

Assuming  $Q = w_i^2$  and taking into account that

$$\bar{Q} = \overline{w_1^2} = \overline{v_1^2} + \overline{u_1^2}; \quad \overline{u_i Q} = 2\overline{v_1 u_1 u_i} + \overline{u_1^2 u_i}; \quad i = 1, 2, 3 \quad (1-5-38)$$

we get

$$m\Delta Q(\overline{V_{i1}^2}) = \rho \frac{d\overline{u_1^2}}{d\tau} + \frac{\partial}{\partial x_i} (\rho \overline{u_1^2 u_i}) + 2\rho \left( \overline{u_1^2 u_i} \frac{\partial u_1}{\partial x_i} \right) \quad (1-5-39)$$

If we assume that

$$\overline{u_i u_j} = 0, \text{ when } i \neq j; \quad \overline{u_1^2 u_i} = 0, \text{ when } i = 1, 2, 3 \quad (1-5-40)$$

and formula (1-5-33) is valid, i.e.  $p = 1/3 \rho \bar{c}^2$ , then from equation (1-5-39) we obtain

$$m\Delta Q(\overline{V_{i1}^2}) = \rho \frac{d\overline{u_1^2}}{d\tau} + 2\rho \frac{\partial v_1}{\partial x_1} \quad (1-5-41)$$

Comparing equation (1-5-41) with equation (1-5-28), we find

$$\rho \frac{d\overline{u_1^2}}{d\tau} + 2\rho \frac{\partial v_1}{\partial x_1} = \left[ \frac{k}{2m} [\rho^2 A_1(v_1''^2 - v_1'^2) + 3\rho A_2(p - \rho \overline{u_1^2})] \right] \quad (1-5-42)$$

We shall make a few assumptions for the transport velocities  $v_i'$  and  $v_i''$  of the impinging molecules:

$$(v_1'')^2 - (v_1')^2 \gg (v_1'' - v_1')^2, \quad i = 1, 2, 3; \quad (1-5-43)$$

$$(v_1'' v_2'' - v_1' v_2') \gg (v_1' - v_1'')(v_2' - v_2'')$$

Taking into consideration that

$$\frac{dp}{d\tau} + p \operatorname{div} \vec{V} = \frac{\bar{c}^2}{3} \frac{d\rho}{d\tau} + \frac{1}{3} \rho \frac{d\bar{c}^2}{d\tau} + p \operatorname{div} \vec{V} \approx \rho \frac{d\overline{u_1^2}}{d\tau} \quad (1-5-44)$$



from equation (1-5-42), we obtain

$$\frac{dp}{d\tau} + p \operatorname{div} \vec{V} + 2\rho \frac{\partial v_1}{\partial x_1} = \sqrt{\frac{k}{2m}} [\rho A_1^2 (v_1'^2 - v_1'^2) + 3\rho A_2 (p - \rho \bar{u}_1^2)] \quad (1-5-45)$$

If the gas flow is adiabatic, then

$$p = a\rho^\gamma; \quad \frac{1}{p} \frac{dp}{d\tau} = -\gamma \operatorname{div} \vec{V} \quad (1-5-46)$$

where  $\gamma = c_p/c_v$  is the ratio of heat capacities at constant pressure and constant volume.

Then, from equation (1-5-45) we obtain

$$\rho \bar{u}_1^2 = p + \frac{A_1}{3A_2} \rho (v_1'^2 - v_1'^2) - 2\eta \left( \frac{\partial v_1}{\partial x_1} - \frac{\gamma - 1}{2} \operatorname{div} \vec{V} \right) \quad (1-5-47)$$

where  $\eta$  is the coefficient of viscosity, which equals

$$\eta = \frac{p}{3\rho A_2} \sqrt{\frac{2m}{k}} \quad (1-5-48)$$

Similarly, we obtain a relation for  $\rho \bar{u}_2^2$ ,  $\rho \bar{u}_3^2$ , i.e.

$$\rho \bar{u}_i^2 = p + \frac{A_1}{3A_2} \rho (v_i'^2 - v_i'^2) - 2\eta \left( \frac{\partial v_i}{\partial x_i} - \frac{\gamma - 1}{2} \operatorname{div} \vec{V} \right) \quad (1-5-49)$$

If in a physically small volume there exist identical transport velocities, i.e.  $v_i' = v_i' = v_i$ , then for a monatomic gas ( $\gamma = 5/3$ ) it is possible to find the Navier-Stokes equations from (1-5-49).

In a similar way the stress tensor components can be determined:

$$\rho \bar{u}_i u_j = -\eta \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) + \frac{A_1}{3A_2} \rho (v_i' v_j' - v_i' v_j'); \quad i \neq j, \quad i, j = 1, 2, 3 \quad (1-5-50)$$

Taking into consideration (1-5-31), we get [1-13]

$$\begin{aligned} & \rho \frac{dv_i}{d\tau} + \frac{A_1}{3A_2} \left\{ \frac{\partial}{\partial x_i} \rho (v_i' v_j' - v_i' v_j') \right\} \\ & = -\nabla p_i + \eta \left[ \nabla^2 v_i + (2 - \gamma) \frac{\partial}{\partial x_i} \operatorname{div} \vec{V} \right] + \rho F_i; \quad i=1, 2, 3, \quad j=1, 2, 3 \end{aligned} \quad (1-5-51)$$

The system of equations (1-5-51) is open and to complete the system, it is necessary to determine  $v_i'$  and  $v_i''$  in terms of  $v_i$ . Professor A. S. Pred-

voditelev assumed that the following relations hold true:

$$v'_j = v_j + A \left[ (x_i - x_{i0}) \frac{\partial v_j}{\partial x_i} \right] \quad (1-5-52)$$

$$v''_j = v_j - A[(x_i - x_{i0})] \frac{\partial v_j}{\partial x_i}; \quad i = 1, 2, 3, \quad j = 1, 2, 3$$

where  $x_{i0}$  are the coordinates of the centre of gravity of a physically small volume, within which the averaging is done.

Since the ratio between the quantities  $v'_j$  and  $v''_j$  is unknown, then the sign of velocity gradient  $\partial v_j / \partial x_i$  in formulae (1-5-52) is determined from the pattern of interaction between the flow and solid surface. If the relations expressed in (1-5-52) are used, then from equation (1-5-51) we obtain the Navier-Stokes generalized equation, which was first proposed by Professor A. S. Predvoditelev [1-14]:

$$\rho \frac{d\vec{v}}{d\tau} - \rho \beta (\vec{v} \cdot \vec{\nabla}) \vec{v} - \rho \beta \vec{v} \operatorname{div} \vec{v} = \rho \vec{F} - \vec{\nabla} p + \eta \nabla^2 \vec{v} + (2 - \gamma) \eta \operatorname{grad} \operatorname{div} \vec{v} \quad (1-5-53)$$

where  $\beta$  is the nonideal continuity parameter which we call the Predvoditelev number Pd

$$\text{Pd} \equiv |\beta| = \frac{2}{3} \frac{AA_1}{A_2} \quad (1-5-54)$$

On the basis of simple relations it is possible to show that the Predvoditelev number is proportional to the Mach number (M) and the Knudsen number (Kn)

$$\text{Pd} = \frac{3}{2} \text{Kn} M \quad (1-5-55)$$

If  $\text{Pd} = 0$  ( $\text{Kn} \ll 1$  and  $M < 1$ ) and  $\gamma = 5/3$ , then equation (1-5-53) reduces to the Navier-Stokes equation.

Thus, the generalized equation (1-5-53) is valid for  $\text{Pd} \gg 0$ ; this takes place for  $\text{Kn} M \gg 0$ , i.e. in the flux of a rarefied gas ( $\text{Kn} \gg 0$ ) at high velocities ( $M > 1$ ).

Using the Maxwell distribution function, it is possible to show that the Predvoditelev number is directly proportional to the Truesdell number Tr

$$\text{Pd} = \beta = \frac{3}{2} \text{Kn} M = 2.2522 \text{Tr} \quad (1-5-56)$$

where Tr is the Truesdell number.

The Truesdell number characterizes the nonlinear relationship between the viscous stress tensor and the deformation rate tensor. Relation (1-5-54) reveals that the nonlinearity effect in such a relationship is similar to that of a nonideal continuity parameter. The Predvoditelev number characterizes the discrete structure of gas. In one of our works [1-17] it is shown that the equation of flow of liquid consisting of a set of hollow vortexes is described by a similar equation of the form (1-5-52), if it is assumed that  $\gamma = 5/3$  (monatomic gas). In this case the coefficient  $\beta$  or the Predvoditelev number characterizes the asymmetry of the viscous stress tensor that appears due to the very pronounced discrete structure of liquid. The discreteness has the following physical pattern: the liquid consists of separate hollow vortexes; at the plane of contact of hollow vortexes the discontinuity of hydrodynamic flow velocity takes place.

The flow equation is derived by using the concept of a derivative of discontinuous functions. In this case the coefficient  $\beta$  characterizes viscosity due to the rotation of an elementary volume of liquid, i.e.  $\beta$  is a characteristic of rotational viscosity.

The rotational viscosity resulting from the asymmetry of a viscous stress tensor makes an additional contribution to momentum transfer. This contribution is similar to the one made by nonlinear terms in the viscous flow law.

Thus, the Predvoditelev number  $Pd$  characterizes the discrete structure of a fluid and intensifies the additional mode of momentum transfer which takes place in such systems.

#### (a) *Equilibrium Pressure in a Moving Fluid*

Let us now return to formulae (1-5-47) and (1-5-48) and consider the Euler gas, i.e., a zero viscosity gas. In this case the normal-pressure tensor components are

$$\overline{\rho u_i^2} = p + \frac{A_1}{3A_2} \rho (v_i'^2 - v_i'^2) \quad (1-5-57)$$

Formula (1-5-57) shows that apparent motions introduce a correction in the normal-pressure tensor.

Hence, a problem arises: do these corrections hold true only for hydrodynamic equations in the A. S. Predvoditelev form or do the apparent motions destroy the thermal motion of atoms or molecules?

The last statement requires clarification of the Clapeyron equation of state.

Assume that a gas defined by formula (1-5-57) is in equilibrium, but its statistical state is anisotropic, i.e., thermal velocities of atoms or molecules correlate them. Determine the pressure in such a gas as follows:

$$\bar{p} = \frac{1}{3} \rho \sum_i \overline{u_i^2} = p + \frac{A_1}{3A_2} \rho \sum_i (v_i'^2 - v_i'^2) \quad (1-5-58)$$

or based on A. S. Predvoditelev's hypothesis (1-5-52)

$$\bar{p} = p - \frac{2AA_1}{9A_2} \sum_i \rho(x_i - x_{i0}) \frac{\partial V^2}{\partial x_i}; \quad i = 1, 2, 3 \quad (1-5-59)$$

Multiply both sides of formula (1-5-59) by a volume of one mole  $V_m$

$$\bar{p}V_m = pV_m \left\{ 1 - \frac{2AA_1}{9A_2} \frac{\rho}{p} \left[ (x_i - x_{0i}) \frac{\partial V^2}{\partial x_i} \right] \right\} \quad (1-5-60)$$

Here an obvious notation  $v^2 = v_1^2 + v_2^2 + v_3^2$  is introduced.

If the Avogadro number is designated as  $N$ , then the number of molecules per unit volume

$$n = \frac{N}{V_m} \quad (1-5-61)$$

The kinetic energy of molecules in thermal motion  $K$  is expressed in terms of pressure  $p$

$$p = \frac{2}{3} nK \quad (1-5-62)$$

Now it is not difficult to understand that the kinetic energy of molecules in the transfer motion  $K_1$  is determined by the equality

$$2nK_1 = \rho(x_i - x_{0i}) \frac{\partial V^2}{\partial x_i} \quad (1-5-63)$$

Since in the equilibrium state  $K = K_1$ , equation (1-5-60) for the equilibrium state will take the form

$$\bar{p}V_m = pV_m(1 - \beta) \quad (1-5-64)$$

But, from experiments it is known that  $pV_m = RT$ , then the last formula is rewritten as

$$pV_m = RT(1 - \beta) \quad (1-5-65)$$

or, expressing the volume of one mole in terms of specific volume and molecular weight, we obtain

$$\bar{p}v = (1 - \beta) \frac{RT}{M_0} \quad (1-5-66)$$

The Maxwell velocity distribution function leads to the following formula for a mean velocity of atoms or molecules in thermal motion

$$\bar{u} = 2 \sqrt{\frac{2}{\pi}} \sqrt{\frac{RT}{M_0}} \quad (1-5-67)$$

Equation (1-5-66) allows the last formula to be rewritten as follows

$$\bar{u} = 2 \sqrt{\frac{2}{\pi}} \frac{\bar{p}}{\rho(1-\beta)} \sqrt{\frac{M_0}{RT}} \quad (1-5-68)$$

i.e. correlation is done in terms of quantity  $RT$ .

The formula for the coefficient of viscosity in such a statistical system will take the form

$$\eta = (1-\beta) \frac{RT}{3A_2 M_0} \sqrt{\frac{2m}{k}} = (1-\beta)\eta_0 \quad (1-5-69)$$

Consequently, the main property of momentum transfer, i.e. the dynamic viscosity  $\eta$  for a fluid of discrete structure depends on  $Pd$ .

Hence, physically, the Predvoditelev number may be also regarded as a quantity characterizing effective viscosity or viscosity of a fluid of discrete structure

$$Pd = 1 - \frac{\eta}{\eta_0} \quad (1-5-70)$$

Therefore, the nonideal continuity coefficient (the Predvoditelev number) may be interpreted as a viscosity correction for a discrete structure system.

### (b) Interaction Between a Gas and a Solid Wall

Let us consider the problem of momentum transfer through a gas layer bounded by two flat parallel walls. The gas layer flows due to the motion of one of these walls in the direction lying in the plane of the wall itself. Assume that in this case the hydrodynamic flow is one-dimensional,  $v_1 = v_1(y)$ . For such a motion the molecular velocity has two components, i.e. thermal and hydrodynamic

$$w_i = v_i + u_i$$

But, in our case  $v_1 = v_1(y)$ ,  $v_2 = v_3 = 0$ . Therefore,  $v_1$  is the velocity of gas layers parallel to solid walls.

Maxwell was the first to define the conditions at the gas-wall interface. All the molecules falling on a wall are divided into two groups. The first-group molecules, whose amount is  $(1-\nu)$ , stick to the wall, i.e. their transport velocity is equal to that of the solid wall  $u^*$ . The second-group molecules, whose amount is  $\nu$ , are diffusionally repelled from the wall, i.e. their transport velocity is equal to the velocity  $v_1$  of the gas layer adjacent to the wall.

In this problem it is necessary to distinguish two types of motions: apparent motion in the  $x$ -axis direction determined by mean velocity  $w_1$ , and

thermal motion in the  $y$ -axis direction determined by the mean velocity of thermal motion  $\bar{w}_2 = \bar{u}_2$ . In this case according to formula (1-5-68)

$$\bar{u}_2 = 2 \sqrt{\frac{2}{\pi}} \frac{\bar{p}}{\rho(1-\beta)} \sqrt{\frac{M_0}{RT}} \quad (1-5-71)$$

Now calculate gas momentum through a unit plane  $xy$  determined by the quantity  $\rho \bar{w}_1 \bar{w}_2$ . On the one hand, this quantity is determined in terms of apparent and thermal motions as follows: the fraction of molecules, which adhere to the wall,  $(1 - v)\rho$ , loses its velocity in the  $x$ -axis direction by a value  $(v_1^* - v)$ . This fraction of molecules in the direction of the  $y$ -axis takes part only in the thermal motion, and has a velocity

$$\frac{1}{2} \bar{u}_2 = \sqrt{\frac{2}{\pi}} \frac{\bar{p}}{\rho(1-\beta)} \sqrt{\frac{M_0}{RT}} \quad (1-5-72)$$

Therefore,

$$\rho \bar{w}_1 \bar{w}_2 = (1 - v)\rho(v_1^* - v) \sqrt{\frac{2}{\pi}} \frac{\bar{p}}{\rho(1-\beta)} \sqrt{\frac{M_0}{RT}} \quad (1-5-73)$$

On the other hand, this value may be expressed in terms of apparent motion

$$\rho \bar{w}_1 \bar{w}_2 = -(1 + v)\eta \frac{\partial v_1}{\partial y} \quad (1-5-74)$$

Here, it is taken into account that in an apparent motion the amount of molecules increases by a value  $v$  due to the reflected molecules. Taking into consideration formula (1-5-69) for the coefficient of viscosity, we have

$$\rho \bar{w}_1 \bar{w}_2 = -(1 + v)(1 - \beta)\eta_0 \frac{\partial v_1}{\partial y} \quad (1-5-75)$$

In this formula the minus sign accounts for the momentum that is transferred in the direction of decreasing velocity.

Equating velocities (1-5-75) and (1-5-73), we obtain

$$-(1 + v)(1 - \beta)\eta_0 \frac{\partial v_1}{\partial y} = (1 - v)(u^* - u) \sqrt{\frac{2M_0}{\pi RT}} \frac{\bar{p}}{(1 - \beta)}$$

or

$$v_1^* - v_1 = \frac{\eta_0(1 - \beta)^2}{2\bar{p}} \sqrt{\frac{2\pi RT}{M_0}} \frac{(1 + v)}{(1 - v)} \frac{dv_1}{dy} \quad (1-5-76)$$

Let us introduce a slip coefficient

$$\delta = \frac{\eta_0(1 - \beta)^2(1 + v)}{2\bar{p}(1 - v)} \sqrt{\frac{2\pi RT}{M_0}} \quad (1-5-77)$$

Now formula (1-5-76) is rewritten as

$$-\delta \frac{dv_1}{dy} = (v_1^* - v_1) \quad (1-5-78)$$

## 1-6 TRANSFER EQUATIONS FOR ASYMMETRIC FLUIDS

A two-phase fluid is a typical model of asymmetric fluids. Certain rheological fluids may also be grouped with these fluids. Sometimes these fluids are called microfluids for which each fluid element has two degrees of freedom:

(1) translational degrees of freedom causing averaged motion (the classical velocity of motion);

(2) rotation and stretch, due to which fluid particles undergo independent intrinsic spins and homogeneous deformations. Sometimes it is considered that two degrees of freedom appear as a result of independent motions and deformation in "three directions" associated with each material point.

The most general theory of transfer processes is the theory developed by A. C. Eringen [1-15] who arrived at the heat- and momentum-transfer equations on the basis of nonlinear thermomechanics of continuum. In particular, a thermodynamic pressure tensor, the coupling of the temperature gradient with the stress-moment constitutive equations, and microgyrations were shown to be present in the heat-conduction equation.

If  $i_{kl}$  ( $i_{kl} = i_{lk}$ ) is the microinertia tensor,  $\pi_{kl}$  is the stress tensor,  $\pi_{klm}$  is the first stress-moment tensor,  $s$  is the specific entropy,  $u_k$  is the velocity of the discontinuity surface and  $v_{kl}$  is the gyration tensor, we may write Eringen equations [1-15] as follows:

(1) Conservation of mass

$$\frac{\partial \rho}{\partial \tau} + (\rho v_k)_{,k} = 0; \quad [\rho(\vec{v} - \vec{u})] \cdot \vec{n} = 0 \quad (1-6-1)$$

(2) Conservation of microinertia

$$\frac{\partial i_{kl}}{\partial \tau} + i_{kl,m} v_m - v_{kr} i_{rl} - v_{lr} i_{rk} = 0; \quad [\rho i_{kl}(\vec{v} - \vec{u})] \cdot \vec{n} = 0 \quad (1-6-2)$$

(3) Balance of momentum

$$\pi_{kl,k} + \rho(f_l - v_l) = 0; \quad [\pi_{kl} - \rho v_l(v_k - u_k)]n_k = 0 \quad (1-6-3)$$

(4) Balance of momentum moments

$$\pi_{klm,k} + \pi_{ml} - \pi_{ml} + \rho f_{lm} - \rho i_{km}(v_{lk} + v_{lr} v_{rk}) = 0 \quad (1-6-4)$$

$$[\pi_{klm} - \rho i_{rm} v_{lr}(v_k - u_k)]n_k = 0$$

## (5) Conservation of energy

$$-\rho e + \pi_{kl} v_{l,k} + \pi_{klm} v_{lm,k} + (\bar{\pi}_{kl} - \pi_{kl}) v_{lk} + q_{k,k} + \rho Q = 0 \quad (1-6-5)$$

$$\left[ q_k + \pi_{kl} v_l + \pi_{klm} v_{lm} - \rho \left( e + \frac{1}{2} v^2 + \frac{1}{2} i_{rs} v_{lr} v_{ls} \right) (v_k - u_k) \right] n_k = 0$$

## (6) Entropy inequality

$$\rho \dot{s} - \left( \frac{q_k}{T} \right)_{,k} + \frac{\rho}{T} Q \geq 0 \quad (1-6-6)$$

$$\left[ \rho s (v_k - u_k) - \frac{q_k}{T} \right] n_k \geq 0$$

where  $\bar{\pi}_{kl}$  is the mean stress tensor ( $\bar{\pi}_{kl} = \bar{\pi}_{lk}$ ), and  $Q$  is the heat source. The remaining notations are conventional. The summation is made over the repeating indices. The subscript after a comma indicates a partial derivative with respect to space rectangular coordinates  $x_k$  and the dot above a quantity indicates material derivative, i.e.

$$v_{k,l} = \partial v_k / \partial x_l; \quad \dot{v}_k = \frac{\partial v_k}{\partial \tau} + v_{k,l} v_l \quad (1-6-7)$$

In reference [1-16] an asymmetric fluid of a rigid structure is considered. The fluid particles have intrinsic angular velocities  $\omega$  different from the angular velocity of a part of the fluid taken as a whole. Thus, concurrent with the ordinary shear viscosity  $\eta$  and bulk viscosity  $\eta_v$ , there exists the rotational viscosity (viscosity due to internal gyration of particles) characterized by three coefficients of viscosity  $\eta_r$ ,  $\eta'_r$  and  $\eta''_r$ . In this case the differential equations of motion for an asymmetric fluid are written as [1-16]

$$\rho \vec{v} = \rho \vec{F} - \vec{\nabla} p + \eta (1 - \gamma/\eta) \nabla^2 \vec{v} + \left( \eta_v + \frac{1}{3} \eta + \gamma \right) \vec{\nabla} \text{div } \vec{v} - 2\gamma \text{curl } \omega \quad (1-6-8)$$

$$(\eta'_r + \eta''_r) \vec{\nabla} \text{div } \omega - \eta_r \nabla^2 \omega + 2\gamma \omega - \gamma \text{curl } \vec{v} + \rho \Pi_v = 0 \quad (1-6-9)$$

where  $\Pi_v$  is the density of volume distributed force moments.

The system of equations (1-6-8) and (1-6-9) contains three components of velocity  $\vec{v}$ , three components of angular velocity  $\omega$ , the pressure  $p(\rho, T)$  and the density  $\rho$ . The coefficient  $\gamma$  is the measure of particle adhesion to the surrounding fluid. If  $\gamma = 0$ , the particle rotates freely with respect to the surrounding fluid (no adhesion), if  $\gamma = \infty$ , the particle rotates together with the adjacent fluid. The continuity equation is independent of the stressed state equation and has a general form.



To solve equations (1-6-8) and (1-6-9) it is necessary to prescribe boundary conditions for a definite problem. Usually in classical hydrodynamics the adhesion conditions are assumed, i.e. it is considered that the fluid velocity at the wall is equal to zero. These conditions may be maintained for translational velocity. But, it is necessary to prescribe wall conditions for angular velocities  $\omega$ . If a particle does not rotate at the wall (the extreme case of strong fluid-wall interaction), then  $(\omega)_s = 0$ . If a particle rotates freely at the wall (the extreme case of weak fluid-wall interaction), then

$$(\Pi_{vi})_s = (n_i \Pi_{ij})_s = 0 \quad (1-6-10)$$

Obviously, in reality there is friction between a rotating particle and the wall that may be considered to be proportional to the difference of angular velocities  $\Delta\omega$  characterizing the above mentioned two extreme cases. In [1-16] the wall boundary conditions are assumed to be

$$(\beta_i \omega)_s = \left[ 2\eta_r'' \operatorname{div} \omega \delta_{ij} + 2\eta_r' \frac{\partial \omega_j}{\partial x_i} + 2\eta_r (\partial \omega_i / \partial x_j) \right]_s \quad (1-6-11)$$

where  $\beta_i$  is the coefficient of rotational surface friction.

#### (a) Fluid Flow in a Cylindrical Tube

For the given boundary conditions a very simple problem of fluid flow from a cylindrical capillary tube was solved. Assuming the velocity field to be symmetric and taking  $\vec{F} = 0$ ,  $M_v = 0$ , we obtain a solution in the form [1-16].

$$v = -\frac{1}{4\eta} \frac{\partial p}{\partial z} \left[ 1 - \frac{r^2}{R^2} + \frac{2}{A^2} \frac{I_0(kr/R) - I_0(k)}{k^{-1}I_1(k) + BI_2(k)} \right] \quad (1-6-12)$$

where

$$A = R \sqrt{\frac{2\eta}{\eta_r}}; \quad k = R \left[ \frac{2\gamma}{\eta_r(1 - \gamma|\eta)} \right]^{1/2} \quad B = \left[ 1 - \frac{\eta_r'}{\eta_r} - \frac{\beta R}{\eta_r} \right]^{-1} \quad (1-6-13)$$

and  $I_0$ ,  $I_1$ ,  $I_2$  are the modified Bessel functions of the first kind and of the zeroth, first and second orders, respectively.

The amount of liquid  $\dot{V}$  flowing out of the capillary tube per unit time will be

$$\frac{\dot{V}}{\dot{V}_0} = 1 - \frac{4}{A^2} \left[ B + \frac{I_1(k)}{kI_2(k)} \right]^{-1} \quad (1-6-14)$$

where  $\dot{V}_0$  is the amount of outflowing liquid from the capillary, which is determined by the Poiseuille formula

$$\dot{V}_0 = -\frac{\pi R^4}{8\eta} \frac{\partial p}{\partial z} \quad (1-6-15)$$

As is seen from formula (1-6-14), the amount of outflowing liquid  $\dot{V}$  will be less than  $\dot{V}_0$ . The smaller the radius of the capillary tube, the greater will be the difference in  $\dot{V}$  and  $\dot{V}_0$ . If  $|\gamma/\eta| \gg 1$ , then  $k = A$ .

Hence

$$\frac{\dot{V}}{\dot{V}_0} = 1 - \frac{4}{A^2} \left[ B + \frac{I_1(A)}{AI_2(A)} \right]^{-1} \quad (1-6-16)$$

The value of  $B$  mainly depends on the friction factor  $\beta$ , which varies between 0 and  $\infty$  ( $0 < \beta < \infty$ ).

In Fig. 1-4 the curves  $\dot{V}/\dot{V}_0 = f(A)$  are plotted for two extreme values of  $\beta$  ( $\beta = 0$  and  $\beta = \infty$ ) and for certain values of  $\gamma$ . It may be seen from the

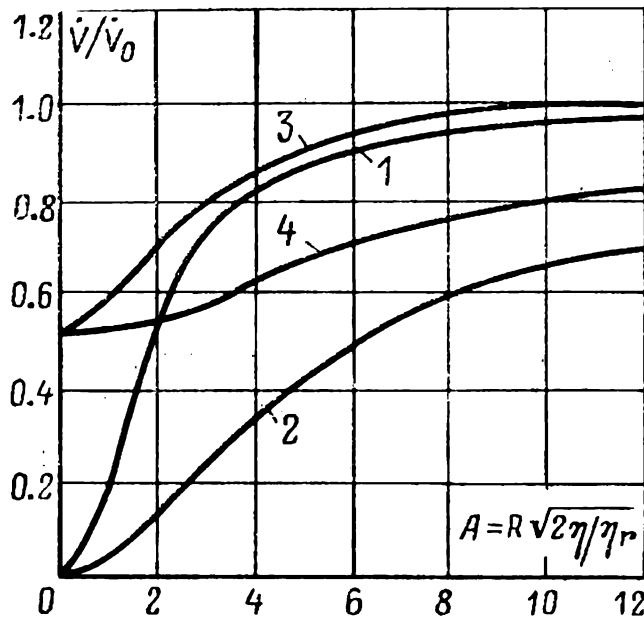


Fig. 1-4. Plot of  $\dot{V}/\dot{V}_0$  versus asymmetry number  $A$  for two limiting values of  $\beta$  (1, 3,  $\beta = 0$  and 2, 4,  $\beta = \infty$ ) and for values  $\gamma/\eta \gg 1$  (curves 1 and 2) and  $\gamma/\eta = 1$  (curves 3 and 4)

figure that in all cases where  $A < 8$  it is not possible to use the classical Poiseuille formula.

### (b) Heat Transfer in a Cylindrical Tube

The Fourier-Kirchhoff equation differs from the ordinary heat transfer equation by an additional term expressing the amount of work done by friction forces (a dissipative function). The general form of the equation is [1-15]

$$c_p \rho \frac{dT}{d\tau} = \lambda \nabla^2 T + \frac{dp}{d\tau} + (\sigma_{ij} D_{ij} + \pi_{ij} \dot{v}_{ij}) \quad (1-6-17)$$

where  $D_{ij}$  is the shear rate tensor and  $\dot{v}_{ij}$  is the microswirl and microbend velocity tensor.

As a particular example, we shall consider heat transfer in a laminar fluid flow in a cylindrical tube.

For a constant pressure gradient ( $\partial p/\partial z = \text{const}$ ) the velocity distribution  $v(r)$  in a cylindrical tube is given by relation (1-6-12).

For a symmetrical problem the heat transfer equation takes the form

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \frac{\partial t}{\partial r} = \frac{v_m C}{\lambda} \left[ 1 - \frac{r^2}{R^2} + \frac{2}{A^2} \times \frac{I_0(kr/R) - I_0(k)}{k^{-1}I_1(k) + BI_2(k)} \right] \quad (1-6-18)$$

where  $v_m$  is the maximum axial velocity in a tube of diameter  $2R$ :

$$v_m = - \frac{1}{4\eta} \frac{\partial p}{\partial z} R^2 \quad (1-6-19)$$

Here, it is assumed that the physical properties have constant values. The temperature distribution along the cylinder length  $z$  for a constant heat flux is expressed in the form

$$t = Cz + T(r) \quad (1-6-20)$$

The equation for function  $T(r)$  is obtained from the solution of the differential equation

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) = \frac{v_m C}{\lambda} \left[ 1 - \frac{r^2}{R^2} + \frac{2}{A^2} \frac{I_0(kr/R) - I_0(k)}{k^{-1}I_1(k) + BI_2(k)} \right] \quad (1-6-21)$$

The solution of this equation for the first-kind boundary conditions is

$$T(r) = \frac{1}{4} \left[ 1 - \frac{2I_0(k)}{M} \right] \frac{r^2}{R^2} - \frac{1}{16} \frac{r^4}{R^4} - \frac{3}{4} - \frac{2I_0(kr/R)}{k^2 M} + \frac{4 + k^2}{2k^2 M} \quad (1-6-22)$$

where

$$M = A^2[k^{-1}I_1(k) + BI_2(k)]$$

The Nusselt number is determined from the formula

$$\text{Nu} = \frac{2R}{T_{mn}} \left( \frac{\partial T}{\partial r} \right)_{r=R} \quad (1-6-23)$$

where  $T_{mn}$  is the mean temperature determined by the formula

$$T_{mn} = \frac{4}{R^2 v_m} \int_0^R r T(r) v dr \quad (1-6-24)$$

For small values of  $k$  ( $\eta_r \neq 0$  and  $\gamma$  is small) approximate values of the velocity  $v$ , temperature  $T$ , and the Nusselt number may be obtained. If in the Bessel function series expansion  $I_0$ ,  $I_1$  and  $I_2$  the values greater than

$(kr/R)^2$  are neglected, then we have

$$v \approx v_m \left( 1 - \frac{r^2}{R^2} \right) (1 + N) \quad (1-6-25)$$

$$T \approx \frac{v_m CR^2}{4\lambda} (1 + N) \left[ \frac{r^2}{R^2} - \frac{1}{4} \frac{r^4}{R^4} - \frac{3}{4} \right] \quad (1-6-26)$$

$$\text{Nu} \approx 4.36(1 - N) \quad (1-6-27)$$

where

$$N = \frac{4\gamma\eta^{-1}}{1 + \gamma\eta^{-1}} \left[ 1 - \frac{\gamma\eta^{-1}}{8(1 + \gamma\eta^{-1})} A^2(1 - 2B) \right]; \quad \gamma > 0 \quad (1-6-28)$$

Formulas (1-6-27) and (1-6-28) imply that the Nusselt number for an asymmetric fluid is a function of  $A$  ( $\gamma\eta^{-1}$ ) and  $B$

$$\text{Nu} = f(A, B, \gamma\eta^{-1}) \quad (1-6-29)$$

This is quite a new result in the heat transfer theory. Analysis of formula (1-6-28) reveals that for  $B > 0.5$  the Nusselt number is smaller compared to that for symmetric fluids, other conditions being equal.

Depending on  $B$  and, consequently, on  $A$ , the Nusselt number for asymmetric fluids may be larger than the Nusselt number for symmetric fluids.

In Fig. 1-5 the Nusselt number is plotted versus the parameter  $A$  for  $B = 0$  and  $B = 1$ . Formula (1-6-22) implies that the fluid temperature at

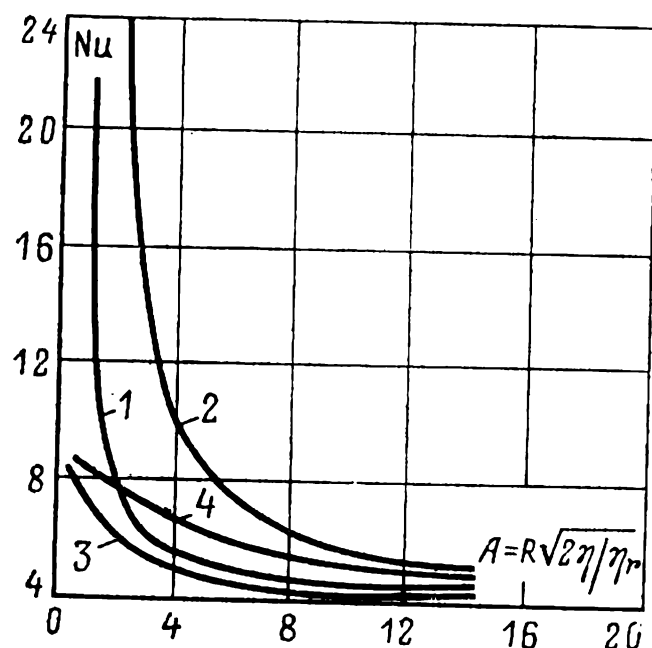


Fig. 1-5. Plot of  $\text{Nu}$  versus  $A$  for the same conditions as in Fig. 1-4

a given point  $(r_1, z_1)$  will differ from the temperature at this point for symmetric fluids.

For  $B > 0.5$  the fluid temperature will be higher than that under ordinary conditions.

Thus, in the convective heat-transfer theory for asymmetric fluids a new criterion, or parameter  $A$ , equal to  $R(2\eta/\eta_r)^{0.5}$  appears. For ordinary (symmetric) fluids this criterion tends to infinity ( $A \rightarrow \infty$ ), since the rotational viscosity coefficient tends to zero ( $\eta_r \rightarrow 0$ ).

### 1-7. HYDRODYNAMICS OF A VORTEX STRUCTURE FLUID

Let us visualize a fluid having a vortex structure, i.e. an ensemble of solid vortices in a state of translational motion. The solution of the Euler equation for vortex flows is known to lead to the Helmholtz theorem on vortex line conservation. This derivation is not, however, confirmed by experiments. The Euler equation does not provide an explanation as to why vortices appear and disappear. The Navier-Stokes equations explain the process of vortex attenuation rather than their formation. Therefore, it is necessary to generalize the Navier-Stokes equation. Prof. N. P. Kasterin [1-18] was the first to direct his attention to the problem and propose a vortex model of fluid.

Let us assume a fluid made up of individual hollow vortices. If two adjacent vortices are considered, the vectors of their angular velocities have the same direction. Then, at the point of their contact the linear velocity of the fluid is not single-valued. The linear velocity due to the first vortex has one direction, the velocity due to the second one has the opposite.

Such a pattern is physically senseless. It is therefore assumed that near hollow vortices, in the area of their contact, there is a thin liquid layer within which the linear velocity varies abruptly from one vortex to another. The concept of discontinuity of the linear velocity of fluid flow formed the basis of Kasterin's hydrodynamic equations.

Thus, the field of velocities of the fluid is assumed to be a discontinuous function  $f(x)$ . For the derivation of hydrodynamic equations, in ref. [1-19]

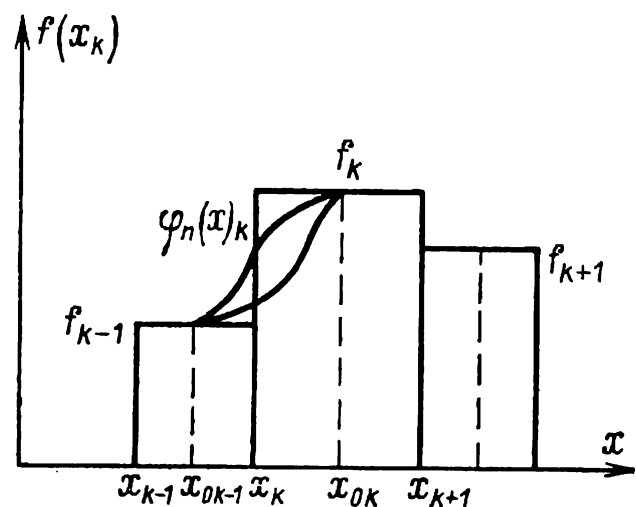


Fig. 1-6. Derivation of equation of elementary volume rotation

the method of finite differences was employed. The essence of this method is that the interval of change in argument  $x$  is divided into  $k$  parts:  $x_k = x_0 + kh$ .

Here, the function  $f(x_k)$  is constant within each grid mesh and changes abruptly at the mesh points (Fig. 1-6). If the coordinates of the centres

of each mesh are designated by

$$x_{0k} = x_k + \frac{1}{2}(x_{k+1} - x_k) \quad (1-7-1)$$

then the discontinuous function within the segment  $(x_{0k-1}, x_{0k})$  may be approximated by a set of smooth functions  $\varphi_n(x)$ . In so doing the functions  $\varphi_n(x)$  must take the values of straight line segments at points  $x_{0k-1}, x_{0k}, \dots$ . The function within the segment  $(x_{0k-1}, x_{0k})$  may behave in any way, since the finite-difference segments do not convolve to points. Evidently, the functions  $\varphi_n(x)$  have inflexions in the segment  $(x_{0k-1}, x_{0k})$ .

According to the theory of generalized functions,  $\varphi_n(x)$  forms a space ( $n = 1, 2, 3$ ) of basic functions for the function  $f = f(x_k)$ . The segment  $(x_{k-1}, x_{k+1})$  is the interval of function  $f(x_k)$ , and  $(x_{0k-1}, x_{0k})$  is the carrier of function  $\varphi_n(x)$ .

These conditions are satisfied if we assume that

$$\begin{aligned} f(x_i) &= \begin{cases} f_{k-1} = \text{const} & \text{at } x_{k-1} \leq x \leq x_k \\ f_k = \text{const} & \text{at } x_k \leq x \leq x_{k+1} \end{cases} \\ \varphi_n(x) &= \begin{cases} f_{k-1} = \text{const} & \text{at } x \leq x_{0k-1} \\ f_k = \text{const} & \text{at } x \geq x_{0k} \end{cases} \\ \varphi_n(x_k) &= \begin{cases} a & \text{if } f_k > f_{k-1} \\ -a & \text{if } f_k < f_{k-1} \end{cases} \end{aligned} \quad (1-7-2)$$

The conditions of contact of  $f(x)$  with straight-line segments amount to equating to zero all the derivatives at points  $x_{0k}, x_{0k-1}, \dots$ , i.e.

$$\frac{d^{(n)}\varphi_n(x)}{dx_n} = 0 \quad \text{at } x_{0k} \leq x \leq x_{0k-1} \quad (1-7-3)$$

The linear continuous functional

$$(f, \varphi) = \int_{x_{0k-1}}^{x_{0k}} f(x_k)\varphi_n(x) dx = f_{k-1} \int_{x_{0k-1}}^{x_k} \varphi_n(x) dx + f_k \int_{x_k}^{x_{0k}} \varphi_n(x) dx \quad (1-7-4)$$

will be called the generalization of the discontinuous function  $f(x_k)$ .

The derivative of the generalized function  $f(x_k)$  is the functional

$$\begin{aligned} (f', \varphi_n) &= \int_{x_{0k-1}}^{x_{0k}} f'(x_k)\varphi_n(x) dx = f\varphi_n \Big|_{x_{0k-1}}^{x_{0k}} \\ &- \int_{x_{0k-1}}^{x_{0k}} f\varphi'_n(x) dx = (f_k - f_{k-1})\varphi_n(x) \end{aligned} \quad (1-7-5)$$

In a similar way, using relation (1-7-3), the following derivatives are calculated:

$$\frac{d^n(f, \varphi_n)}{dx^n} = (-1)^{n-1}(f_k - f_{k-1})\varphi_n^{(n-1)}(x_k) \quad (1-7-6)$$

If the absolute value of the first finite-difference derivative is designated by

$$\frac{df_{k-1}}{dx_{0k-1}} = \frac{f_k - f_{k-1}}{(x_{0k} - x_{0k-1})} \quad (1-7-7)$$

then formula (1-7-6) becomes

$$\frac{d^n(f, \varphi_n)}{dx^n} = \alpha_{n-1} \frac{df_{k-1}}{dx_{0k-1}} = \alpha_{n-1} \frac{f_k - f_{k-1}}{(x_{0k} - x_{0k-1})} \quad (1-7-8)$$

where

$$\alpha_{n-1} = (-1)^{n-1}(x_{0k} - x_{0k-1})\varphi_n^{(n-1)}(x_k) \quad (1-7-9)$$

From formula (1-7-9) it is evident that the coefficient  $\alpha_{n-1}$  determines the direction of discontinuity of the discontinuous function, therefore  $\alpha_{n-1} > 0$  or  $\alpha_{n-1} < 0$ .

The obtained relations enable the equation of motion for a vortex-structure fluid to be derived.

Let the hydrodynamic velocity vector  $\vec{v}(\vec{r}_k)$  be described by the discontinuous function  $f(x_k)$ . In a fluid flow the elementary mesh  $(\vec{r}_k - \vec{r}_{k-1})$  is isolated. Here  $\vec{r}$  is the position vector with coordinates  $x, y, z$  (Fig. 1-7).

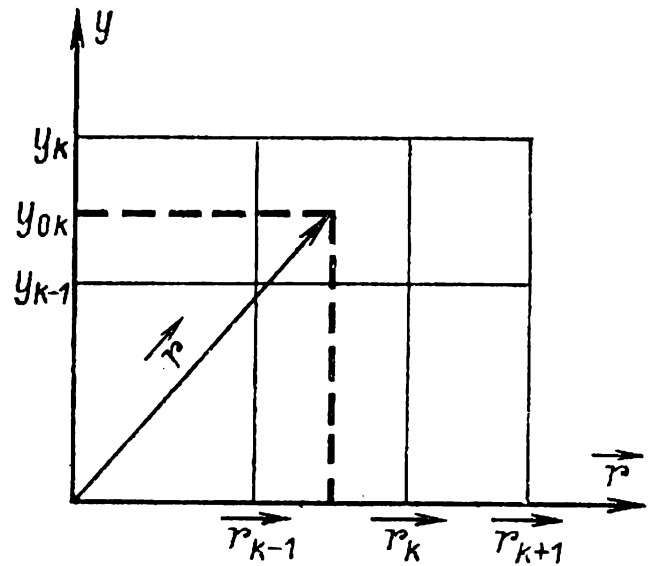


Fig. 1-7. Elementary mesh in a fluid in motion

Expanding the velocity  $v_k$  of mesh  $(\vec{r}_k, \vec{r}_{k+1})$  into a Taylor series by using formula (1-7-8), we get

$$\vec{v}_k = \vec{v}_{k-1} + (\vec{r}_{0k} - \vec{r}_{0k-1})\alpha_{n-1} \frac{\partial \vec{v}_{k-1}}{\partial \vec{r}_{0k-1}} \quad (1-7-10)$$

Let  $b_k$  be the ratio of the dimensions of two grid meshes

$$b_k = \frac{\vec{r}_{k+1} - \vec{r}_k}{\vec{r}_k - \vec{r}_{k-1}} \quad (1-7-11)$$

then

$$\vec{v}_k = \vec{v}_{k-1} + \beta(\vec{r}_{0k-1} - \vec{r}_{k-1}) \frac{\partial \vec{v}_{k-1}}{\partial \vec{r}_{0k-1}} \quad (1-7-12)$$

where

$$\beta = (1 + b_k)\alpha_{n-1} \quad (1-7-13)$$

From formula (1-7-9) it follows that the coefficient  $\beta$  may be positive or negative depending on  $\alpha_{n-1}$ . The change in velocity from  $\vec{v}_{k-1}$  to  $\vec{v}_k$  will result in additional rotation of all points of the grid mesh around the centre of gravity of the mesh with angular velocity (Fig. 1-8)

$$\vec{\omega}_{k-1} = \alpha_{n-1} \text{curl } \vec{v}_{k-1} \quad (1-7-14)$$

Relationship (1-7-14) is of particular interest.

Usually the angular velocity is defined in terms of linear velocity by  $\vec{\omega} = 1/2 \text{curl } \vec{v}$ . In the present case this relationship is more general. The coefficient  $\alpha_{n-1}$  is similar to the coefficient of adhesion of a fluid particle

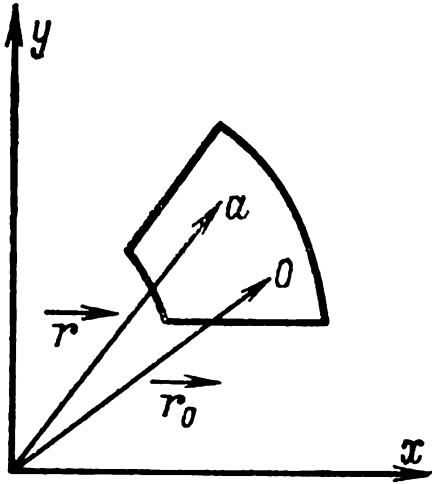


Fig. 1-8. Calculation of rotation of an elementary volume

to the surrounding fluid in the theory of asymmetric hydrodynamics, but unlike the latter the coefficient  $\alpha_{n-1}$  may be greater or less than zero. This additional rotation will change the mean linear velocity of all points of the neighbouring mesh by a value  $\vec{v}'_k$ , i.e.

$$\begin{aligned} \vec{v}'_k &= \alpha_{n-1} [\text{curl } \vec{v}_{k-1} \times (\vec{r}_{0k} - \vec{r}_{0k-1})] \\ &= \beta [\text{curl } \vec{v}_{k-1} \times (\vec{r}_{0k} - \vec{r}_{0k-1})] \end{aligned} \quad (1-7-15)$$

Let us also take into account the rate  $\vec{v}''_k$  of volume expansion for mesh deformation

$$\vec{v}''_k = \beta(\vec{r}_{0k-1} - \vec{r}_{k-1}) \text{div } \vec{v}_{k-1} \quad (1-7-16)$$



Thus, the total velocity of the grid mesh  $k$  will be

$$\vec{v} = \vec{v}_k + \vec{v}'_k + \vec{v}''_k \quad (1-7-17)$$

The kinetic energy of the grid mesh of mass  $m$  is

$$E_k = \frac{mv^2}{2} \approx m \left\{ \frac{1}{2} v_{k-1}^2 + \beta(\vec{r}_{0k-1} - \vec{r}_{k-1}) \left[ \vec{v}_{k-1} \frac{\partial \vec{v}_{k-1}}{\partial \vec{r}_{0k-1}} + \text{curl } \vec{v}_{k-1} \times \vec{v}_{k-1} + \vec{v}_{k-1} \text{div } \vec{v}_{k-1} \right] \right\} \quad (1-7-18)$$

The potential energy of the grid mesh in the presence of a uniform field of surface forces  $\vec{F}$  is

$$E_p = -\vec{F} \vec{r}_{0k} \quad (1-7-19)$$

Then the projections of vector  $\vec{F}_i$  will be

$$\vec{F}_i = \frac{m}{\rho} \frac{\partial \vec{P}_i}{\partial x_i}; \quad i = 1, 2, 3 \quad (1-7-20)$$

where  $\rho$  is the density of the grid mesh and  $\vec{P}$  is the stress of the grid mesh.

Formula (1-7-19) now takes the form

$$E_p = -\frac{m}{\rho} \left( \frac{\partial P_i}{\partial x_i} \right) \vec{r}_{0k}; \quad i = 1, 2, 3 \quad (1-7-21)$$

The Lagrangian equation will be used, in which  $\vec{r}_{0k-1}$  and  $\vec{v}_{k-1}$  are the generalized coordinates ( $\vec{v}_{k-1} = d\vec{r}_{0k-1}/d\tau$ )

$$\frac{\partial H}{\partial \vec{r}_{0k-1}} - \frac{d}{d\tau} \frac{\partial H}{\partial \vec{v}_{k-1}} = 0 \quad (1-7-22)$$

where  $H = E_p - E_k$  is the kinetic potential.

Let us find the necessary relations

$$\begin{aligned} \frac{\partial H}{\partial \vec{r}_{0k-1}} = & -m \left\{ \frac{1}{\rho} \frac{\partial P_i}{\partial x_i} + \beta \left[ \vec{v}_{k-1} \frac{\partial \vec{v}_{k-1}}{\partial \vec{r}_{0k-1}} + (\text{curl } \vec{v}_{k-1} \times \vec{v}_{k-1}) + \vec{v}_{k-1} \text{div } \vec{v}_{k-1} \right] \right\} \end{aligned} \quad (1-7-23)$$

$$\frac{d}{d\tau} \frac{\partial H}{\partial \vec{v}_{k-1}} = m \frac{d\vec{v}_{k-1}}{d\tau} = m \left[ \frac{\partial \vec{v}_{k-1}}{\partial \tau} + (\vec{v}_{k-1} \vec{\nabla}) \vec{v}_{k-1} \right] \quad (1-7-24)$$

Then

$$\frac{\partial \vec{v}_{k-1}}{\partial \tau} + (1 - \beta)(\vec{v}_{k-1} \cdot \vec{\nabla}) \vec{v}_{k-1} - \beta \vec{v}_{k-1} \operatorname{div} \vec{v}_{k-1} = \frac{1}{\rho} \frac{\partial P_i}{\partial x_i} \quad (1-7-25)$$

The above equation is valid for any adjacent meshes, and the subscript  $(k-1)$  may, therefore, be dropped. Then, using the Stokes formula for a viscous stress tensor, we arrive at

$$\rho \frac{d\vec{v}}{d\tau} = - \vec{\nabla} p + \eta \nabla^2 \vec{v} + \left( \eta_v - \frac{1}{3} \eta \right) \vec{\nabla} \operatorname{div} \vec{v} + \beta (\vec{v} \cdot \vec{\nabla}) \vec{v} + \beta \operatorname{div} \vec{v} \quad (1-7-26)$$

This equation is identical with the Predvoditelev equation, only the coefficient  $\beta$  has a different physical significance.

In the Predvoditelev equation for a rarefied gas, the coefficient  $\beta$  is the nonideal continuity parameter; it describes the change in the hydrodynamic velocity within a physically small volume, over which all micro-quantities are averaged on the basis of gas-kinetic concepts.

In the present case, the coefficient  $\beta$  determines the choice of the basic function, which approximates the discontinuous function, but according to formula (1-7-13)  $\beta$  characterizes the discrete structure of a fluid (i.e. the transition from one hollow vortex to another as per the law of averaging discontinuous functions), as well as the interaction between the hollow vortex and the surrounding fluid that separates these hollow vortices.

The common points in the two cases are the discrete structure of a fluid and the averaging procedure involving expansion of an unknown function into the Taylor series. The latter derivation is however more general.

In particular, equation (1-7-26) is valid for incompressible ( $\operatorname{div} \vec{v} \neq 0$ ) and inviscid ( $\eta = 0$ ) fluids.

The coefficient  $\beta$  in the present case depends on the vortex structure itself. If  $\beta=2$ , then from equation (1-7-26) Kasterin's equation is obtained. For  $\eta = 0$  and  $\operatorname{div} \vec{v} = 0$ , we obtain from equation (1-7-26) the solution in the form of the Bernoulli formula

$$\frac{1}{2} (1 - \beta) \rho v^2 + p = p_0 + \frac{1}{2} \rho v_0^2 \quad (1-7-27)$$

where  $p_0$  and  $v_0$  are the pressure and velocity in the main flow around a body.

If the pressure and the velocity behind the body in the flow are designated by  $p_\infty$  and  $v_\infty$ , respectively, we get

$$(p_\infty - p_0) = \frac{1}{2} \rho [v_0^2 - (1 - \beta) v_\infty^2] \quad (1-7-28)$$

If  $\beta = 0$ , we obtain the classical Euler formula

$$p_{\infty} - p_0 = \frac{1}{2} \rho(v_0^2 - v_{\infty}^2) \quad (1-7-29)$$

For an inviscid fluid ( $\eta = 0$ ),  $v_0 = v_{\infty}$ , which leads to D'Alembert's paradox ( $p_{\infty} = p_0$ ).

Assuming  $\beta = 2$ , we get Kasterin's relation

$$p_{\infty} - p_0 = \frac{1}{2} \rho(v_0^2 + v_{\infty}^2) \quad (1-7-30)$$

which was experimentally verified by Milovich [1-20].

Relations (1-7-28) and (1-7-29) show that the drag force is proportional to the dynamic pressure. This fact agrees with the Zhukovsky formulas for the pressure force of flow created by solid vortices. The incompleteness of Euler equations leads to the fact that for abrupt changes in the gradients of hydrodynamic quantities in a fluid flow, the equations in the Euler form do not have unique solutions [1-21]. The same difficulties appear when calculating flows with local shock waves.

The flow of a discrete fluid in the form of a set of hollow vortices described by equation (1-7-26) is also of interest for approximate solutions of asymmetric hydrodynamics.

The generalized Navier-Stokes equation for asymmetric hydrodynamics is given in Section 1-6 (equation 1-6-8). The last term of equation (1-6-8) is found from the additional equation describing transport of micro-momenta moment, i.e.

$$2\gamma \operatorname{curl} \vec{\omega} = f(\gamma, \vec{v}, K, A)$$

Earlier it is mentioned that there is a certain analogy between the coefficients  $\gamma$  and  $\beta$ . Equations (1-6-8) and (1-7-26) differ from the Navier-Stokes equation by the additional terms. Unlike equation (1-6-8), equation (1-7-26) contains only one additional coefficient  $\beta$ , while to solve equation (1-6-8) it is necessary to know three coefficients of rotational viscosity, the coefficient  $\gamma$  and the field of angular velocities of fluid particles.

This difference is quite clear, since in asymmetric hydrodynamics the mechanism of momentum transfer by translational and rotational diffusion is described in detail. The common feature is that a fluid (liquid) has a discrete structure and, consequently, to correctly describe such a fluid as a homogeneous one the mathematical formulas of the discontinuous function theory should be used. It is of interest to investigate the interaction between such a fluid and solid surface. Usually fluid particles are assumed to adhere to a solid surface, i.e. the translational velocity of the fluid at the solid surface equals zero;  $\vec{v} = \vec{v}_0$ , where  $\vec{v}_0$  is the velocity of the solid surface (boundary velocity).

The mechanism of interaction between a hollow vortex and a solid surface is not clear. In the extreme case, when the solid surface and the fluid are in strong interaction, the angular velocity of a vortex is equal to the surface velocity itself

$$(\omega)_s = \frac{1}{2} \text{curl } \vec{v}_0 \quad (1-7-31)$$

In another extreme case (extremely weak interaction), the hollow vortex rotates freely relative to the solid surface

$$(M_i)_s = 0 \quad (1-7-32)$$

where  $M_i$  is the density of hollow vortex moments.

In a general case, the angular velocity drop  $\Delta\omega_j$  ( $\Delta\omega_j = \omega_j - 1/2 \text{curl } v_0$ ) may be assumed proportional to  $(M_i)_s$ . The proportionality factor will characterize surface friction. In our model the hollow vortex is a part of the discrete-structure fluid. Therefore, on the basis of relations (1-7-14), we may conclude that at the solid surface the fluid velocity is not zero, i.e. the adhesion condition does not hold

$$(v_i - v_0)_s = \delta \frac{\partial v_i}{\partial x_j} \quad (1-7-33)$$

where  $\delta$  is the slip factor, which is directly proportional to  $(1-\beta)^2$ , i.e.  $\delta \sim (1-\beta)^2$  (1-7-34)

The above relation is similar to formula (1-5-77) obtained by a molecular-kinetic method. The qualitative relationships obtained for a solid boundary with a vortex structure fluid are also similar to conclusions derived from the solutions of equations of asymmetric hydrodynamics.

The smaller the linear dimensions of the system, the greater is the deviation from the results of classical (symmetric) hydrodynamics. Nonclassical results may be obtained if in the formulas of general hydrodynamics (the amount of fluid flowing out of tubes, drag forces, viscosity), the actual dimension is substituted by the effective one  $l_{eff}$  ( $l_{eff} = l + \Delta$ ), where  $\Delta$  depends on fluid properties. The latter is equivalent to the fact that by keeping  $l = \text{const}$ , we assume fluid slip conditions at the solid surface.

## 1-8. INHOMOGENEOUS TURBULENCE HEAT TRANSFER

Both laminar and turbulent flows are known to exist. The photo showing the motion of smoke from a cigarette is well known [1-22]. First, smoke rises in a straight column which then becomes wavy and curly and finally disappears completely, blending with the air. In the beginning the flow is laminar, then it becomes turbulent. Hydrodynamic agitation of the fluid is a peculiar feature of turbulent flows. This increases the rate of momentum, heat and mass transfer in comparison to a laminar flow.

If at certain points the velocities of turbulent flow are measured at different times, we get the relation  $v = f(\tau)$  in the form of a graph shown in Fig. 1-9.

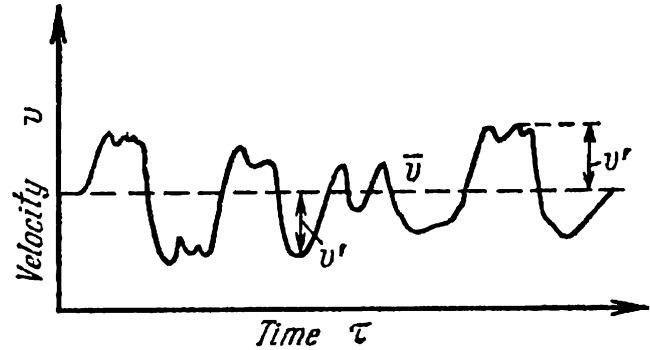


Fig. 1-9. Velocity fluctuations in a turbulent flow

Inhomogeneous turbulence is one occurring when there is velocity gradient of the main flow.

According to the Reynolds concept (1895), velocity may be expressed as the sum of mean velocity  $\bar{v}$  and fluctuation velocity  $v'$

$$\bar{v} = \bar{\bar{v}} + \bar{v'}; \quad v_i = \bar{v}_i + v'_i \quad (1-8-1)$$

A turbulent flow is termed steady when  $\bar{v}_i$  is independent of time.

For the fields of vector and scalar quantities the Reynolds' concept may be extended to all properties, i.e. any value may be presented as a mean value and its fluctuations.

In accordance with the averaging rules we have

$$\overline{v_i v_j} = \bar{v}_i \bar{v}_j + \overline{v'_i v'_j} \quad (1-8-2)$$

i.e.  $\overline{v'_i v'_j} \neq 0$  and  $\overline{v_i'^2} \neq 0$ .

The averaging rules allow one to derive transfer equations with turbulent flow.

#### (a) Momentum Transfer Equations

The continuity equation for a turbulent flow may be written as follows:

$$\frac{\partial \rho}{\partial \tau} + \text{div } \rho \bar{v} = \frac{\partial \bar{\rho}}{\partial \tau} + \text{div } (\bar{\rho} \bar{v}) + \text{div } (\rho' \bar{v}') = 0 \quad (1-8-3)$$

as  $\bar{\rho}' = 0$ .

Thus, for a turbulent flow the continuity equation is supplemented by  $\text{div } (\rho' \bar{v}')$ .

If the density is constant ( $\bar{\rho} = \text{const}$ ;  $\rho' = 0$ ), then from (1-8-3) we get

$$\text{div } \bar{v} = 0 \quad (1-8-4)$$

Then

$$\text{div } \bar{v} = \text{div } \bar{v} + \text{div } \bar{v}' = 0$$

hence

$$\operatorname{div} \vec{v}' = 0 \quad (1-8-5)$$

Before deriving the Reynolds equations for a turbulent flow, let us write down the Navier-Stokes equation which for  $v_x$  is expressed as

$$\rho \frac{dv_x}{d\tau} = -\frac{\partial p}{\partial x} + \eta \nabla^2 v_x \quad (1-8-6)$$

Here we assume that the fluid is incompressible and no external forces act on it. For total derivative  $dv_x/d\tau$  we may write the expression

$$\begin{aligned} \frac{dv_x}{d\tau} &= \frac{\partial v_x}{\partial \tau} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \\ &= \frac{\partial v_x}{\partial \tau} + \frac{\partial v_x^2}{\partial x} + \frac{\partial v_x \partial v_y}{\partial y} + \frac{\partial v_x v_z}{\partial z} \end{aligned} \quad (1-8-7)$$

that is readily checked by differentiation and from the continuity equation

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

If  $v_x$ ,  $v_y$ ,  $v_z$  are substituted by corresponding expressions from (1-8-1), we get

$$\begin{aligned} \rho \left[ \frac{\partial(\bar{v}_x + v'_x)}{\partial \tau} + \frac{\partial(\bar{v}_x + v'_x)(\bar{v}_x + v'_x)}{\partial x} + \frac{\partial(\bar{v}_x + v'_x)(\bar{v}_y + v'_y)}{\partial y} \right. \\ \left. + \frac{\partial(\bar{v}_x + v'_x)(\bar{v}_z + v'_z)}{\partial z} \right] = \frac{\partial(\bar{p} + p')}{\partial x} \\ + \eta \left[ \frac{\partial^2(\bar{v}_x + v'_x)}{\partial x^2} + \frac{\partial^2(\bar{v}_x + v'_x)}{\partial y^2} + \frac{\partial^2(\bar{v}_x + v'_x)}{\partial z^2} \right] \end{aligned}$$

after averaging we get

$$\begin{aligned} \rho \frac{d\bar{v}_x}{d\tau} &= \rho \left[ \frac{\partial \bar{v}_x}{\partial \tau} + \bar{v}_x \frac{\partial \bar{v}_x}{\partial x} + \bar{v}_y \frac{\partial \bar{v}_x}{\partial y} + \bar{v}_z \frac{\partial \bar{v}_x}{\partial z} \right] \\ &= -\frac{\partial \bar{p}}{\partial x} + \eta \nabla^2 \bar{v}_x - \rho \left[ \frac{\partial \overline{v_x'^2}}{\partial x} + \frac{\partial \overline{v'_x v'_y}}{\partial y} + \frac{\partial \overline{v'_x v'_z}}{\partial z} \right] \end{aligned} \quad (1-8-8)$$

Equation (1-8-8) differs from equation (1-8-6) in that the actual values  $v_x$ ,  $p$  have been substituted by averaged values  $\bar{v}_x$ ,  $\bar{p}$ , and an additional term

appears to account for the fluctuation characteristics (the expression in square brackets on the right hand side of the equation).

The above relationships may be obtained by a more general procedure. The average value with respect to time of any quantity  $\bar{B}(\tau)$  is defined by

$$\bar{B}(\tau) \equiv \frac{1}{\Delta\tau} \int_{\tau}^{\tau+\Delta\tau} B(\tau') d\tau' \quad (1-8-9)$$

where  $\bar{B}(\tau)$  may be a scalar, vector or a tensor quantity.

Use of (1-8-9) in the continuity equation

$$\frac{1}{\Delta\tau} \int_{\tau}^{\tau+\Delta\tau} \left( \frac{\partial \rho}{\partial \tau'} + \text{div } \rho \bar{v} \right) d\tau' = 0 \quad (1-8-10)$$

yields

$$\frac{\partial \bar{\rho}}{\partial \tau} + \text{div } (\bar{\rho} \bar{v}) = 0 \quad (1-8-11)$$

This is an averaged over time continuity equation which is identical with (1-8-3) if formula (1-8-2) is accounted for. The use of averaging rules in the motion equation

$$\frac{1}{\Delta\tau} \int_{\tau}^{\tau+\Delta\tau} \left[ \frac{\partial \rho \bar{v}}{\partial \tau'} + \text{div } (\rho \bar{v} \bar{v}) + \bar{\nabla} p - \text{div } \sigma - \rho \bar{F} \right] d\tau' = 0 \quad (1-8-12)$$

gives

$$\frac{\partial \bar{\rho} \bar{v}}{\partial \tau} + \text{div } (\bar{\rho} \bar{v} \bar{v}) + \bar{\nabla} \bar{p} + \text{div } (\bar{\sigma} + \sigma_t) + \bar{\rho} \bar{F} = 0 \quad (1-8-13)$$

where the turbulent tensor of the Reynolds stresses equals

$$\sigma_t = \bar{\rho} \bar{v} \bar{v} - \bar{\rho} \bar{v} \bar{v} \quad (1-8-14)$$

In case of an incompressible fluid

$$\rho (\partial \bar{v} / \partial \tau + \bar{\nabla} \bar{v} \cdot \bar{v}) = - \bar{\nabla} \bar{p} + \text{div } (\bar{\sigma} + \sigma_t) + \rho \bar{F} \quad (1-8-15)$$

where

$$\sigma_t = \rho (\bar{v} \bar{v} - \bar{v} \bar{v}) - \rho (\bar{v} - v)(\bar{v} - v) \quad (1-8-16)$$

(b) *Turbulent Friction*

The quantities  $\rho \overline{v'_i v'_j}$  are the characteristics of a turbulent flow, which are called Reynolds turbulent stresses. In Cartesian coordinates the turbulent stress tensor is of the form

$$\rho \begin{pmatrix} \overline{v_x'^2} & \overline{v_x' v_y'} & \overline{v_x' v_z'} \\ \overline{v_y' v_x'} & \overline{v_y'^2} & \overline{v_y' v_z'} \\ \overline{v_z' v_x'} & \overline{v_z' v_y'} & \overline{v_z'^2} \end{pmatrix} \quad (1-8-17)$$

To illustrate this let us consider friction in a two-dimensional turbulent flow ( $v_z = 0$ ). Let the distribution of a longitudinal averaged component of velocity  $\overline{v_x}(y)$  close to the wall be represented by the curve shown in Fig. 1-10. Then the pulsating flow momentum  $\rho v'_x$  of an individual fluid

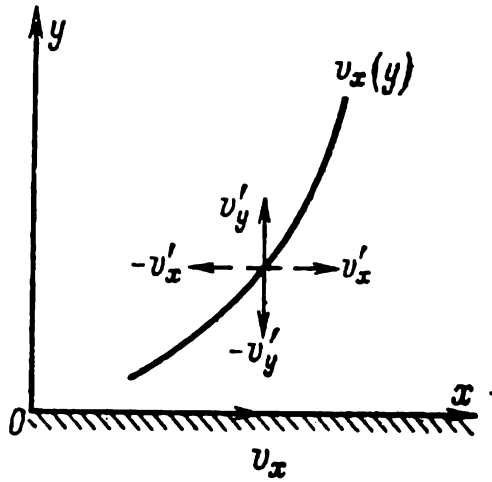


Fig. 1-10. Distribution of averaged longitudinal velocity in a turbulent flow past a solid

mole will move upward and downward with fluctuating transverse velocity  $v'_y$ . These fluctuations will add turbulent stress  $\rho \overline{v'_x v'_y}$  to the internal molecular stress. The turbulent shear stress thus equals

$$(\sigma_{xy})_t = \rho \overline{v'_x v'_y}. \quad (1-8-18)$$

At any plane of a turbulent flow the total shear stress is the sum of internal molecular stress  $\sigma_m$  and the turbulent stress  $\sigma_t$ ,

$$\sigma_{ij} = (\sigma_{ij})_m + (\sigma_{ij})_t = -\nu\rho \left( \frac{\partial \overline{v}_i}{\partial \xi_j} + \frac{\partial \overline{v}_j}{\partial \xi_i} \right) + \rho \overline{v'_i v'_j} \quad (1-8-19)$$

(c) *Energy and Mass Transfer Equations*

If enthalpy diffusion is neglected, the Fourier-Kirchhoff equation in the absence of mass sources may be written as

$$c_p \rho \frac{\partial T}{\partial \tau} = \lambda \nabla^2 T + \frac{dp}{d\tau} + \eta \Phi_v \quad (1-8-20)$$



Let us use the averaging formula (1-8-9)

$$c_p \rho \frac{\partial \bar{T}}{\partial \tau} + c_p \rho \operatorname{div} (\bar{v} T) = \operatorname{div} (\lambda \nabla \bar{T}) + \eta \bar{\Phi}_v \quad (1-8-21)$$

$$\bar{\Phi}_v = (\bar{\Phi}_v)_m + (\bar{\Phi}_v)_t$$

Specific heat  $c_p \rho$  and thermal conductivity  $\lambda$  are assumed constant ( $\rho = \text{const}$ ;  $\operatorname{div} \bar{v} = 0$ ).

Since

$$\bar{v} T = \bar{v} \bar{T} - \bar{v} \bar{T} + \bar{v} T = \bar{v} \bar{T} - (\bar{v} \bar{T} - \bar{v} T) \quad (1-8-22)$$

equation (1-8-21) may be written in a different form

$$c_p \rho \left( \frac{\partial \bar{T}}{\partial \tau} + \bar{v} \nabla \bar{T} \right) = \operatorname{div} (\lambda \nabla \bar{T}) + \operatorname{div} (q_t) + \eta \bar{\Phi}_v \quad (1-8-23)$$

where the turbulent transfer vector

$$q_t = c_p \rho (\bar{v} \bar{T} - \bar{v} T) \quad (1-8-24)$$

The dissipative function of turbulent transfer

$$(\bar{\Phi}_v)_t = \frac{1}{2} \sum_{i,j} [2\bar{v}_{i,j}^2 + (\bar{v}'_{i,j} + \bar{v}'_{j,i})^2] \quad (1-8-25)$$

In a similar way, the mass transfer in a turbulent flow takes place due to molecular and turbulent diffusion

$$\vec{j}_k = \vec{j}_{km} + \vec{j}_{kt} = -D\rho \nabla \bar{\rho}_{k0} + \rho \overline{\rho'_{k0} \vec{v}'} \quad (1-8-26)$$

where  $\rho_{k0}$  is the relative concentration ( $\rho_{k0} = \bar{\rho}_{k0} + \rho'_{k0}$ ).

Substitution of (1-8-26) into the mass transfer equation followed by averaging gives

$$\rho \frac{d\bar{\rho}_{k0}}{d\tau} = D\rho \nabla^2 \bar{\rho}_{k0} - \rho \operatorname{div} \overline{\rho'_{k0} \vec{v}'} \quad (1-8-27)$$

As the number of unknown functions appearing in nonlinear equations is more than the number of equations that describe them, then the turbulent transfer problem consists in obtaining the transfer equations in a closed form.

#### (d) Earlier Phenomenological Transfer Theories

During the earlier stages of turbulent transfer studies the phenomenological theories were aimed at describing only the averaged fields. First of all we shall dwell on the Boussinesque theory [1-23].

**Boussinesque's theory.** The main points of this theory are as follows. The turbulent flows of vector and scalar quantities are similar in structure to the corresponding molecular flows, i.e. they are proportional to the gradients of a corresponding quantity. Let us now illustrate this on the example of momentum and heat transfer.

Consider heat and momentum transfer in a boundary layer flowing around an infinitely long flat plate. The molecular momentum and heat transfer is described by the following equations

$$(\sigma_{xy})_m = -\eta \frac{\partial v_x}{\partial y}; \quad q_m = -\lambda \frac{\partial T}{\partial y} \quad (1-8-28)$$

Hence

$$q_m = -(\sigma_{xy})_m \frac{\lambda}{\eta} \frac{\partial T}{\partial v_x} = -(\sigma_{xy})_m \frac{a}{\nu} c_p \frac{\partial T}{\partial v_x} \quad (1-8-29)$$

Now we shall consider the momentum and heat transfer through a reference surface in a boundary layer (Fig. 1-11). Through a unit area of this

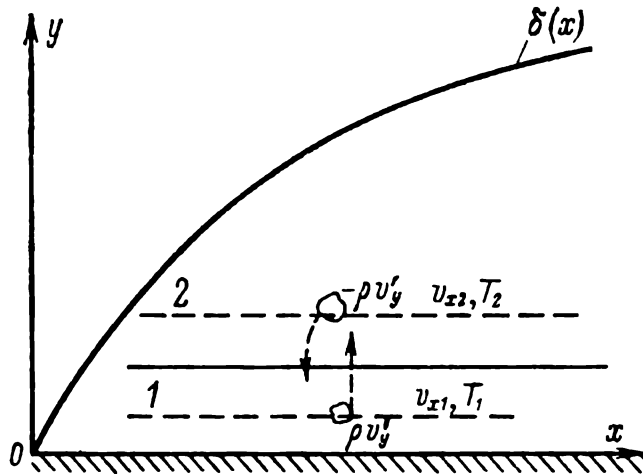


Fig. 1-11. Approximate schematic of turbulent heat transfer

surface will pass per unit time a fluid mass  $\rho v'_y$ , the velocity and temperature of which before transfer were  $v_{x1}$  and  $T_1$ , respectively. This quantity of fluid and enthalpy  $c_p \rho v'_y T_1$  will be transferred to surface 2. In the steady state the same quantity of the fluid and enthalpy will be transferred by a turbulent exchange from plane 2 to plane 1. The enthalpy difference  $(c_p \rho v'_y T_2 - c_p \rho v'_y T_1)$  will be equal to the turbulent heat flux

$$q_t = \rho c_p v'_y (T_2 - T_1) \quad (1-8-30)$$

If  $v_{x2} > v_{x1}$ , then the transition of fluid particles (moles) from layer 1 to layer 2 is accompanied by an increase in the velocity, and vice versa. Turbulent transfer thus involves a momentum change that results in turbulent stress

$$(\sigma_{xy})_t = \rho v'_y (v_{x2} - v_{x1}) \quad (1-8-31)$$

Hence

$$q_t = c_p(\sigma_{xy})_t \frac{\Delta T}{\Delta v_x} \approx c_p(\sigma_{xy})_t \frac{dT}{dv_x} \quad (1-8-32)$$

This relationship was originally obtained by Reynolds [1-22] and is called the Reynolds analogy.

Comparison of (1-8-32) and (1-8-29) may result in the formal conclusion that molecular and turbulent transfers may be described in a similar way if the Prandtl number equals unity ( $Pr = \nu/a = 1$ ).

$$(\sigma_{xy})_t = -\nu_t \rho \frac{\partial v_x}{\partial y} \quad (1-8-33)$$

$$q_t = -c_p \rho a_t \frac{\partial T}{\partial y} \quad (1-8-34)$$

where  $\nu_t$  and  $a_t$  are the coefficients of turbulent viscosity and turbulent thermal diffusivity, respectively.

Let us use  $\nu_\varepsilon$  and  $a_\varepsilon$  ( $a_\varepsilon \equiv a_t$ ) to denote turbulent kinematic viscosity and turbulent thermal diffusivity. Here, the subscript  $t$  is substituted by  $\varepsilon$  and the subscript indicating laminar transfer is omitted.

Mass transfer by turbulent motion is generally called turbulent diffusion.

By analogy with molecular diffusion, we may write

$$(j_k)_\varepsilon = -\rho D_\varepsilon \overrightarrow{\nabla} \bar{\rho}_{k0} \quad (1-8-35)$$

where  $D_\varepsilon$  is the turbulent diffusion coefficient.

Thus, according to the Boussinesque theory [1-23] the turbulent transfer is an analog of molecular transfer. This analogy is, certainly, a rough approximation; it may be adopted as a design diagram for which turbulent transfer coefficients  $\nu_\varepsilon$ ,  $a_\varepsilon$ ,  $D_\varepsilon$  are found experimentally.

In a general case transfer equations are written as follows

$$\bar{\sigma} = \rho(\nu + \nu_\varepsilon) \bar{\dot{\varepsilon}}; \quad \sigma_{ij} = \rho(\nu + \nu_\varepsilon) \dot{\varepsilon}_{ij} \quad (1-8-36)$$

$$q = \rho c_p(a + a_\varepsilon) \overrightarrow{\nabla} \bar{T}; \quad j_k = \rho(D + D_\varepsilon) \overrightarrow{\nabla} \bar{\rho}_{k0} \quad (1-8-37)$$

where  $\bar{\dot{\varepsilon}}$  is the strain rate tensor.

**Prandtl's theory.** Prandtl [1-24] made an attempt to relate  $\nu_\varepsilon$  and  $a_\varepsilon$  with characteristics of turbulence. According to the kinetic theory of gases the coefficient of kinematic viscosity is proportional to the product of root-mean-square velocity of thermal motion and mean free path length. The

turbulent momentum of a liquid mole may similarly be considered as a transfer substance (that retains its properties up to a definite distance).

Let us now take an example similar to the one considered above (a flat plate in a turbulent flow).

The distribution of an averaged longitudinal velocity in a boundary layer is shown in Fig. 1-12. Turbulent stress  $(\sigma_{xy})_s$  is defined by  $\rho \overline{v'_x v'_y}$ .

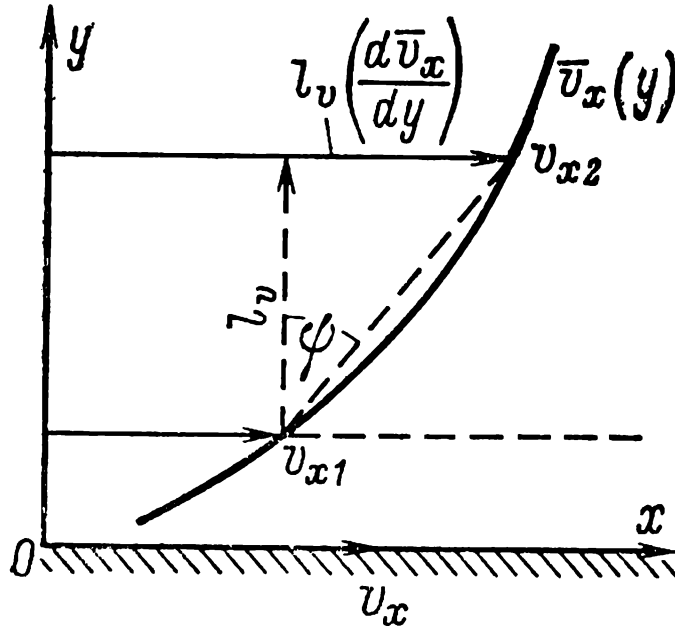


Fig. 1-12. Derivation of the Prandtl formula

According to Fig. 1-12 we have

$$\bar{v}_{x2} - \bar{v}_{x1} = l_v \tan \psi = -l_v \left| \frac{d\bar{v}_x}{dy} \right| \quad (1-8-38)$$

Based on the continuity equation for a plane flow

$$\frac{\partial \bar{v}_x}{\partial x} = -\frac{\partial \bar{v}_y}{\partial y} \quad (1-8-39)$$

it may be assumed that

$$v_{y2} - v_{y1} = l_v \left| \frac{d\bar{v}_x}{dy} \right| \quad (1-8-40)$$

Then

$$(\sigma_{xy})_s = \rho \overline{v'_x v'_y} = -l_v^2 \rho \left| \frac{d\bar{v}_x}{dy} \right| \frac{d\bar{v}_x}{dy} \quad (1-8-41)$$

Comparison of the expression for  $(\sigma_{xy})_e$  with formula (1-8-26) yields

$$v_e = v_t = l_v^2 \left| \frac{d\bar{v}_x}{dy} \right| \quad (1-8-42)$$

where  $l_v$  is the parameter of length proportional to the root-mean-square value of the distance over which turbulent liquid moles can be identified. Parameter  $l_v$  is generally called the mixing length of momentum transfer. This analogy may be used when considering the turbulent flow of a scalar substance (concentration or temperature) as a transferable substance. In so doing the turbulent diffusion coefficient takes the form

$$a_e = D_e = l_m^2 \left| \frac{d\bar{v}_x}{dy} \right| \quad (1-8-43)$$

where  $l_m$  is the parameter of length characterizing the distance over which the turbulent flow of the scalar substance can be identified.

It should be pointed out that parameters  $l_v$  and  $l_m$  have no definite physical meaning: they are used to relate theoretical and experimental averaged values  $\bar{v}$ ,  $\bar{T}$ ,  $\bar{\rho}_{k0}$ .

**Taylor's theory.** The Taylor transfer theory [1-25] is based on the assumption that in a turbulent flow a vorticity has the properties of a transportable quantity. In complete consistency with the momentum transfer theory, the vortex flow  $\omega'_v = \frac{\partial v'_2}{\partial \xi_1} - \frac{\partial v'_1}{\partial \xi_2}$  is proportional to the gradient of averaged vorticity. In this expression the turbulent vorticity factor

$$\varepsilon_\omega = l_\omega^2 \left| \frac{d\bar{v}_x}{dy} \right| \quad (1-8-44)$$

is used as a proportionality factor, i.e.

$$\overline{\rho v'_y \omega'_v} = -\rho \frac{d}{dy} \overline{v'_x v'_y} = \rho l_\omega^2 \left| \frac{d\bar{v}_x}{dy} \right| \frac{d\bar{\omega}}{dy} \quad (1-8-45)$$

where  $\bar{\omega} = -\frac{d\bar{v}_x}{dy}$  is the averaged vorticity.

The two theories, based on the "mixing length" concept, have the following disadvantages:

1. They fail to describe transfer processes at points where  $\frac{d\bar{v}_x}{dy} = 0$  and  $\frac{d\bar{T}}{dy} = 0$ . At these points  $v_e = a_e = \varepsilon_\omega = 0$ , whereas the turbulent transfer coefficients should be finite quantities determined by turbulent transfer of flows  $\overline{v'_i v'_i}$  and  $\overline{v'_i T'}$  with velocity of fluctuation motion.

2. They do not account for the pressure fluctuation effect on transfer processes, although it is a very essential effect as the momentum transfer can occur under the action of pressure gradient even when an "individual" liquid mole is immovable.

3. The concept of "eddy" viscosity and thermal conductivity is purely formal.

4. The concept of a "mixing length" is very ambiguous as well as the concept of "individuality" of turbulent moles.

Many more disadvantages of phenomenological turbulence theories based on the "mixing length" hypothesis can be listed. But it should be borne in mind that all these defects are inherent in the considered theories which do not provide a correct description of transfer mechanisms. The fruitfulness of the theories may be attributed to the fact that they have made it possible to obtain semi-theoretical relationships which have been put into practice.

The briefly discussed turbulence theories are deductive: certain assumptions about fluctuating momentum flows, scalar substance or vorticity are made. Based on these assumptions the averaged velocity and scalar quantity profiles are derived (deduced) by the use of averaging transfer equations. It is obvious that in this case the averaged characteristics, such as velocity or scalar quantity profiles, are deduced from doubtful physical hypotheses.

Reichard [1-26] was the first who noticed this fact. He criticized the deductive transfer theories and suggested an inductive theory.

**Reichard's theory** was developed for free turbulent flows. The main points of this theory are as follows. Having noticed that in the cross sections of the flow mixing zone the distribution of total longitudinal velocity follows the Gaussian curve, Reichard proposed that the turbulent transfer process is a statistical process and is exactly similar to the molecular transfer process. Hence, the differential equation governing the behaviour of  $\bar{\omega}_1'^2$  should be identical to the molecular diffusion equation. For this the motion equation should be so transformed that a diffusion equation is obtained. Thus, by neglecting the terms containing pressure and viscosity, the projection of the motion equation on the direction of flow may be written as

$$\frac{\partial \omega_1'}{\partial \tau} + \omega_1' \frac{\partial \omega_1'}{\partial x} + \omega_2' \frac{\partial \omega_1'}{\partial y} = 0 \quad (1-8-46)$$

Using the incompressibility equation, the above equation may be reduced to

$$\frac{\partial \omega_1'}{\partial \tau} + \frac{\partial \omega_1'^2}{\partial x} + \frac{\partial \omega_1' \omega_2'}{\partial y} = 0 \quad (1-8-47)$$

With the assumption of steady turbulent motion, averaging of the above equation gives

$$\frac{\partial \overline{\omega_1'^2}}{\partial x} + \frac{\partial \overline{\omega_1' \omega_2'}}{\partial y} = 0 \quad (1-8-48)$$

On the other hand if  $\overline{\omega_1'^2}$  follows the Gaussian law, then it should satisfy the equation

$$\frac{\partial \overline{\omega_1'^2}}{\partial x} = L \frac{\partial^2 \overline{\omega_1'^2}}{\partial y^2} \quad (1-8-49)$$

where  $L$  is a mathematical expression of the diffusion coefficient and is measured in units of length. It is deprived of a distinct physical meaning. Thus, for the transformation of (1-8-47) to (1-8-48) it should be assumed that

$$\overline{\omega_1' \omega_2'} = -L \frac{\partial \overline{\omega_1'^2}}{\partial y^2} \quad (1-8-50)$$

This relation was called by Reichard the momentum transfer law which reads as follows: "The intensity of momentum transfer, corresponding to component  $\omega_x'$ , in the transverse direction with velocity  $\omega_y'$  is proportional to the momentum flow variation  $\overline{\omega_x'^2}$  in the same direction" (a similar "law" may also be obtained for the transfer of a scalar quantity). This law is unsatisfactory as equation (1-8-49) "gives preference" to the  $x$ -axis compared to the  $y$ -axis, which is unrealistic. The Reichard theory, like previous ones, fails to describe the actual mechanism of turbulent transfer and does not even make any attempt to provide better understanding of this mechanism.

In conclusion it should be mentioned that all the described semi-empirical turbulent theories provide no idea of the interrelation between the averaged and fluctuating transfer characteristics. This problem is however of fundamental significance because of the necessity of better understanding of the turbulent transfer mechanism. In fact we often deal with such problems of turbulent transfer, in which the determination of the components of the Reynolds stress tensor and fluctuation flows of a scalar quantity is necessary not only for obtaining averaged transfer equations in a closed form, but often is the main aim of the research.

To these problems may be added, in particular, the problems concerning heat and mass transfer within the boundary layer from the external turbulent flow, electromagnetic wave propagation in media having systematic and random non-uniformities of permittivity, etc. In so doing, the semi-empirical relationships (1-8-33) for shear turbulent stresses and turbulent cross-flows of a scalar quantity (1-8-34), obtained on the basis of phenomenological theory of "mixing length", prove to be inadequate.

**The author's theory.** We shall primarily dwell on turbulent heat transfer, as it is the most simple case of turbulent transfer. For simplicity, let us assume that the fluid is incompressible ( $\rho = \text{const}$ ). The averaged  $c_p \rho \overline{vT}$  is usually substituted by

$$c_p \rho \overline{vT} = c_p \overline{\rho v T} - c_p \rho (\overline{vT} - \overline{v} \overline{T}) = c_p \overline{\rho v T} - c_p \rho \overline{v' T'} \quad (1-8-51)$$

An alternative procedure is also possible. The quantity  $\overline{\vec{v}T}$  is a vector; this may be substituted by  $\overline{\vec{v}T}$  through the use of the second-order tensor  $A$

$$\overline{\vec{v}T} = A \cdot \overline{\vec{v}T} \quad (1-8-52)$$

Then the Fourier-Kirchhoff equation for turbulent heat transfer becomes

$$\begin{aligned} c_p \rho \frac{\partial \overline{T}}{\partial \tau} + (c_p \rho \overline{\nabla T}) \cdot (\overline{\vec{v}} \cdot A) + c_p \rho \overline{T} \text{tr} (\overline{\nabla \vec{v}} A) \\ + c_p \rho \overline{T} \overline{\vec{v}} \text{div} A = \text{div} (\lambda \overline{\nabla T}) \end{aligned} \quad (1-8-53)$$

For an incompressible fluid,  $\text{div} \overline{\vec{v}} = 0$ . Let us assume that

$$A = \alpha I \quad (1-8-54)$$

where  $\alpha$  is a scalar quantity.

Then

$$\text{div} (A \cdot \overline{\vec{v}T}) = \overline{\vec{v}} \text{grad} (\alpha T) + \alpha \overline{T} \text{div} \overline{\vec{v}} \quad (1-8-55)$$

and equation (1-8-53) takes the form

$$c_p \rho \left[ \frac{\partial T}{\partial \tau} + \overline{\vec{v}} \cdot \overline{\nabla} (\alpha T) \right] = \text{div} (\lambda \overline{\nabla T}) \quad (1-8-56)$$

Similarly, dyad  $\overline{\vec{v}\vec{v}}$  is to be substituted by dyad  $\overline{\vec{v}\vec{v}}$  as per the relation

$$\rho \overline{\vec{v}\vec{v}} = B \cdot \rho \overline{\vec{v}\vec{v}} \quad (1-8-57)$$

Then we get

$$\text{div} \rho \overline{\vec{v}\vec{v}} = B \cdot \text{div} (\rho \overline{\vec{v}\vec{v}}) + \rho \overline{\vec{v}\vec{v}} \cdot \text{div} B \quad (1-8-58)$$

The turbulent momentum transfer equation will then take the form

$$\frac{\partial \rho \overline{\vec{v}}}{\partial \tau} + B \cdot \text{div} (\rho \overline{\vec{v}\vec{v}}) = - \overline{\nabla p} + \text{div} \overline{\sigma} - \rho \overline{\vec{v}\vec{v}} \cdot \text{div} B + \rho \overline{F} \quad (1-8-59)$$

Equation (1-8-59) differs from the Navier-Stokes equation. It may be written in a different form if the following relation is used

$$\begin{aligned} \text{div} (\rho \overline{\vec{v}\vec{v}} \cdot B) &= \overline{\nabla} \rho (\overline{\vec{v}\vec{v}} : B) + \rho [(\overline{\nabla \vec{v}} / \vec{v}) : B] \\ &+ \rho \overline{\vec{v}} (\overline{\nabla \vec{v}}) : B + \alpha \overline{\vec{v}\vec{v}} \cdot \text{div} B \end{aligned} \quad (1-8-60)$$



(e) *Phenomenological Statistical Transfer Theory  
with Inhomogeneous Turbulence*

The distinguishing feature of the statistical phenomenological turbulent transfer theory is that the turbulent fields of considered quantities are treated as random functions of space and time. Here, the statistical characteristics of fields, i.e. the distribution of probabilities for the values of the fields, are used to describe turbulent transfer processes. Phenomenological hypotheses are employed to obtain equations in the closed form that describe distribution of these statistical characteristics in space and time.

This approach to the turbulence transfer problem is formulated in the following statement: the transfer characteristics of a quantity are expressed in terms of statistical functions of the velocity field; the initial and boundary conditions are expressed using phenomenological hypotheses for some properties of a fine structure turbulence. This approach to the inhomogeneous turbulence transfer problem is rather novel and until recently it has been used only to discuss momentum transfer. Kolmogorov [1-27] was the first to create fundamentals of the statistical and phenomenological approach to the inhomogeneous turbulence problem. In his work turbulence is characterized by its level and scale. (An almost similar idea was later suggested by Prandtl [1-28]). The Kolmogorov-Prandtl ideas are most extensively used in Rotta's theory [1-29].

**Rotta's Theory of Turbulent Momentum Transfer.** The Reynolds equations, containing components of the turbulent stress tensor  $\sigma_{ij} = \rho \overline{v'_i v'_j}$ , are supplemented by a set of equations describing changes in these stresses. For the derivation of motion equations the general method proposed by Keller and Friedman [1-30] for writing equations of moments may be used

The equations read

$$\begin{aligned} \frac{\partial \overline{v'_i v'_j}}{\partial \tau} + \sum_k \bar{v}_k \frac{\partial}{\partial \xi_k} \overline{v'_i v'_j} + \sum_k \left( \overline{v'_i v'_k} \frac{\partial v_j}{\partial \xi_k} + \overline{v'_j v'_k} \frac{\partial \bar{v}_i}{\partial \xi_k} \right) \\ + 2\nu \sum_k \frac{\partial \overline{v'_i}}{\partial \xi_k} \frac{\partial \overline{v'_j}}{\partial \xi_k} - \frac{1}{\rho} p' \left( \frac{\partial \overline{v'_i}}{\partial \xi_j} + \frac{\partial \overline{v'_j}}{\partial \xi_i} \right) - \nu \nabla^2 \overline{v'_i v'_j} \\ + \sum_k \frac{\partial}{\partial \xi_k} \left[ \overline{v'_i v'_j v'_k} + \frac{1}{\rho} (\overline{v'_j p'} \delta_{ik} + \overline{v'_i p'} \delta_{jk}) \right] = 0 \end{aligned} \quad (1-8-61)$$

The terms appearing in equation (1-8-61) characterize respectively the total change of the Reynolds stresses per unit time, generation of the stresses due to the averaged velocity field, viscous dissipation, fluctuation energy exchange in various directions due to pressure fluctuations (total kinetic energy of fluctuations remaining the same), viscous and turbulent diffusion.

Equation (1-8-61) may be supplemented by terms to correlate higher order moments. For specific problems, it is necessary to take only a small

number of momental equations, namely, lower momental equations, and to try to complete the system of the first and higher momental equations by making some phenomenological assumptions that would allow the moments which remain indefinite to be expressed in terms of the main moments, whose number equals the number of considered differential equations for moments.

The averaged velocity and pressure, single-point second moments of velocity fluctuations and the turbulence scale may be chosen as main functions describing a turbulent flow at a point. All the remaining moments, which appear in the equations for the main functions, may be approximately expressed in terms of main moments.

If the set of equations is restricted by second moment equations (1-8-61), then to obtain a complete set of equations consisting of the Reynolds equations and Reynolds stress equations, it is necessary to make phenomenological assumptions for the following statistical characteristics which cannot be found from the given set of equations:

pressure fluctuation correlations with fluctuation velocity gradients

$$p_v = \frac{1}{\rho} p' \left( \frac{\partial v'_i}{\partial \xi_k} + \frac{\partial v'_j}{\partial \xi_k} \right) \quad (1-8-62)$$

with the dissipation function

$$\Phi_v = 2\nu \sum_k \frac{\partial v'_i}{\partial \xi_k} \cdot \frac{\partial v'_j}{\partial \xi_k} \quad (1-8-63)$$

and with the turbulent diffusion of the Reynolds stresses

$$D_v = \sum_k \frac{\partial}{\partial \xi_k} \left[ \overline{v'_i v'_j v'_k} + \frac{1}{\rho} p' (\overline{v'_j \delta_{ik}} + \overline{v'_i \delta_{jk}}) \right] \quad (1-8-64)$$

Here, use is made of the following hypotheses:

1. The redistribution process of fluctuations energy in different directions is similar to the effect of elastic collision of molecules in an ideal gas which is accounted for in the Boltzmann kinetic equation. This analogy yields the following approximation of correlation (1-8-62):

$$p_v = -k_v \frac{E^{1/2}}{L_v} \left( \overline{v'_i v'_j} - \frac{2}{3} E \delta_{ij} \right) \quad (1-8-64a)$$

where  $E = \frac{1}{2} \sum_k \overline{v_k'^2}$  is the total kinetic energy of fluctuations;

$L_v = \frac{3\pi}{8E} \int_0^\infty \frac{F(k, \tau)}{k} dk$  is the integral scale of turbulence;

$k_v$  is the empirical constant;  
 $F(k, \tau)$  is the energy spectrum of turbulence;  
 $k$  is the wave number.

2. The dissipation of kinetic energy of fluctuations into heat is the result of superposition of dissipation at large and small Reynolds numbers

$$Re_E = \frac{E^{1/2} L_v}{\nu}$$

$$\Phi_v = \frac{2}{3} A_v \frac{E^{3/2}}{L_v} \delta_{ij} + A_{1v} \nu \frac{\overline{v'_i v'_j}}{L_v^2} \quad (1-8-65)$$

where  $A_v, A_{1v}$  are empirical constants.

3. The diffusion of fluctuations energy (kinetic energy plus pressure energy) is assumed to be a transferrable quantity for which the Prandtl gradient expression is introduced

$$D_v = \sum_k \frac{\partial}{\partial \xi_k} \left( D_\varepsilon \frac{\partial \overline{v'_i v'_j}}{\partial \xi_k} \right) \quad (1-8-66)$$

where  $D_\varepsilon$  is the turbulent diffusivity.

With consideration for approximations (1-8-64a)-(1-8-66) the system of motion equations becomes complete with respect to the main turbulence properties such as averaged velocity and Reynolds stresses (the turbulence scale appears as a parameter). To get the turbulence scale, either (as Rotta has done) an appropriate differential equation similar in structure to (1-8-61) is written, or a scale-coordinate function is given empirically.

The turbulence theory under consideration involves a number of empirical constants. Some of them are determined from the available experimental data on fine turbulent structure, and the remaining may be estimated.

Thus, the considered turbulence theory, though it makes use of statistical quantities, is in fact semi-empirical. Besides it involves a larger number of empirical constants compared to the Prandtl-Boussinesque theory. But, being rather a complicated one and requiring vast experimental material on statistical quantities, it lacks the earlier mentioned serious disadvantages of the mixing length theory. As regards empirical coefficients, they can be readily determined due to the highly developed state of aerodynamic research. Their great advantage is that they are universal for different wall flows. Finally, it should be pointed out that the considered theory does not contradict Prandtl's phenomenological theory. In particular, it may be easily demonstrated that from the second moment equations an expression similar to the Prandtl relation (1-8-41) is obtained for the Reynolds shear stresses accurate within a constant. To this end it is sufficient to eliminate the diffusion terms in equations (1-8-61) for a fully developed steady-state boundary layer flow and to assume  $\frac{E^{1/2} L_v}{\nu} \gg 1$ .

The phenomenological mixing length theory may thus be considered a particular case of a more general theory using velocity fluctuation moment equations. This particular case holds only for the flow in the turbulent core. Therefore for less accurate engineering calculations where it is necessary to know only the profile of averaged velocity in the inner part of a wall flow, preference should be given to the Prandtl theory. If higher accuracy of turbulence is required, especially when it is necessary to have a more or less detailed discussion of various factors controlling the turbulent transfer picture over the entire region of a turbulent boundary layer, the use of the considered theory is undoubtedly justified [1-51].

**Theory of Turbulent Transfer of Scalar Substance.** The exact knowledge of the turbulent momentum transfer is necessary in order to study the problems of heat and mass transfer in turbulent wall flows. For this aim it is necessary to use the advantages of the dynamic theory, which employs single-point moment equations of velocity fluctuations to improve the semi-empirical theory of transfer of scalar substance (heat and mass) in turbulent shear flows. This theory is based only on the assumption of an analogy between the scalar substance transfer and the momentum transfer. The averaged transfer equation of a scalar substance, which contains components  $\overline{v'_i T'}$  of fluctuating heat fluxes, is supplemented by a system of equations describing the behaviour of these fluxes in space. These equations are derived from the transfer equation (1-8-6) and the averaged transfer equations, and are of the form

$$\begin{aligned} \frac{\partial}{\partial \tau} \overline{v'_i T'} + \sum_k \overline{v_k} \frac{\partial}{\partial \xi_k} \overline{v'_i T'} + \sum_k \overline{v'_i v'_k} \frac{\partial \overline{T}}{\partial \xi_k} + \sum_k \overline{v'_k T'} \frac{\partial \overline{v_i}}{\partial \xi_k} - a \nabla^2 \overline{v'_i T'} \\ - \frac{1}{\rho} \overline{p'} \frac{\partial \overline{T}}{\partial \xi_i} + 2a \sum_k \frac{\partial \overline{v'_i}}{\partial \xi_k} \cdot \frac{\partial \overline{T'}}{\partial \xi_k} + \sum_k \frac{\partial}{\partial \xi_k} \left( \overline{v'_i v'_k T'} + \frac{1}{\rho} \overline{p' T'} \right) = 0 \end{aligned} \quad (1-8-67)$$

(for simplicity, here the Prandtl molecular number is assumed equal to unity) [1-24].

In this equation the first two terms show total change per unit time in the fluctuating flow of a scalar quantity (accurate within a constant), the third term is the direct generation  $\overline{v'_i T'}$  from the averaged field  $\overline{T}$ , the fourth one is the production of fluctuating flows of a scalar quantity due to interaction of the fluctuating motion and bulk flow; the subsequent terms define molecular diffusion, change of  $\overline{v'_i T'}$  due to the dependence of pressure fluctuations upon the fluctuation gradient  $T'$ , viscous "dissipation" and diffusion due to turbulent energy transfer of the fluctuating motion.

To obtain equations (1-8-67) in a closed form, phenomenological hypotheses for the following statistical properties are necessary:

1. Correlations of pressure fluctuations with fluctuation gradients of the scalar quantity

$$\bar{p}_t = \frac{1}{\rho} \cdot \frac{\partial \overline{T}}{\partial \xi_i} p' \quad (1-8-68)$$

2. Correlations characterizing "smearing" of fluctuating heat fluxes due to the thermal resistance of fluid

$$\Phi_{vt} = 2a \sum_k \frac{\partial \overline{v'_i}}{\partial \xi_k} \cdot \frac{\partial T'}{\partial \xi_k} \quad (1-8-69)$$

3. Turbulent diffusion of fluctuating heat fluxes

$$D_{vt} = \sum_k \frac{\partial}{\partial \xi_k} \left[ \overline{v'_k(v'_i T')} + \frac{1}{\rho} \overline{p' T'} \right] \quad (1-8-70)$$

Here the following hypotheses are used:

1. The process of levelling fluctuating heat fluxes in various directions is similar to the process of levelling kinetic energy in the same directions. Taking into account relation (1-8-64a) the following approximation of correlation (1-8-68) should be used:

$$p_{vt} = -k_\gamma \frac{E^{1/2}}{L_{vt}} \overline{v'_i T'} \quad (1-8-71)$$

where  $L_{vt}$  is the integral scale defined by special correlation, velocity is a scalar quantity.

2. The "smearing" effect of fluctuating heat fluxes is the result of superposition of the above effects when  $Re_E \rightarrow 0$  and  $Re_E \rightarrow \infty$ , i.e.

$$\Phi_{vt} = A_{1vt} a \frac{\overline{v'_i T'}}{L_{vt}^2} \quad (1-8-72)$$

(here  $\lim_{Re_E \rightarrow \infty} \Phi_{vt} = 0$  is taken into account).

3. The turbulent transfer of fluctuating scalar quantity flows occurs due to eddy diffusion, i.e. for (1-8-70) a gradient expression of the form

$$D_{vt} = \sum_k \frac{\partial}{\partial \xi_k} \left( D_{\epsilon t} \frac{\partial}{\partial \xi_k} \overline{v'_i T'} \right) \quad (1-8-73)$$

may be introduced.

With allowance made for approximations (1-8-71)-(1-8-73), the Fourier-Kirchhoff equations are completed with respect to the main characteristics of turbulent transfer, namely, the averaged value of scalar quantity  $\overline{T}$  and fluctuating fluxes  $\overline{v'_i T'}$ .

The theory of scalar quantity transfer under consideration as well as the momentum transfer theory contains a number of empirical constants. It may be demonstrated [1-31] that for  $k_\tau$  and  $A_{1\tau}$ , certain values may be assigned, which can then be verified by experimental data related to statistical characteristics of turbulent heat transfer.

The proposed transfer theory of a scalar quantity in turbulent inhomogeneous flows involves the use of equations for statistical moments of fluctuating quantities. It should be pointed out that the greater the number of involved equations (for higher order moments) the more complete will be the description of transfer processes. Equations governing turbulent transfer of a scalar quantity may be obtained in a closed form by introducing certain phenomenological approximations which enable us to get rid of "new" moments, i.e. those moments which are inconsistent with the chosen set of equations. A final measure of the validity of approximations to be introduced is the experiment. Therefore, the proposed theory is essentially semi-empirical.

However, the hypothetic relations employed in the theory, between the "known" and "unknown" quantities, refer to fluctuating characteristics in contrast to purely heuristic relations between averaged and fluctuating quantities used in Prandtl-Bussinesque's theory; incidentally, the latter relationships are based on the assumption of similar modes of turbulent momentum and scalar substance transfer. But an analogy between momentum-transfer and heat-transfer processes exists only when  $\bar{v}_i = \alpha T$ , where  $\alpha$  is a proportionality factor; then the averaged equations of momentum and scalar substance transfer, which in the general case involve "motive force"  $F_i$ , become identical. This is possible if the following conditions

$$-\frac{1}{\rho} \cdot \frac{\partial p}{\partial \xi_i} + F_i = \alpha F_i, \quad \text{Pr} = 1, \quad \text{Pr}_\tau = 1$$

are fulfilled, where  $F_i$  is the density of external forces.

In a general case the above conditions cannot be satisfied. The advantage of the transfer theory involving equations for statistical moments of fluctuations is that the theory is free of the above restrictions. It is also of great importance that the theory accounts for the effect of external turbulent flows on transfer processes within the boundary layer. Since the equations for the second moments contain turbulent diffusion terms, it is possible to calculate transfer characteristics right up to the external boundary layer, and thus to account for (through boundary conditions) external flow turbulence. Another advantage of the theory is the possibility to take into account the effect of pressure fluctuations on the behaviour of scalar quantity fluctuating flows, which is impossible in the phenomenological theory based on the "mixing length" concept.

The theory under consideration, when applied to particular problems of wall flows, does not envisage the use of multilayer flow models neces-

sary for phenomenological transfer theories. In fact by the use of equations for second moments the characteristics of averaged and fluctuation fields over the whole region of flow can be calculated. In so doing, the possibility of calculating fluctuating fluid characteristics is, undoubtedly, an advantage of the method in those cases, when the task resides in the determination of these characteristics.

It should be pointed out that the theory of scalar substance transfer should not be considered a contradiction of the Prandtl-Boussinesque theory. Similar to the earlier considered theory of momentum transfer, it may be demonstrated that the relationship used in the phenomenological transfer theory for turbulent scalar substance flows may be obtained from the equations for corresponding second moments when  $Re_E \gg 1$  and turbulent diffusion is neglected [1-31].

Thus, the phenomenological transfer theory of Prandtl-Boussinesque may in this sense be considered as a particular case of a more general theory, employing equations for fluctuating scalar substance flows, which holds only in the turbulent core range. Therefore, for engineering calculations not intended to provide a detailed picture of scalar substance turbulent transfer processes, but meant to deal only with the averaged scalar substance field, at least in the central part of the wall flow (the profile in the immediate vicinity of the wall may be found by using a two-layer model), it is apparently expedient to make use of Prandtl-Boussinesque's theory. But, in cases, when a more detailed consideration of different factors determining the scalar substance turbulent transfer picture in the range of near-wall turbulent flows (including those cases when the aim of the problem is to determine the scalar substance fluctuating field characteristics) is necessary, the use of the considered theory of transfer is justified.

The considered theory was used for numerical solution of a number of turbulent transfer problems, such as turbulent heat transfer in a plane duct at a constant wall heat flux [1-31], heat transfer in a circular rotating tube [1-32], turbulent heat transfer during natural convection in a narrow vertical cell. In these problems the fluctuating heat flux distributions were originally calculated for the whole space of the near-wall turbulent region.

In spite of the evident advantages of the statistical phenomenological theory compared to the Prandtl-Boussinesque pure phenomenological theory, it may be easily seen that the suggested theory is not free of empirical relationships appearing due to the introduction of phenomenological approximations of certain statistical characteristics.

A question arises: is it not possible to leave out the earlier mentioned phenomenological approximations based on the statistical description of turbulence, and to obtain a mathematical description of "unknown" statistical characteristics that lead to a closed system of equations for superior moments, i.e. is it not possible to get rid of a very large number of empirical constants? A natural way to achieve this is, apparently, an attempt to introduce differential equations for these "extra" statistical characteris-

tics, i.e. to impart the transfer theory of inhomogeneous turbulence a purely statistical meaning\*. The main points of this theory are briefly discussed below.

### (f) *Correlation Model of Inhomogeneous Turbulence*

Let us formulate correlation models of incomplete statistical description of momentum and scalar substance transfer processes in inhomogeneous turbulence without the use of semi-empirical closing relations (which in view of a great number of equations involved would have contained a tremendous number of empirical constants). Unlike the majority of semi-empirical models, the proposed models fulfill the necessary conditions of Galilean and tensor invariance of equations. They are universal as regards their use for any geometric configurations in a general case of unsteady turbulent flows for all Prandtl numbers (within the framework of incompressibility concept).

The theory is based on a statistical approach to turbulent transfer of momentum and scalar quantities in terms of one-point and two-point moments of hydrodynamic fields (velocity, pressure and temperature, impurity concentration) considered as random functions of space and time. The Navier-Stokes equations of motion for incompressible viscous fluid and the scalar substance transfer equation are constitutive equations

$$\frac{\partial u_i}{\partial \tau} + u_k \frac{\partial u_i}{\partial x_k} = - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \gamma \frac{\partial^2 u_i}{\partial x_k^2}$$

$$\frac{\partial \Gamma}{\partial \tau} + u_k \frac{\partial \Gamma}{\partial x_k} = \chi \frac{\partial^2 \Gamma}{\partial x_k^2}$$

The correlation model for incomplete statistical description of inhomogeneous turbulence is formulated in [1-33] as a system of a finite number of "coupling equations" for the moments

$$\frac{\partial \bar{v}_i}{\partial \tau} + v_k \frac{\partial \bar{v}_i}{\partial x_k} + \frac{\partial}{\partial x_k} \overline{u_i u_k} + \frac{\partial \bar{p}}{\partial x_i} - \nu \frac{\partial^2 \bar{v}_i}{\partial x_k^2} = 0 \quad (1-8-74)$$

$$\frac{\partial}{\partial \tau} \overline{u_i u_k} + v_l \frac{\partial}{\partial x_l} \overline{u_i u_k} + \frac{\partial}{\partial x_l} \overline{u_i u_k u_l} + \dots = 0 \quad (1-8-75)$$

$$\frac{\partial}{\partial \tau} \overline{u_i u_k u_l} + v_m \frac{\partial}{\partial x_m} \overline{u_i u_k u_l} + \frac{\partial}{\partial x_m} \overline{u_i u_k u_l u_m} + \dots = 0 \quad (1-8-76)$$

where bars over quantities show probabilistic averaging (mathematical expectation). The chains of equations are broken using Millionshchikov's quasi-

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\* Such an attempt was made in [1-33].



normal hypothesis for the fourth-order moments. The idea of completing these equations relative to unknown moments amounts to their kinematic transformation by the use of the Burgers variables with subsequent application of the proposed quasi-homogeneity hypothesis [1-33].

The final set of inhomogeneous turbulence equations contains differential equations for the following characteristics: the first, second and third central moments of the velocity field ( $\bar{v}_i$ ,  $\overline{u_i u_j}$ ,  $\overline{u_i u_j u_k}$ ), the second and third mixed moments of velocity and pressure ( $\overline{u_i p}$ ,  $\overline{u_i u_j p}$ ), the second order tensor of turbulence microscales  $l_{ij}$ . This set of equations is complete with an accuracy of two "homogeneous" statistical coefficients which, in case of isotropy, change into known statistical coefficients.

The correlation model for incomplete statistical description of scalar quantity transfer in case of inhomogeneous turbulence has been formulated in [1-34] as a system of a finite number of meshing equations for the first moment of a scalar quantity field and mixed moments of a higher order

$$\frac{\partial \bar{\Gamma}}{\partial \tau} + \bar{v}_i \frac{\partial \bar{\Gamma}}{\partial x_i} + \frac{\partial}{\partial x_i} \overline{u_i \Gamma} - x \frac{\partial^2 \bar{\Gamma}}{\partial x_i^2} = 0 \quad (1-8-77)$$

$$\frac{\partial}{\partial \tau} \overline{u_i \Gamma} + v_k \frac{\partial \overline{u_i \Gamma}}{\partial x_k} + \frac{\partial}{\partial x_k} \overline{u_i u_j \Gamma} + \dots = 0 \quad (1-8-78)$$

$$\frac{\partial}{\partial \tau} \overline{u_i u_j \Gamma} + v_k \frac{\partial \overline{u_i u_j \Gamma}}{\partial x_k} + \frac{\partial}{\partial x_k} \overline{u_i u_j u_k \Gamma} + \dots = 0 \quad (1-8-79)$$

As in the previous case the chains may be broken by using Millionshchikov's quasi-normal hypothesis [1-34]

$$\overline{u_i u_j u_k \Gamma} = \overline{u_i u_j} \cdot \overline{u_k \Gamma} + \overline{u_i u_k} \cdot \overline{u_j \Gamma} + \overline{u_i \Gamma} \cdot \overline{u_j u_k}$$

The equations are completed with respect to unknown moments by using the quasi-homogeneity hypothesis. The finite set of equations defining the following characteristics of scalar substance transfer

$$\bar{\Gamma}, \overline{u_i \Gamma}, \overline{u_i u_j \Gamma}, \overline{\gamma p}, \overline{u_i \gamma p}$$

is complete with an accuracy of two statistical coefficients which did not appear before.

Similar to the previous models, a correlation model for scalar substance fluctuations field with inhomogeneous turbulence is formed

$$\frac{\partial \bar{\gamma}^2}{\partial \tau} + v_i \frac{\partial \bar{\gamma}^2}{\partial x_i} + \frac{\partial}{\partial x_i} \overline{u_i \gamma^2} + \dots = 0 \quad (1-8-80)$$

$$\frac{\partial}{\partial \tau} \overline{u_i \gamma^2} + v_k \frac{\partial \overline{u_i \gamma^2}}{\partial x_k} + \frac{\partial}{\partial x_k} \overline{u_i u_k \gamma^2} + \dots = 0 \quad (1-8-81)$$

where Millionshchikov's hypothesis [1-35]

$$\overline{u u_k \gamma^2} = \overline{u_i u_k} \gamma^2 + 2 \overline{u_i \gamma} \cdot \overline{u_k \gamma}$$

and the quasi-homogeneity hypothesis are also used. The final set of equations describe the following characteristics of the scalar quantity field, namely, the second and third moment  $\overline{\gamma^2}$ ,  $\overline{u_i \gamma^2}$ ,  $\overline{p \gamma^2}$ , and also a microscale of scalar quantity  $\lambda$  fluctuations. The set of equations contains two "homogeneous" statistical coefficients which in case of isotropy are statistical coefficients of the isotropic scalar field.

The proposed models for incomplete statistical description of transfer processes in case of inhomogeneous turbulence may be used for numerical solution of "model" boundary-value problems of wall and main-flow turbulence.

### 1-9 ELEMENTS OF NONLINEAR THERMOMECHANICS IN CONTINUA

At present three phenomenological theories of transfer processes in continua are available, namely, (1) thermodynamics of irreversible processes (TIP), (2) entropy-free thermodynamics of transfer (ETIT); (3) nonlinear field theory of mechanics (NFT).

The first theory was developed by Onsager and de Groot, and it is the generalization of classical thermodynamics. ETIT is based on Meixner's works [1-35] who uses Klausius' classical definition of entropy. NFT was developed by Truesdell and his school of thought. The theory describes nonlinear laws of transfer processes in a most general form for different material structures. The fundamentals of this theory will be presented in this section.

All three theories are based on the laws of conservation of mass, momentum, moment of momentum and energy. It is assumed that three types of mechanical interaction take place, namely, (1) mechanical interaction of contact forces between the elements of a body; (2) mechanical interaction of contact forces appearing on the body surface; (3) body forces acting at a distance from the surrounding medium. For the description of thermal effects the concepts of temperature  $T(z, \tau)$ , which is positive at each point  $z$  of space and at any time  $\tau$ , and of specific entropy  $s(z, \tau)$  are used. Here it is useful to consider the concepts of a body and its motion. A body is defined as a certain reference configuration containing particles  $z$ . The motion of a body is known if the position  $f(Z, \tau)$  occupied by particle  $Z$  at any time  $\tau$  is known. It is assumed that function  $f$  is differentiable as many times as necessary. It should be mentioned that two different particles  $Z$  and  $Y$  cannot occupy the same position  $f(Z, \tau) \neq f(Y, \tau)$  if  $Z \neq Y$ . Cartesian coordinates may be taken instead of material coordinates  $(Z, \tau)$  as independent variables. Then the equation  $z = f(Z, \tau)$  will be reversed to express  $Z$  in terms of  $z$  and  $\tau$ , and used to describe scalar, vector and tensor fields as a function of the spatial coordinates  $(z, \tau)$ .

To distinguish the gradients taken with respect to the variables  $z$  and  $Z$ , we shall introduce the following designations

$$\text{grad } \varphi(\vec{z}, \tau) = \frac{\partial \varphi(\vec{z}, \tau)}{\partial z_i}; \quad (1-9-1)$$

$$\text{grad } \psi(\vec{Z}, \tau) = \frac{\partial \psi(\vec{Z}, \tau)}{\partial Z_i}; \quad i = 1, 2, 3$$

the time derivatives will then be

$$\dot{\varphi}(\vec{z}, \tau) = \frac{\partial \varphi(\vec{z}, \tau)}{\partial \tau} + \vec{v}(\vec{z}, \tau) \text{grad } \varphi(\vec{z}, \tau) \quad (1-9-2)$$

$$\dot{\psi}(\vec{Z}, \tau) = \frac{\partial \psi(\vec{Z}, \tau)}{\partial \tau} \quad (1-9-3)$$

With the above assumptions about the function  $f(Z, \tau)$ , the determinant of shear tensor  $G(\cdot)$  is always greater than zero

$$G = \text{grad } f(Z, \tau) \quad (1-9-4)$$

If at any moment the body configuration is a reference configuration, then the determinant of tensor  $G$  should be positive.

The motion of body  $\mathcal{B}$  is a uniparametric succession of configurations with positive time

$$\vec{z} = f(\vec{Z}, \tau); \quad \vec{Z} = f^{-1}(\vec{z}, \tau) \quad (1-9-5)$$

Localization is an important concept. The curves  $\mathcal{C}$  and  $\mathcal{D}$  are transformed into  $\mathcal{C}'$  and  $\mathcal{D}'$  curves as per (1-9-5). If  $dZ$  is an element of curve  $\mathcal{C}$ , then the substitution of variables results in

$$\int_{\mathcal{C}} [\dots] dZ^K = \int_{\mathcal{C}'} [\dots] Z_{,k}^K dz^k \quad (1-9-6)$$

where

$$Z_{,k}^K \equiv \partial Z^K / \partial z_k$$

The spatial field  $dz$  is defined in terms of the material field  $dZ$  by

$$dZ^K = Z_{,k}^K dz^k \quad (1-9-7)$$

whose solution for  $dz$  is

$$dz = z_{,K}^k dZ^K \quad (1-9-8)$$

In equation (1-9-8)  $dz$  is a field that describes a deformed material whose determination requires knowledge of the deformation in the neighbour-

hood of  $z$ . The quantity  $z_{,K}^k$  is a component of the deformation gradient  $G$ . The tensor  $G$  may be divided into the rotation tensor  $R$  and pure deformation in the form of symmetric stretch tensor  $U$  (right positive definite stretch tensor)

$$G = RU = VR \quad (1-9-9)$$

where  $V$  is the left positive definite stretch tensor.

It is often more convenient to use the right  $C$  and left  $B$  Cauchy-Green stretch tensors

$$C = G^T G = U^2; \quad B = GG^T = V^2 \quad (1-9-10)$$

The components of the Cauchy-Green tensor are rational functions of  $G$  components, whereas the components  $U$  and  $V$  are not rational functions.

If  $b$  is a reference configuration at time  $\tau_0$  and  $\beta(\tau)$  and  $\beta(\theta)$  are the configurations at times  $\tau$  and  $\theta$ , then the relative deformation gradient is determined by the use of usual transformations [1-36]

$$G(\theta) = G_{(\tau)}(\theta)G(\tau) \quad (1-9-11)$$

where

$$G(\theta): b \rightarrow \beta(\theta); \quad G(\tau): b \rightarrow \beta(\tau); \quad G_{(\tau)}(\theta): \beta(\tau) \rightarrow \beta(\theta)$$

Thus the relative gradient of deformation rate at time  $\tau$  can be presented as

$$L(\tau) \equiv \dot{G}_{(\tau)}(\tau) = \left. \frac{\partial}{\partial \theta} G_{(\tau)}(\theta) \right|_{\theta=\tau} = \dot{G}_{(\tau)} G(\tau)^{-1} \quad (1-9-12)$$

which is simply the spatial velocity gradient  $\text{grad } \vec{z}$ .

Similarly, the spatial gradient of the  $n$ -th acceleration

$$L_{(n)}(\tau) = \frac{\partial z^{(n)}}{\partial z} = G_{(\tau)}^{(n)}(\tau) = \text{grad } z^{(n)}; \quad n = 1, 2, \dots \quad (1-9-13)$$

for

$$L_0 = I; \quad L_1 = L \quad (1-9-14)$$

Using polar expansion of the relative deformation gradient and differentiating with respect to time, we find the stretching rate  $D$  and the spin  $W$

$$D = \dot{U}_{(\tau)}(\tau) = \frac{1}{2} (L + L^T) \quad (1-9-15)$$

$$W = \dot{R}_{(\tau)}(\tau) = \frac{1}{2} (L - L^T) \quad (1-9-16)$$

where  $D$  is obviously the symmetric part of  $L$  sometimes called the deformation rate tensor.

A useful relationship is easily derived

$$\text{tr } D = \text{tr } L = \text{div } \vec{z} \quad (1-9-17)$$

If  $\rho$  is the mass density of the body,  $\rho_0$  is the mass density of the body in the reference configuration and  $\mathcal{B}$  is an area occupied by a part of the body, then according to the law of conservation of mass we have

$$\int_{\mathcal{B}_0} \rho_0(Z) dV(Z) = \int_{\mathcal{B}} \rho(z, \tau) dV(z) \quad (1-9-18)$$

where  $\mathcal{B}_0$  is the area occupied by the reference configuration. After replacing the variable  $z = f(Z, \tau)$  in the integrand on the right-hand side, we find that  $\rho_0$  and  $\rho$  are related by the equation

$$\rho_0 = \rho \det G \quad (1-9-19)$$

If we differentiate with respect to time the equality (1-9-19) and use the identity

$$\text{grad } v = \dot{G}G^{-1} \quad (1-9-20)$$

and formula (1-9-2), we obtain the continuity equation

$$\frac{\partial \rho}{\partial \tau} + \text{div}(\rho \vec{v}) = 0 \quad \text{or} \quad \frac{\partial \rho}{\partial \tau} + \text{tr grad } \rho \vec{v} = 0$$

If there is a contact force field  $\vec{\pi}(z, \tau)$ , calculated per unit area, and a body force field  $\vec{F}(z, \tau)$ , then from the momentum balance equation we get

$$\int_{(A)} \Pi \cdot \vec{n} dA + \int_{(V)} \rho \vec{F} dV = \int_{(V)} \rho \ddot{z} dV \quad (1-9-21)$$

where  $\Pi$  is the stress tensor related to the stress vector  $\vec{\pi}$  by the equation

$$\vec{\pi}(z, n, \tau) = \Pi(z, \tau) \cdot \vec{n} \quad (1-9-22)$$

where  $\vec{n}$  is a normal to the surface  $(A)$  confining the volume  $(V)$ . From equation (1-9-21) the Cauchy equation of motion is obtained

$$\rho \vec{v} = \text{div } \Pi + \rho \vec{F} \quad (1-9-23)$$

where  $\rho$  is the density of the body of reference configuration.

If  $\vec{q}$  is the heat flux vector, i.e.

$$q(\vec{z}, \vec{n}, \tau) = -\vec{q}(\vec{z}, \tau) \cdot \vec{n} \quad (1-9-24)$$

and  $u$  is the internal energy,  $Q$  is the heat source, then we may write

$$\frac{d}{d\tau} \int_{(V)} \rho \left( u + \frac{1}{2} \vec{v} \cdot \vec{v} \right) dV = \int_{(A)} (\vec{v} \cdot \Pi \cdot \vec{n} - \vec{q} \cdot \vec{n}) dA \quad (1-9-25)$$

$$+ \int_{(V)} \rho (\vec{v} \cdot \vec{F} + Q) dV$$

Using the Gauss-Ostrogradsky theorem, we get the energy transfer equation in the differential form

$$\rho \dot{u} = -\operatorname{div} \bar{q} + \operatorname{tr} (\Pi^T \operatorname{grad} \bar{v}) + \rho Q \quad (1-9-26)$$

The second aim of NFT is to express a great variety of materials by constitutive equations. The theory of constitutive equations is based on the following principles:

1. determinism (the body stress depends on the history of the body motion);
2. local action (stress in a particle is controlled only by the motion close to this particle);
- (3) independence of material properties of the frame of reference;
- (4) equipresence (the quantity present in one of the constitutive equations as an independent variable is present in all other equations unless it is forbidden by the laws of physics or invariance conditions). Application of these principles to the solution of specific problems of nonlinear thermomechanics of continua is described in the works of C. Truesdell and his school.

We shall consider the equipresence principle in more detail. The Curie principle for isotropic materials which is the main principle of irreversible thermodynamics follows from the equipresence principle as a particular case.

According to this principle the main quantities which characterize a thermodynamic process such as stress tensor  $\Pi$ , heat flux  $\bar{q}$ , internal energy,  $u$ , and entropy  $s$ , must be functions of one and the same set of variables  $(z, \dot{z}, G, \dot{G}, T, \dot{T}, \vec{\nabla} T, \vec{\nabla} \dot{T})$ , i.e.

$$\left. \begin{array}{l} \Pi \\ \bar{q} \\ u \\ s \end{array} \right\} = \Phi_{(\cdot)}(z, \dot{z}, G, \dot{G}, T, \dot{T}, \vec{g}, \vec{\dot{g}}) \quad (1-9-27)$$

where  $\vec{g} = \operatorname{grad} T \equiv \vec{\nabla} T$  is the temperature gradient,  $\Phi_{(\cdot)}$  is the functional of  $\Pi$ ,  $\bar{q}$ ,  $u$ , and  $s$ .

In case of isotropic materials the independent variables  $z$  and  $\dot{z}$  do not appear in the constitutive equations (1-9-27) which should satisfy the basic principles of NFT discussed above. Besides, the constitutive equations for thermomechanical processes should satisfy the main inequality of thermomechanics — the Clausius-Duhem inequality

$$\rho \dot{s} - \left( \frac{\rho Q}{T} - \operatorname{div} \frac{\bar{q}}{T} \right) \geq 0 \quad (1-9-28)$$

The above inequality contains the quantity  $\rho Q$  dependent on the heat source  $Q$ . It may be eliminated using the energy transfer equation. If the

Piola-Kirchhoff nonsymmetric tensor is designated by  $\mathcal{P}$

$$\mathcal{P} = \frac{1}{\rho} \Pi (G^T)^{-1} \quad (1-9-29)$$

then equation (1-9-26) may be written as

$$\rho \dot{u} = -\operatorname{div} \bar{q} + \rho Q + \rho \mathcal{P} : \dot{G} \quad (1-9-30)$$

After eliminating  $\rho Q$  from equation (1-9-30) and carrying out operation  $\operatorname{div} \frac{\bar{q}}{T}$ , we get the Clausius-Duhem inequality in the following form

$$\dot{s} \geq \frac{\dot{u}}{T} - \frac{1}{T} \operatorname{tr} \mathcal{P}^T \dot{G} - \frac{1}{\rho T^2} \bar{q} \cdot \bar{g} \quad (1-9-31)$$

Coleman demonstrated that

$$\left. \begin{aligned} \dot{s} &\geq 0, \quad \text{at} \quad \dot{G} = \dot{u} = \bar{g} = 0 \\ \dot{u} &\leq 0, \quad \text{at} \quad \dot{G} = \dot{s} = \bar{g} = 0 \end{aligned} \right\} \quad (1-9-32)$$

Thus, we see that in a typical case of stress relaxation under constant temperature, the specific internal energy decreases and the entropy increases. The equipresence principle is of great importance as it takes account of the interrelation between transfer processes. This principle does not however contradict the classical description of these processes. For some model materials under small deviations from equilibrium the number of independent variables decreases and in a particular case the momentum, heat and mass transfer is described by separate variables. This we shall illustrate with an example.

Let the constitutive equations for a certain material be of the form

$$\left. \begin{aligned} \Pi \\ \bar{q} \\ u \\ s \end{aligned} \right\} = \Phi_{(\cdot)}(T, \bar{g}, G, \dot{G}) \quad (1-9-33)$$

If the above equations satisfy the Clausius-Duhem inequality, the variables  $\bar{g}$  and  $\dot{G}$  drop out from the equations for  $u$  and  $s$ . But the variables  $\dot{G}$  and  $\bar{g}$  will appear in the constitutive equations for  $\Pi$  and  $\bar{q}$ .

If the constitutive equations for  $\Pi$  and  $\bar{q}$  satisfy the principle of independence of material properties of the frame of reference, the statical part of the stress tensor will then be defined only in terms of thermodynamic variables ( $G, \dot{G}, \bar{g}$  drop out). In reference [1-37] it is shown that for a simple liquid (the isotropy group of all the four constitutive equations is a unimodu-

lar group), the constitutive equations are of the following form

$$\Pi = - \left( \frac{\partial u}{\partial V_0} \right) \mathbf{I} + \Phi_\sigma(s, \bar{\mathbf{g}}, V_0, \mathbf{D}) \quad (1-9-34)$$

$$\bar{\mathbf{q}} = \Phi_q(s, \bar{\mathbf{q}}, V_0, \mathbf{D}) \quad (1-9-35)$$

$$u = \Phi_u(s, V_0); \quad T = \frac{\partial \Phi_u(s, V_0)}{\partial s} \quad (1-9-36)$$

where  $\Phi_\sigma$  and  $\Phi_q$  are isotropic functions,  $V_0$  is the specific volume ( $V_0 = 1/\rho$ ).

Constitutive equations (1-9-34) through (1-9-36) imply that the separation of effects is only partial. If the constitutive functions  $\Phi_\sigma$  and  $\Phi_q$  are expanded in the vicinity of  $\mathbf{D} = 0$  and  $\bar{\mathbf{g}} = 0$ , then the constitutive equations (1-9-34) and (1-9-35) will become

$$\Pi = - \frac{\partial u}{\partial V_0} \mathbf{I} + \left( \eta_V - \frac{2}{3} \eta \right) \text{tr } \mathbf{D} + 2\eta \mathbf{D} \quad (1-9-37)$$

$$\bar{\mathbf{q}} = -\lambda \text{grad } T \quad (1-9-38)$$

with an error of

$$O(\sqrt{\text{tr } \mathbf{D}^2 + \bar{\mathbf{g}}^2})$$

i.e. we obtain fully separated classical equations of momentum and heat transfer for linear-viscosity and linear thermal conductivity liquid.

For oriented fluids and microfluids the Clausius-Duhem inequality should be generalized to the form [1-15]

$$\begin{aligned} \rho \dot{s} = & - \frac{\rho}{T} (\psi + Ts) + \frac{1}{T} \pi_{kl} V_{l,k} + \frac{1}{T} (\pi_{kl} - \pi_{lk}) v_{lk} \\ & + \frac{1}{T} \pi_{klm} v_{lm,k} + \frac{1}{T^2} q_k T_{,k} \geq 0 \end{aligned} \quad (1-9-39)$$

where  $\psi$  is the Gibbs free energy and the other notations are the same as in Sec. 1-6.

Using the inequality (1-9-39) Eringen proved the existence of a thermodynamic stress tensor and the effect of microgyration on the heat conduction [1-15] transfer process. In particular, for thermomicropolar fluids with linear constitutive equations, the heat flux  $\bar{\mathbf{q}}$  will be expressed as

$$\mathbf{q} = -\lambda \mathbf{T}_{,k} + \beta \epsilon_{klm} v_{l,m} \quad (1-9-40)$$

where  $\beta$  is a constant coefficient.

Let us now illustrate the usefulness of the application of the methods of nonlinear thermomechanics to transfer processes. We shall consider non-isothermal diffusion in a gaseous mixture.



The following notations will be used

$$\rho_{k0} = \rho_k / \rho; \quad \sum_k \rho_{k0} = 1; \quad f_1 = \sum_k \rho_{k0} f_k; \quad u_1 = \sum_k \rho_{k0} u_k;$$

$$s = \sum_k \rho_{k0} s_k; \quad \Pi_1 = \sum_k \Pi_k; \quad q_1 = \sum_k q_k$$

The subscript  $k$  indicates that the quantity refers to the  $k$ -th component of the mixture.

The transfer equations for the  $k$ -th component [1-8] may then be written as

$$\partial \rho_k / \partial \tau + \operatorname{div} \rho_k \dot{z}_k = \dot{\rho}_k + \rho_k \operatorname{div} \dot{z}_k - \rho \dot{I}_k = 0 \quad (1-9-41)$$

$$\rho_k \ddot{z}_k - \operatorname{div} \Pi_k - \rho_k \vec{f}_k + \rho \dot{I}_k \ddot{z}_k - \rho \vec{m}_k = 0 \quad (1-9-42)$$

$$\rho \dot{M}_k = \Pi_k - \Pi_k^T \quad (1-9-43)$$

$$\rho \dot{I}_k \left( u_k - \frac{1}{2} \dot{z}_k^2 \right) + \rho_k \dot{u}_k + \rho \vec{m}_k \dot{z}_k - \operatorname{tr} (\Pi^T \operatorname{grad} \dot{z}_k) \quad (1-9-44)$$

$$- \operatorname{div} \vec{q}_k - \rho Q_k - \rho_k \dot{I}_{lk} = 0$$

where  $\dot{I}_k$ ,  $\dot{I}_{lk}$ ,  $\vec{m}_k$ ,  $\dot{M}_k$  are the sources of mass, energy, magnetic moment and torsional moment, respectively.

For a gaseous mixture the sum of all the sources is zero

$$\sum_k \dot{I}_k = 0; \quad \sum_k \dot{I}_{lk} = 0; \quad \sum_k \dot{m}_k = 0; \quad \sum_k \dot{M}_k = 0 \quad (1-9-45)$$

The diffusional velocity is defined by the relation

$$\vec{w}_k = \ddot{z}_k - \ddot{z} \quad (1-9-46)$$

Then using the general concepts for gaseous mixtures, we may write [1-38]

$$\Pi = \Pi_1 - \sum_k \rho_k \vec{w}_k \vec{w}_k \quad (1-9-47)$$

$$\vec{f} = \vec{f}_1; \quad u = u_1 + \frac{1}{2} \sum_k \rho_{k0} w_k^2 \quad (1-9-48)$$

$$\vec{q} = \vec{q}_1 + \sum_k \left[ \Pi_k^T \cdot \vec{w}_k - \rho \left( u_k + \frac{1}{2} w_k^2 \right) \vec{w}_k \right] \quad (1-9-49)$$

$$Q = Q_1 + \sum_k \rho_{k0} \vec{f}_k \cdot \vec{w}_k \quad (1-9-50)$$

Using relations (1-9-47)-(1-9-50), we may deduce from equations (1-9-41)-(1-9-44) the general equations of momentum, heat and mass transfer.

The methods used in [1-38] for the determination of thermodynamic motive forces in a binary gaseous mixture resulted in the following transfer equations

$$\vec{j}_1 = \rho_1 \vec{w}_{12} = -\frac{\rho_1}{\beta_1} T \left( \gamma - \frac{Q^*}{T} \right) \text{grad } \log_e T - \frac{\rho_1^2}{\beta_1} \vec{X}_1 \quad (1-9-51)$$

$$\vec{q} = -\left( \lambda T + \frac{Q^* T}{\beta_1} \right) \left( \frac{Q^*}{T} - \gamma \right) \text{grad } \log_e T + \frac{\rho_1 Q^*}{\beta_1} \vec{X}_1 \quad (1-9-52)$$

where

$$\vec{X}_1 = \text{grad } \mu_1 - \vec{F}_1 + \frac{1}{\rho_1} \text{div } \Pi_1 + \frac{1}{\rho_1} \frac{\partial \rho_1 \vec{v}_1}{\partial \tau} + \frac{1}{\rho_1} \text{div } (\rho_1 \vec{v}_1 \vec{v}_1) \quad (1-9-53)$$

where  $Q^*$  is the isothermal heat of transfer,  $\beta_1$  and  $\gamma$  are constants,  $\mu_1$  is the chemical potential. These equations imply that heat and mass transfer processes depend not only on the thermodynamic forces  $\text{grad } \mu_1$  and  $\text{grad } T$ , but also on the tensor  $\Pi$  and velocity  $\vec{v}_1$ . Assuming  $\gamma = 0$ , the well-known equality of Onsager kinetic coefficients or the reciprocity principle in irreversible thermodynamics is obtained.

We shall present another example taken from [1-39] where a heat convection problem is considered in a fully developed laminar flow in a cylindrical tube with the second kind boundary conditions (the heat flux at the surface is constant). The classical solution of this problem gives a constant value for Nu

$$\text{Nu} = 4.36 = \text{const} \quad (1-9-54)$$

In this case heat flux is found from the formula

$$\vec{q} = -\lambda \text{grad } T + \beta_1 D \text{grad } T \quad (1-9-55)$$

The solution resulted in the formula

$$\text{Nu} = 4.36 - K_D \quad (1-9-56)$$

where  $K_D$  is a dimensionless quantity that takes account of the effect of the hydrodynamic motion on molecular heat transfer (heat conduction)

$$K_D = \frac{\beta_1}{2\rho c_p R^2} \quad (1-9-57)$$

where  $R$  is the tube radius and  $\beta_1$  is a constant.

Formula (1-9-56) is a novelty in thermal engineering. Thus, the study of nonlinear mechanics of continua is not only of general scientific interest but also yields important practical results.

## 1-10. DISTINGUISHING FEATURES OF RHEOLOGICAL HYDRODYNAMICS

The definition of a simple fluid given by nonlinear mechanics of continua reads: "A simple fluid is a material whose mechanical behaviour is determined experimentally by uniform deformation".

Generally it is assumed that fluids obeying the viscous flow law are Newtonian fluids.

According to Stokes' law, which states that viscous stresses at any point of a continua depend only on the relative motion of the fluid close to this point, the viscous stress tensor and the strain rate tensor in a very simple case are related by

$$\sigma_{ij} = p_{ij} - (-p\delta_{ij}) = \eta \dot{D}_{ij} \quad (1-10-1)$$

where  $\eta$  is the shear viscosity coefficient. The bulk viscosity is eliminated according to Stokes' hypothesis.

Equation (1-10-1) is the initial equation for completing the set of the Navier-Stokes hydrodynamic equations.

The viscosity coefficient  $\eta$  may be calculated from different formulas of the kinetic gas theory. The choice depends on the interaction potential of molecular collisions. However the viscosity may be found experimentally. For this we shall rewrite the equation (1-10-1) for a one-dimensional shear flow as

$$\sigma_{12} = -\eta \frac{dv_1}{dx_2} \quad (x_2 \equiv y) \quad (1-10-2)$$

If the process proceeds at a constant temperature, then  $\eta = \text{const.}$  In this case, by plotting the stresses  $\sigma_{12}$  on the abscissa and the strains  $dv_1/dx_2$  on the ordinate, we obtain a straight line whose slope equals the coefficient of viscosity. Hence, if pure shear conditions are simulated in the experiment and shear rates and stresses corresponding to them are measured, then following the above procedure we may determine the coefficient of viscosity. The instruments used for this purpose are known as viscometers, and the equations such as (1-10-2) are called in rheology the flow curves.

If in equation (1-10-2) forces are substituted for stresses, then it will exactly correspond to the Newtonian equation for a material point. In so doing it is necessary that the elementary volume of liquid should be identical with the material point whose mass is accounted for in the coefficient of viscosity. In this sense the fluids described by equations (1-10-1) and (1-10-2) are called Newtonian fluids.

Inversion of equation (1-10-2) to (1-10-1) is possible only by making an additional assumption of isotropism of space. Therefore, equation (1-10-2) should be treated as an empirical formula, whereas the relationship (1-10-1) may be made a law.

But, numerous viscometric measurements have revealed that very few fluids follow rheological formula (1-10-2) and for many liquids the flow curves are nonlinear. In this case the empirical formula (1-10-2) may be corrected so that it would approximate to a certain degree of accuracy nonlinear flow curves. For example, instead of (1-10-2) we may write

$$\sigma_{12} = -\eta \left( \frac{dv_1}{dx_2} \right)^n \quad (1-10-3)$$

Now, with suitable choice of  $\eta$  and  $n$  a nonlinear flow curve may be approximated with sufficient accuracy within a definite range of shear rates. However relationship (1-10-1) cannot be generalized on the basis of the empirical formula (1-10-3). First, for  $n \neq 1$  the coefficient  $\eta$  is not viscosity, if only because its dimension depends on  $n$ . Second, even though the assumption of isotropism of space is made, we cannot substitute tensor  $\dot{D}_{ij}$  for  $dv_1/dx_2$  in formula (1-10-3) as the operation of raising a matrix to a power is not equivalent to raising each element of the matrix to a power. Thus, we see a contradiction to the principle statements of the kinetic theory.

In rheology the discrepancy with tensor analysis rules was eliminated by introducing the effective viscosity concept

$$\eta = \eta_1 |2I_2|^{\frac{n-1}{2}} \quad (1-10-4)$$

where  $I_2$  is the second invariant of the deformation rate tensor.

Now relationship (1-10-1) is substituted by the so-called power law

$$\sigma_{ij} = \eta_1 |2I_2|^{\frac{n-1}{2}} \dot{D}_{ij} = \eta_{eff} \dot{D}_{ij} \quad (1-10-5)$$

Here a question may immediately arise: what is the effective viscosity from the viewpoint of the kinetic theory of gases or liquids? It may be answered that the power law is valid for liquids and the kinetic theory for liquids has not been developed yet. But, it may be mentioned here that the Navier-Stokes equations were used by Navier and Maxwell for gases, but these equations hold for liquids as well, and the difference amounts only to different forms of the interaction potential of colliding atoms and molecules.

Thus, all attempts to generalize relation (1-10-1) on the basis of empirical formulas (1-10-2) and (1-10-3) will inevitably lead to results conflicting with the fundamentals of existing theories.

C. Truesdell was the first to introduce a reasonable generalization of Stokes' hypothesis mentioned above.

Following C. Truesdell, we present a definition of the generalized Stokes fluid [1-40].

The Stokes fluid is a continuous medium, which may be described by material constants  $\eta_0$  and  $T_0$  called natural viscosity and characteristic temperature, whose dimensions are

$$[\eta_0] = M_m L^{-1} \tau^{-1}; \quad [T_0] = \theta \quad (1-10-6)$$

and the viscous stress tensor depends upon the deformation rate tensor  $D_{ij}$  and quantities  $\eta_0, T_0, P_m, p, T$

$$\begin{aligned} \sigma_{ij} &= f(\eta_0, T_0, p_m, P, T, D_{ij}), \\ \sigma_{ij} &= 0, \quad \text{if } D_{ij} = 0 \end{aligned} \quad (1-10-7)$$

For an isotropic medium, using the known Cayley-Hamilton identity, equation (1-10-7) may be written as

$$\sigma_{ij} = E_0 \delta_{ij} + E_1 D_{ij} + E_2 D_{ik} D_{kj} \quad (1-10-8)$$

and

$$E_0, E_1, E_2 = f(\eta_0, T_0, p_m, P, T, I_1, I_2, I_3)$$

where  $I_1, I_2, I_3$  are the invariants of the deformation rate tensor.

With an accuracy of quadratic terms, following C. Truesdell, we may write equation (1-10-8) in the form

$$\begin{aligned} \sigma_{ij} &= \eta_0 [F_{0100} I_1 \delta_{ij} + F_{1000} D_{ij}] + \frac{\eta_0^2}{P_0} [F_{0200} I_1^2 \delta_{ij} \\ &+ F_{1100} I_1 D_{ij} + F_{0010} I_2 \delta_{ij} + F_{2000} D_{ik} D_{kj}] \end{aligned} \quad (1-10-9)$$

Introducing a shear rate tensor according to formulas

$$\begin{aligned} \overset{\circ}{D} &= D_{ij} - \frac{1}{3} I_1 \delta_{ij} \\ \overline{\overset{\circ}{D}_{ik} \overset{\circ}{D}_{kj}} &= D_{1k} D_{kj} - \frac{1}{3} (I_1^2 - 2I_2) \delta_{ij} \end{aligned} \quad (1-10-10)$$

and if

$$\begin{aligned} 3F_{0100} + F_{1000} &= 0 \\ 3F_{0200} + F_{1100} + F_{2000} &= 0 \\ 3F_{0010} - 2F_{2000} &= 0 \end{aligned} \quad (1-10-11)$$

then instead of (1-10-9) we shall arrive at

$$\sigma_{ij} = \eta_0 F_{1000} \overset{\circ}{D}_{ij} + \frac{\eta_0^2}{P_0} [F_{1100} I_1 \overset{\circ}{D}_{ij} + F_{2000} \overset{\circ}{D}_{ik} \overset{\circ}{D}_{kj}] \quad (1-10-12)$$

Formula (1-10-12) is the initial formula for completing the hydrodynamics equations according to C. Truesdell.

The total stress tensor has the form

$$P_{ij} = -p\delta_{ij} + \sigma_{ij} \quad (1-10-13)$$

Here  $P$  is the hydrostatic or equilibrium pressure defined in terms of the thermal motion of atoms or molecules.

The tensor  $\sigma_{ij}$  adds to equilibrium pressure caused by the apparent motion.

The visible and thermal motions co-exist within the validity limits of the Navier-Stokes equations. C. Truesdell assumed implicitly that even within the validity limits of formula (1-10-9) the conditions of co-existence of these motions are not disturbed. Therefore, he determined the hydrostatic pressure in the same manner as in a classical case, that, in its turn, led to conditions (1-10-11).

In one of his recent works, using criterion  $(\eta_0 \dot{D}_{ij}/p)$  C. Truesdell defined the applicability limit of the Navier-Stokes equations.

Indeed, within the limits of the Stokes hypothesis the formula for a total stress tensor will be as follows:

$$P_{ij} = -p\delta_{ij} + \eta \dot{D}_{ij} \quad (1-10-14)$$

Let us isolate an elementary volume in a moving fluid and determine its stressed state by formula (1-10-14). If  $(\eta_0 \dot{D}_{ij}/p) < 1$ , then the character of the stressed state in a moving fluid will be such as in the equilibrium state. If the above complex is larger than unity, then in the volume under consideration the character of the stressed state will vary, i.e. the cubic compression may change into a cubic stress. C. Truesdell named this effect the upper limit of applicability of the Navier-Stokes equations.

We are apt to consider that this criterion primarily indicates the disturbance of the conditions of the co-existence of visible and thermal motions. This results in redefinition of pressure in a flow.

For this we rewrite formula (1-10-9) as

$$\begin{aligned} \sigma_{ij} = & \eta_0 \left[ F_{0100} + \frac{1}{3} F_{1000} \right] I_1 \delta_{ij} + \frac{\eta_0^2}{p_0} \left[ \left( F_{0200} + \frac{1}{3} F_{1100} + \frac{1}{3} F_{2000} \right) \right. \\ & \times I_1^2 \delta_{ij} + \left. \left( F_{0010} - \frac{2}{3} F_{2000} \right) I_2 \delta_{ij} \right] + \eta_0 F_{1000} \dot{D}_{ij} \\ & + \frac{\eta_0^2}{p_0} F_{1100} I_1 \dot{D}_{ij} + F_{2000} \overline{\dot{D}_{ik} \dot{D}_{kj}} \end{aligned} \quad (1-10-15)$$

The pressure in a flow is determined by the formula

$$\bar{p} = -\frac{1}{3}(p_{11} + p_{22} + p_{33}) = p - \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \quad (1-10-16)$$

The additional in term (1-10-16) is found from (1-10-15)

$$\begin{aligned} \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \eta_0 \left( F_{0100} + \frac{1}{3} F_{1000} \right) I_1 + \frac{\eta_0^2}{p_0} \left[ \left( F_{0200} \right. \right. \\ \left. \left. + \frac{1}{3} F_{1100} + \frac{1}{3} F_{2000} \right) I_1^2 + \left( F_{0010} - \frac{2}{3} F_{2000} \right) I_2 \right] \end{aligned} \quad (1-10-17)$$

To use formula (1-10-17) both for compressible and incompressible materials, we assume that

$$\begin{aligned} 3F_{0100} + F_{1000} &= 0 \\ 3F_{0200} + F_{1100} + F_{2000} &= 0 \end{aligned} \quad (1-10-18)$$

Now formula (1-10-16) takes the form

$$\bar{p} = p - \left( F_{0010} - \frac{2}{3} F_{2000} \right) \frac{\eta^2}{p_0} I_2 \quad (1-10-19)$$

In this case the coefficients  $F_{0010}$  and  $F_{2000}$  are the arbitrary temperature functions in accordance with C. Truesdell's concept.

Formula (1-10-19) leads to the following general form of the equation for an equilibrium pressure

$$\bar{p} = \left( 1 - F_{0010} + \frac{2}{3} F_{2000} \right) \frac{\rho RT}{M_0} \quad (1-10-20)$$

Comparing (1-10-20) with (1-5-66), we find that they are similar. This is explained by the fact that a rheological liquid may be considered as a discrete system consisting of a Newtonian fluid with an addition of a small amount of a polymer. The molecules of a polymer liquid serve as structural particles. Of course, this model is a very rough approximation of a real structure, but it is known from experiment that an addition of a small amount of a polymer to Newtonian fluids causes asymmetry of a viscous stress tensor.

Now according to hypothesis (1-10-18) the formula for viscous stresses should be rewritten as:

$$\begin{aligned} \sigma_{ij} = \eta_0 F_{1000} \overset{\circ}{D}_{ij} + \frac{\eta_0^2}{p_0} \left[ \left( F_{0010} - \frac{2}{3} F_{2000} \right) I_2 \delta_{ij} \right. \\ \left. + F_{1100} I_1 \overset{\circ}{D}_{ij} + F_{2000} \overset{\circ}{D}_{ik} \overset{\circ}{D}_{kj} \right] \end{aligned} \quad (1-10-21)$$

Considering the problem on momentum transfer through a liquid layer bounded by two flat solid parallel walls, and using formulas (1-10-20) and (1-10-21), it may be shown that the boundary conditions near a wall take the form

$$v^* - v = - \frac{(1 + \nu)F_{1000} \left( 1 - F_{0010} + \frac{2}{3} F_{2000} \right)}{2(1 - \nu)} \sqrt{\frac{2\pi RT}{M_0}} \frac{\eta_0}{\bar{p}} \frac{\partial v}{\partial y} \quad (1-10-22)$$

A slip factor may be introduced into the above formula, as this was done by Maxwell

$$\delta = \frac{\eta_0}{\bar{p}} \frac{(1 + \nu)F_{1000} \left( 1 - F_{0010} + \frac{2}{3} F_{2000} \right)}{2(1 - \nu)} \sqrt{\frac{2\pi RT}{M_0}} \quad (1-10-23)$$

If  $F_{1000} = 1$ ,  $F_{0010} = F_{2000} = 0$ , then formula (1-10-13) takes the form of Maxwell's formula.

It is convenient to analyse the conditions of slip or adhesion by formula (1-10-22). If  $(\eta_0/\bar{p})(\partial v/\partial y)$  is significantly less than unity, then the velocities of the adjacent liquid layer and the wall always coincide, i.e. adhesion of liquid takes place.

Adhesion may also occur if  $(\eta_0/\bar{p})(\partial v/\partial y) \approx 1$  and

$$1 - F_{0010} + \frac{2}{3} F_{2000} = 0 \quad (1-10-24)$$

The slip of liquid near a solid wall begins if  $(\eta_0/\bar{p})(\partial v/\partial y) \approx 1$  and the right-hand side of formula (1-10-22) is essentially a negative value. The last statement is equivalent to the constraint

$$F_{1000} \left( 1 - F_{0010} + \frac{2}{3} F_{2000} \right) > 0 \quad (1-10-25)$$

We will not dwell upon the case where  $(\eta_0/\bar{p})(\partial v/\partial y) \approx 1$  and

$$F_{1000} \left( 1 - F_{0010} + \frac{2}{3} F_{2000} \right) < 0 \quad (1-10-26)$$

The above considerations lead to a more precise definition of a normal pressure tensor and a viscous stress tensor. Therefore, when writing hydrodynamic equations a problem arises: which of the above tensors should be used to explain more precisely the earlier given equations. The answer should be sought in a comparison of our considerations and experimental results.



The phenomenon of liquid slip along a solid wall was found experimentally as long ago as 1860 by Helmholtz and Piotrovsky. Interest in this may again be generated in connection with the study of rheological properties of the so-called non-Newtonian fluids. Thus, when a discrete-structure fluid flows it is necessary to take into account the phenomenon of slip along a solid wall provided the Tr number is close to unity.

From the analysis of numerous rheological curves  $\sigma = f(\dot{\epsilon})$ , the following empirical formula was obtained [1-41]

$$\left(\frac{1}{\sigma}\right)^{\frac{1}{n}} = (\sigma_0)^{\frac{1}{n}} + (\eta\dot{\epsilon})^{\frac{1}{m}} \quad (1-10-27)$$

where  $n$  and  $m$  are experimental constants,  $\sigma_0$  is the yield point constant, and  $\dot{\epsilon}$  is the deformation rate ( $\dot{\epsilon} = d\epsilon/d\tau$ ). If  $n = m = 1$  and  $\sigma_0 = 0$ , then we get the relation for a Newtonian fluid. Assuming  $n = m = 1$ , we obtain a relation for the Shvedov-Bingham fluid. For  $n = 1$ , formula (1-10-27) reduces to Bukley-Herschel's formula.

For pure viscous fluids ( $\sigma_0 = 0$ ), formula (1-10-27) changes to a power-law rheological equation with the exponent  $n/m$ . Formula (1-10-27) in dimensionless variables may be rewritten as

$$\sigma^* = (1 + A\dot{\epsilon}^{\frac{1}{m}})^n \quad (1-10-28)$$

where

$$\sigma^* = \sigma/\sigma_0; \quad \dot{\epsilon}^* = \frac{\eta}{\sigma_0} \dot{\epsilon}; \quad A = \sigma_0^{\frac{n-m}{nm}} \quad (1-10-29)$$

If  $(d^2\sigma^*/d\dot{\epsilon}^{*2}) < 0$ , formula (1-10-28) describes the behaviour of pseudo-plastic fluids; when  $(d^2\sigma^*/d\dot{\epsilon}^{*2}) > 0$  the fluids are referred to as dilatant.

For incompressible fluids formula (1-10-28) may be generalized with the help of the Prager-Hohenemer method [1-42] for a three-dimensional deformation of a nonlinear viscoplastic fluid and may be written in the form

$$\sigma_{ij} = 2 \left[ \sigma_0^{\frac{1}{n}} - \frac{1}{m} + \eta^{\frac{1}{m}} \right] A^{\frac{n}{m}-1} \gamma_{ij} \quad (1-10-30)$$

where  $\gamma_{ij} = \frac{1}{2} (v_{i,j} + v_{j,i})$  is the deformation-rate tensor of a fluid in motion;  $A = (2\gamma_{ij}\gamma_{ji})^{1/2}$  is the intensity of deformation rates.

As far as the author is aware, a closed system of equations describing motion of nonlinear viscoplastic fluids has not been reported in the literature. Usually the energy- and momentum-transfer equations are solved

on the basis of the boundary-layer equations. For certain purely viscous rheological fluids boundary-layer equations in the simple cases of flow past solids [1-43] were derived and solved.

Oldroyd [1-44] was the first to obtain the equation for the velocity boundary layer of linear viscoplastic fluids. An analysis of equations for a boundary layer of the Shvedov-Bingham viscoplastic fluid flowing past an arbitrary surface is presented in [1-45].

The motion equation for an incompressible fluid in Cartesian coordinates takes the form

$$\partial v_i / \partial \tau + v_j v_{i,j} = F_i - \frac{1}{\rho} p_{,i} + \sigma_{ij,i}; \quad v_{i,i} = 0; \quad i, j = 1, 2, 3 \quad (1-10-31)$$

where  $\sigma_{ij}$  is determined by the relations

$$\sigma_{ij} = 2Bv_{i,i}; \quad \sigma_{ij} = B(v_{i,j} + v_{j,i}) \quad \text{when } i \neq j \quad (1-10-32)$$

$$B = \left[ \sigma^n D^{-\frac{1}{m}} + \eta^{\frac{1}{m}} \right] A^{\frac{n}{m} - 1} \quad (1-10-33)$$

$$A^2 = 2(v_{i,i})^2 + (v_{i,j} + v_{j,i})^2 \quad (1-10-34)$$

We shall consider a plane boundary layer assuming that the hydrodynamic characteristics are constant. The viscous stress tensor is defined by relation (1-10-30).

The following dimensionless quantities will be introduced

$$\begin{aligned} x_1^* &= x/L; \quad x_2^* = y/L; \quad v_2^* = v_x/v_\infty; \\ v_1^* &= v_y/v_\infty; \quad p^* = (p - p_0)/p_0 \end{aligned} \quad (1-10-35)$$

Then equations (1-10-31) through (1-10-34) become

$$\partial v_i^* / \partial x_i^* = 0; \quad i = 1, 2 \quad (1-10-36)$$

$$\begin{aligned} \kappa^{\frac{n}{m}+1} \operatorname{Re} \left( v_j^* \frac{\partial v_j^*}{\partial x_j^*} \right) &= - \varepsilon^{\frac{n}{m}+1} \frac{L^{n/m} p_0}{\eta^{n/m} v_\infty} \frac{\partial p^*}{\partial x^*} \\ &+ 2\kappa^2 \frac{\partial}{\partial x_i^*} \left\{ \left[ S^{\frac{1}{n}} \kappa^{\frac{1}{m}} + A^{\frac{1}{m}} \right] \frac{1}{A} \frac{\partial v_i^*}{\partial x_i^*} \right\} \\ &+ \kappa^2 \frac{\partial}{\partial x_i^*} \left\{ \left[ S^{1/n} \kappa^{\frac{1}{m}} + A^{\frac{1}{m}} \right] \frac{1}{A} \left( \frac{\partial v_1^*}{\partial x_2^*} + \kappa^2 \frac{\partial v_2^*}{\partial x_1^*} \right) \right\}; \quad i, j = 1, 2 \end{aligned} \quad (1-10-37)$$

where

$$A = \left[ 2\kappa^2 \left( \frac{\partial v_i^*}{\partial x_i^*} \right) + \left( \frac{\partial v_1^*}{\partial x_2^*} + \kappa^2 \frac{\partial v_2^*}{\partial x_1^*} \right)^2 \right]; \quad i = 1, 2 \quad (1-10-38)$$

$$\text{Re} = \frac{\rho L^{\frac{n}{m}} v_\infty^{2-\frac{n}{m}}}{\eta^{n/m}} \quad (1-10-39)$$

is the generalized Reynolds number and

$$S = \frac{\sigma_0 L^{\frac{n}{m}}}{\eta^{\frac{n}{m}} v_\infty^{n/m}} \quad (1-10-40)$$

is the dimensionless parameter of nonlinear viscoplasticity.

When  $m = n$  the complex  $S$  equals the Ilyushin number and expresses the relationship between plastic and viscous dissipations.

For different relations between  $A$ ,  $\text{Re}$  and  $S$ , the boundary layers may be of different types. The most typical cases will be considered.

1. If  $\kappa^{n/m+1} \text{Re} \approx 1$  and  $S^{1/n} \kappa^{1/m} \ll 1$ , the terms accounting for the plasticity effect may be neglected. In this case  $\kappa = \text{Re}^{-\frac{m}{n+m}}$ . If  $p^*$  is assumed to be equal to  $\rho v_\infty^2$  ( $p^* = \rho v_\infty^2$ ), then we get the boundary-layer equation for a pure viscous power-law fluid

$$v_1 v_{1,1} + v_2 v_{1,1} = -p_{,1} + \frac{n}{m} (v_{1,2})^{\frac{n}{m}-1} \left( \frac{\partial^2 v_1}{\partial x_2^2} \right) \quad (1-10-41)$$

$$v_{1,1} + v_{2,2} = 0; \quad p_{,2} = 0 \quad (1-10-42)$$

where the superscript  $*$  is omitted.

The similarity solutions depending on one argument play a significant role in the asymptotic boundary layer theory. Two types of similarity boundary conditions are possible:

(a) Falkner-Scan's velocity law

$$v_\infty = x^a \quad (1-10-43)$$

which corresponds to a wedge flow.

Then, the similarity variables become \*

$$\xi \eta = y x^{-\frac{\left(\frac{n}{m} - 2\right)a+1}{1+n/m}}; \quad f = \psi x^{-\frac{\left(2\frac{n}{m} - 1\right)a+1}{m+1}} \quad (1-10-44)$$

\* Ordinary similarity variables are designated by  $\eta$ ,  $\psi$ ,  $f$ , but we shall follow the notations used in [1-2].

When  $a = 1/3$  (a rectangular wedge in a flow symmetric to its face) the transformation (1-10-44) is independent of rheological parameters.

(b) Goldshtik's velocity law

$$v_\infty = e^{kx} \quad (1-10-45)$$

then the similarity variables will be

$$\xi = y \exp\left(-\frac{2 - n/m}{n/m + 1} kx\right); \quad f = \psi \exp\left(-\frac{1 - 2n/m}{1 + n/m} kx\right) \quad (1-10-46)$$

The constant  $k$  may assume any real values other than zero.

For the boundary-value problems, the boundary layer equations will be of the form:

$$(f')^2 - \frac{(2m/n - 1)a + 1}{a(n/m + 1)} ff'' = 1 + \frac{1}{a} \frac{d}{d\xi} (f'')^{n/m} \quad (1-10-47)$$

$$(f')^2 - \frac{1 - 2n/m}{1 + n/m} ff'' = 1 + \frac{1}{k} \frac{d}{d\xi} (f'')^{n/m} \quad (1-10-48)$$

2. Let us assume that  $\kappa^{1+n/m} \text{Re} \approx 1$  and  $\kappa^{\frac{1}{m}} \rho^{\frac{1}{n}} \approx 1$ . Then assuming  $\kappa = \text{Re}^{-\left(\frac{1}{1+n/m}\right)}$  and  $p_0 = \rho v_\infty^2$ , we get the following boundary layer equations

$$v_{1,1} + v_{2,2} = 0; \quad p_{,2} = 0 \quad (1-10-49)$$

$$v_1 v_{1,1} + v_{2,2} = -p_{,1} + \left[ S^{\frac{1}{n}} \text{Re}^{-\frac{m}{(n+m)n}} + (v_{1,2})^{\frac{1}{m}} \right]_{,2} \quad (1-10-50)$$

Equation (1-10-50) takes into account the plasticity and viscosity. The greatest difficulty in solving (1-10-50) resides in the determination of  $S$ . The point is that for viscoplastic bodies the classical hydrodynamic equality  $-p_{,1} = \rho v_\infty^2 v_{\infty,1}$  cannot be used. In [1-45] it is assumed that  $p = \text{const}$  for all bodies, different from a plate, when they are blown by the Shvedov-Bingham fluid flow.

One of the ways to overcome this difficulty is to use the methods of calculating a finite-thickness boundary layer. In [1-41] the integral boundary-layer equations have been obtained and solved by the Karman-Pohlhausen method. In particular, the problem has been solved for the boundary conditions

$$\text{when } x_2 = 0; \quad v_1 = 0; \quad v_2 = 0 \quad (1-10-51)$$

$$\text{when } x_2 = \delta(x_1); \quad v_1 = 1; \quad v_{1,2} = 0; \quad \sigma = \sigma_0 \quad (1-10-52)$$

In this case the integral equation has the form

$$-\frac{d}{dx_1} \left( \delta \int_0^1 v_1 (1 - v_1) d\xi \right) = N^{\frac{1}{n}} + \frac{1}{\delta^{1/m}} \left( \frac{\partial v_1}{\partial \xi} \right)_{\xi=0}^{\frac{1}{m}} - N \quad (1-10-53)$$

where  $N$  is the nonlinear viscoplasticity number

$$N = \left( \frac{\sigma_v}{\rho v_\infty^2} \right) \text{Re}^{\frac{n}{m+n}} = \rho \text{Re}^{-\frac{m}{m+n}} \quad (1-10-54)$$

When  $m = n$ , the integration of equation (1-10-53) is completed.

For a particular case, when  $n = 2$  (the Casson fluid), the following relation is obtained

$$\kappa = \frac{3.39}{16 \cdot 280 N^2} \left[ \frac{1}{3} \beta^3 - \frac{1}{2} \beta^2 + \beta - \log_e(1 + \beta) \right] = C \sum_{n=4}^{\infty} (-1)^n \frac{1}{n} \beta^n \quad (1-10-55)$$

where

$$C = \frac{3.39}{16 \cdot 280 N^2} ; \quad \beta = \left( \frac{8 N \delta}{\xi} \right)^{\frac{1}{3}} \quad (1-10-56)$$

If only the first term in (1-10-55) is taken, then

$$\delta = \sqrt[3]{\frac{280}{13} \kappa} \quad (1-10-57)$$

Relation (1-10-57) is the Skelland formula [1-46] for a plate in a longitudinal Shvedov-Bingham fluid flow. Therefore, the first term of series (1-10-55) accounts for linear rheological properties of fluid and the subsequent terms account for the nonlinear viscoplasticity.

In [1-41] the case  $\kappa^{\frac{n}{m} + 1} \text{Re} \ll 1$  and  $S^n \kappa^{\frac{1}{m}} \sim \frac{1}{\kappa^2}$  has been considered. Now the inertia forces may be neglected and  $\kappa$  may be taken equal to

$$\kappa = S^{-\frac{m}{n(1+2m)}} \quad (1-10-58)$$

The characteristic pressure is taken as  $p_0 = \kappa \sigma_0$ . This case corresponds to a slow flow adjacent to the wall where the plastic forces dominate over the viscous ones. This case has been analyzed in detail in [1-41].

All the above solutions of hydrodynamic equations for rheological fluids were obtained with the assumption that the viscous stress tensor is symmetrical ( $\sigma_{ij} = \sigma_{ji}$ ).

Many rheological fluids do not however satisfy the symmetry condition since these are oriented mediums. Therefore they should be considered as asymmetric fluids.

### 1-11. ON HYPERBOLIC HEAT- AND MASS-TRANSFER EQUATIONS

Mass and energy (heat) transfer is governed by differential parabolic equations. They are derived on the basis of mass and energy conservation laws and also from the Fick and Fourier hypotheses on the relations between mass and heat flows and temperature and concentration gradients.

Hyperbolic equations, in which the wave propagation velocity (sound and light velocities) appear as a basic parameter, describe the propagation of sound and electromagnetic waves. These equations are derived from conservation laws without making any additional assumptions, which is explained as follows. Parabolic equations are not invariant with respect to the sign of the variable  $\tau$ , i.e. the substitution of  $-\tau$  for the time  $\tau$  changes the equation itself. This is evident from the heat-conduction equation

$$\frac{\partial T}{\partial \tau} = a \nabla^2 T \quad (1-11-1)$$

The irreversible processes of heat propagation are described by the heat-conduction parabolic equations.

The hyperbolic equation describing wave propagation

$$\frac{\partial^2 \varphi}{\partial \tau^2} = c^2 \nabla^2 \varphi \quad (1-11-2)$$

where  $c$  is the wave propagation velocity invariant relative to the (algebraic) sign of variable  $\tau$ ; the equation, therefore, describes the reversible wave propagation process. Consequently, if the differential equation is invariant relative to the transformation of inversion with respect to time, then the process described by this equation is reversible. The heat transfer process is an irreversible process of heat propagation.

The difference between the physics of heat and light propagation has a mathematical explanation. Any physical process takes place in space and time. Any transfer process is therefore characterized by a certain physical quantity which is a function of space and time. For example, the heat transfer process is characterized by temperature propagation in space and time.

For the description of transfer of a physical scalar substance under consideration (temperature, mass concentration, energy) the first and second

order differential invariants are used. The first order differential invariant of functions  $\varphi_1$  and  $\varphi_2$  is  $H_{\varphi_1, \varphi_2}$ , which is defined by the relation

$$H_{\varphi_1, \varphi_2}^2 = \frac{\partial \varphi_1}{\partial x} \cdot \frac{\partial \varphi_2}{\partial x} + \frac{\partial \varphi_1}{\partial y} \cdot \frac{\partial \varphi_2}{\partial y} + \frac{\partial \varphi_1}{\partial z} \cdot \frac{\partial \varphi_2}{\partial z} \quad (1-11-3)$$

If the functions  $\varphi_1$  and  $\varphi_2$  are equal, i.e. ( $\varphi_1 = \varphi_2 = \varphi$ ), then

$$H_\varphi = \left[ \left( \frac{\partial \varphi}{\partial x} \right)^2 + \left( \frac{\partial \varphi}{\partial y} \right)^2 + \left( \frac{\partial \varphi}{\partial z} \right)^2 \right]^{1/2} = |\vec{\nabla} \varphi|. \quad (1-11-4)$$

Thus

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial n} \cos(n, x); \quad \frac{\partial \varphi}{\partial y} = \frac{\partial \varphi}{\partial n} \cos(n, y); \quad \frac{\partial \varphi}{\partial z} = \frac{\partial \varphi}{\partial n} \cos(n, z) \quad (1-11-5)$$

$$\cos^2(n, x) + \cos^2(n, y) + \cos^2(n, z) = 1 \quad (1-11-6)$$

$$|\vec{\nabla} \varphi| = \frac{\partial \varphi}{\partial n} \quad (1-11-7)$$

The first order differential invariants are related by the following equation

$$H_{\varphi_1, \varphi_2}^2 = H_{\varphi_1} H_{\varphi_2} \cos(n_{\varphi_1}, n_{\varphi_2}) \quad (1-11-8)$$

This relationship determines the angle in a three-dimensional manifold.

The second order differential invariant of the function is

$$H_\varphi'' = \nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \quad (1-11-9)$$

The first and second order differential invariants are related by the Green theorem

$$\int_{(V)} H_{\varphi_1, \varphi_2}^2 dV = \int_{(S)} \varphi_2 \frac{\partial \varphi_1}{\partial n} dS - \int_{(V)} \varphi_2 \nabla^2 \varphi_1 dV \quad (1-11-10)$$

which changes to the Gaussian formula if  $\varphi_2$  is constant or its normal is orthogonal to the normal of function  $\varphi_1$

$$\int_{(S)} \frac{\partial \varphi}{\partial n} dS = \int_{(V)} \nabla^2 \varphi dV \quad (1-11-11)$$

Formulas (1-11-10) and (1-11-11) are integrated over a closed surface  $S$  bounding volume  $V$ .

Let us now consider the transfer of heat. An isothermal surface in the Euclidian space is described by the equation

$$T(x, y, z, \tau) = \text{const} \quad (1-11-12)$$

To derive differential equations, main physical quantities which should be independent of the coordinate system are chosen. By the use of these quantities the differential invariants are closed. The first differential invariant of isothermal surface  $H_T$  is equal to the absolute value of temperature gradient

$$H_T = \sqrt{\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 + \left(\frac{\partial T}{\partial z}\right)^2} = \frac{\partial T}{\partial n} = |\vec{\nabla} T| \quad (1-11-13)$$

This differential invariant may be used to relate the propagation velocity of isotherm  $w_T$  and the local derivative of temperature with respect to time  $\partial T/\partial \tau$ . Differentiation of equation (1-11-12) gives

$$\frac{\partial T}{\partial n} dn + \frac{\partial T}{\partial \tau} d\tau = 0 \quad (1-11-14)$$

Let us denote

$$w_T = \frac{dn}{d\tau} \quad (1-11-15)$$

and call it the propagation velocity of an isothermal surface. Then

$$w_T = -\frac{\partial T}{\partial \tau} \left(\frac{\partial T}{\partial n}\right)^{-1} = -\frac{1}{H_T} \cdot \frac{\partial T}{\partial \tau} \quad (1-11-16)$$

The law of conservation of energy must be used to close differential invariants.

The energy flux is defined in terms of temperature or rather temperature gradient. These quantities are related by the Fourier hypothesis

$$\mathbf{q} = -\lambda \vec{\nabla} T \quad (1-11-17)$$

In this case the energy (heat) flux should be independent of the choice of the frame of reference.

The heat transfer equation is obtained from the conservation equation

$$\frac{\partial T}{\partial \tau} = a \nabla^2 T + \dot{I}_q / c_p \rho \quad (1-11-18)$$

where  $a$  is the thermal diffusivity or heat diffusion coefficient ( $a = \lambda / c_p \rho$ ) which is directly proportional to the isothermal propagation velocity

$$a = \frac{\vec{\nabla} T}{\nabla T} w_T \quad (1-11-19)$$



This is however not the only way of describing the heat transfer process. For some particular cases a complete solution of differential invariants may be obtained without using the Fourier hypothesis [1-47] which will be described in more detail.

The isothermal surface  $T(x, y, z, \tau)$  belongs to a class of medium state surfaces. There is another class of medium state surfaces described by the equation

$$\varphi(x, y, z, \tau, p) = \text{const} \quad (1-11-20)$$

where  $p$  is a free parameter.

A continuous change of the parameter results in many intersecting surfaces which form a new enveloping surface. To find its equation, it is necessary to eliminate parameter  $p$  from the equation

$$\frac{\partial \varphi}{\partial p} = 0 \quad (1-11-21)$$

This enveloping surface will be the locus of the characteristics and is called the Monge surface. The characteristics are tangential curves of the enveloping surface to each of the enveloped ones. The Monge surface in its physical meaning characterizes simultaneous processes. A light wave front, which is an enveloping surface for secondary waves (the Huygens-Fresnel principle), may serve as an example. According to the Huygens-Fresnel principle, in order to find a new front of the light wave each point of the front wave should be considered an independent source of spherical waves. The enveloping surface for all the secondary waves forms a new front of the light wave.

A. S. Predvoditelev extended the range of physical characteristics of the Monge surface which should cover simultaneous processes consistent with conservation laws. Such a surface may be a phase transition surface or a chemical conversion surface, i.e. at any point of the surface the processes should be simultaneous. This fact sharply distinguishes a second class surface  $\varphi(x, y, z, \tau, p)$  from a first class surface  $T(x, y, z, \tau)$ .

Indeed, the isothermal surface  $T(x, y, z, \tau)$  divides the fluid into disturbed and non-disturbed regions. Transition from one region to the other is accompanied by continuous variations of  $T(x, y, z, \tau)$  and its derivatives.

A surface of the second class also divides the fluid into two regions, but transition from one region to the other is accompanied by a discontinuity and the interface fluid acquires specific properties.

The compatibility conditions of the fluid state, divided by the Monge surface, were obtained by Hugoniot and Hadamard.

Let us consider a fluid state function of the second class  $\Phi(x, y, z, \tau)$ . Then the following relations hold for the transition through the Monge

surface

$$\delta \frac{\partial \Phi}{\partial x} = \frac{\varepsilon_1}{H_f} \cdot \frac{\partial f}{\partial x}; \quad \delta \frac{\partial^2 \Phi}{\partial x \partial y} = \frac{\varepsilon_2}{H_f^2} \cdot \frac{\partial f}{\partial x} \cdot \frac{\partial f}{\partial y}; \quad \delta \frac{\partial^2 \Phi}{\partial x^2} = \frac{\varepsilon_2}{H_f^2} \left( \frac{\partial f}{\partial x} \right)^2 \quad (1-11-22)$$

$$\delta \frac{\partial \Phi}{\partial \tau} = \frac{\varepsilon_1}{H_f} \cdot \frac{\partial f}{\partial \tau} = -\varepsilon_1 w_f; \quad \delta \frac{\partial^2 \Phi}{\partial \tau^2} = \frac{\varepsilon_2}{H_f^2} \left( \frac{\partial f}{\partial \tau} \right)^2 = \varepsilon_2 w_f^2 \quad (1-11-23)$$

where  $f$  is the Monge function;  $\varepsilon_1$  and  $\varepsilon_2$  are discontinuity constants of the first and second orders, respectively;  $w_f$  is the propagation velocity of the Monge surface;  $\delta$  is the Appel delta indicating transition through the Monge surface.

From relations (1-11-22) and (1-11-23) we get

$$\delta \nabla^2 \Phi = \varepsilon_2; \quad \delta \frac{\partial^2 \Phi}{\partial \tau^2} = \varepsilon_2 w_f^2 \quad (1-11-24)$$

Hence

$$w_f^2 \delta \nabla^2 \Phi = \delta \frac{\partial^2 \Phi}{\partial \tau^2} \quad (1-11-25)$$

For the Appel delta  $\delta$  the following commutation formulas hold true

$$d\delta = \delta d; \quad \frac{\partial}{\partial n} \delta = \delta \frac{\partial}{\partial n} \quad (1-11-26)$$

The quantity  $w_f^2$  is a continuous function of time and coordinates at any point of the space, the operation  $\delta$  is therefore reversible. The following differential equation will result

$$\frac{\partial^2 \Phi}{\partial \tau^2} = w_f^2 \nabla^2 \Phi + f(x, y, z, \tau) \quad (1-11-27)$$

where  $f(x, y, z, \tau)$  is any continuous function.

In deriving hyperbolic equation (1-11-27) the conservation laws were used for the determination of the Monge surface. In so doing secondary processes should occur which compensate for energy dissipation.

However, differential equations of heat conduction in materials with fading memory are of the greatest interest.

In Ref. [1-49] the specific heat flux is defined by the relation

$$\vec{q}(\tau) = - \int_0^\infty k(\theta) g(\tau - \theta) d\theta \quad (1-11-28)$$

where  $g \equiv \text{grad } T$  is the temperature gradient. The specific heat flux is thus independent of the temperature gradient at a given time moment. Besides, relation (1-11-28) is valid for isotropic materials.

If the temperature gradient  $g$  is independent of time  $\theta$ , then denoting by  $\lambda$  the quantity

$$\lambda = \int_0^{\infty} k(\theta) d\theta \quad (1-11-29)$$

we get the formula for the Fourier heat-conduction law  $\vec{q} = -\lambda \text{ grad } T$ , where  $\lambda$  is the thermal conductivity in a state of equilibrium.

If we assume that

$$k(\theta) = \frac{\lambda}{\tau_r} \exp\left(-\frac{\theta}{\tau_r}\right) \quad (1-11-30)$$

where  $\tau_r$  is the Maxwell-Cattaneo relaxation time, then the heat flux expression, taking account of the finite heat propagation velocity, is obtained in the form

$$\vec{q} = -\lambda \text{ grad } T - \tau_r \frac{d\vec{q}}{dt} \quad (1-11-31)$$

The relaxation time  $\tau_r$  is inversely proportional to the square of the finite heat propagation velocity  $w_q$  and directly proportional to thermal diffusivity  $a$  ( $\tau_r = a/w_q^2$ ).

Let us use the linearized fundamental equation for heat flux  $q$  and internal energy  $e$  in the form [1-49]

$$\vec{q} = - \int_0^{\infty} k(\theta) g(\tau - \theta) d\theta = \int_0^{\infty} k'(\theta) \vec{g}^{\tau}(\theta) d\theta \quad (1-11-32)$$

$$e = e_0 + CT - \int_0^{\infty} \beta'(\theta) \vec{T}^{\tau}(\theta) d\theta \quad (1-11-33)$$

$$\frac{d}{d\theta} \vec{g}^{\tau}(\theta) = \vec{g}^{\tau}(\theta) = g(\tau - \theta) \quad (1-11-34)$$

where  $C$  is the heat capacity of a solid;  $\beta(\theta)$  is a relaxation function of internal energy. Then we get the following relations for time derivatives of the

heat flux  $\vec{q}$  and internal energy  $\dot{e}$

$$\vec{q} = -k(0)g - \int_0^{\infty} k'(\theta)g(\tau - \theta) d\theta \quad (1-11-35)$$

$$\dot{e} = C\dot{T} + \beta(0)T + \int_0^{\infty} \beta'(\theta)T(\tau - \theta) d\theta \quad (1-11-36)$$

If the energy conservation equation is used, then the following generalized linear heat-conduction equation is obtained

$$\begin{aligned} C \frac{\partial^2 T(\vec{x}, \tau)}{\partial \tau^2} + \beta(0) \frac{\partial T(\vec{x}, \tau)}{\partial \tau} + \int_0^{\infty} \beta'(\theta) \frac{\partial T(\vec{x}, \tau - \theta)}{\partial \tau} d\theta \\ = k(0) \nabla^2 T(\vec{x}, \tau) + \int_0^{\infty} k'(\theta) \nabla^2 T(\vec{x}, \tau - \theta) d\theta + \dot{Q}(\vec{x}, \tau) \end{aligned} \quad (1-11-37)$$

where  $\dot{Q}(\vec{x}, \tau)$  is an external heat source. Equation (1-11-37) differs from the hyperbolic differential heat-conduction equation by additional integral terms taking account of relaxation functions of thermal conductivity and internal energy. Relations (1-11-35) and (1-11-36) show that relaxation times for thermal stress and internal energy are different.

In [1-48] for an isotropic fluid the linearized constitutive equation for a heat flux is used

$$\vec{q}(\tau) = -\lambda(0) \vec{\nabla} T(\tau) - \int_0^{\infty} \lambda'(\theta) \vec{\nabla} T(\tau - \theta) d\theta$$

which accounts for the dependence of heat flux on the temperature gradient at a given moment. If we assume that  $\lambda'(\theta) = 0$ , then we get the Fourier classical heat-conduction equation. For  $\lambda(0) = 0$ , a relationship similar to (1-11-28) is obtained. The quantity  $\lambda(0)$  is known as the instantaneous coefficient of thermal conductivity. In this case the differential heat-conduction equation will be of the form

$$\begin{aligned} C(0) \frac{\partial T(\vec{x}, \tau)}{\partial \tau} + \int_0^{\infty} \alpha'(\theta) \frac{\partial T(\vec{x}, \tau - \theta)}{\partial \tau} d\theta = \lambda(0) \nabla^2 T(\vec{x}, \tau) \\ + \int_0^{\infty} \lambda'(\theta) \nabla^2 T(\vec{x}, \tau - \theta) d\theta + \dot{Q}(\vec{x}, \tau) \end{aligned} \quad (1-11-38)$$

In the derivation of equation (1-11-38) the linearized constitutive equation for internal energy was used

$$e = e_0 + C(0)T + \int_0^{\infty} \alpha'(\theta)T(\tau - \theta) d\theta \quad (1-11-39)$$

where  $\alpha'(\theta)$  is the relaxation function of internal energy;  $C(0)$  is the instantaneous heat capacity.

Differential equation (1-11-38) was solved for a half-space when the kernels of integral relations  $\alpha(\theta)$  and  $\lambda(\theta)$  are power or exponential functions of time  $\theta$ . The integral relations in heat-conduction equation (1-11-38) do not involve great difficulties when solving it by the Laplace integral transformation method, since these relations are integrated with respect to time from 0 to  $\infty$  [1-50]. Temperature waves in materials with memory are of special interest; they have their own peculiarities; the rates of their propagation and attenuation coefficients are different from similar relationships of the classical heat-conduction theory.

The hyperbolic heat-conduction equation follows from the above equation as a particular case.

If we assume that  $\beta'(\theta) = 0$  and  $k'(\theta) = 0$ , then

$$\frac{\partial T(x, \tau)}{\partial \tau} + \tau_{rq} \frac{\partial^2 T(\vec{x}, \tau)}{\partial \tau^2} = a(0) \nabla^2 T(\vec{x}, \tau) \quad (1-11-40)$$

where

$$a(0) = k(0)\beta(0); \quad \tau_{rq} = c/k(0)$$

Equation (1-11-40) differs from the Fourier heat-conduction equation by the additional term which characterizes the wave process of heat propagation.

The finite heat propagation velocity in a solid may be accounted for by a dimensionless number called the Veron number  $Ve$

$$Ve = \frac{a}{l} \sqrt{\frac{\rho_0}{k(0)}} \quad (1-11-41)$$

where  $l$  is the characteristic dimension, and  $a$  is the thermal diffusivity.

For heat-transfer processes, characterized by the displacement of phase or chemical conversion front with velocity  $w_f$ , the Veron number is

$$Ve = \frac{a}{w_f l} \quad (1-11-42)$$

Since relation (1-11-41) holds for a finite heat propagation velocity  $w_q (w_q = \sqrt{a/\tau_{rq}})$ , the Veron number here equals  $Ve = a/w_q l$ .

## 2-1. DIFFERENTIAL EQUATION OF HEAT CONDUCTION

The Fourier-Kirchhoff differential equation describes heat transfer in flowing fluids. If the diffusional heat conduction and heat transfer due to diffusion are neglected, then in the absence of an external force field the equation will take the form

$$c_p \rho \frac{dT}{d\tau} = \text{div} (\lambda \text{grad } T) + I_q + \eta \Phi_v \quad (2-1-1)$$

where  $I_q$  is the source of heat and  $\Phi_v$  is the Rayleigh dissipation function. For a solid body the material derivative  $dT/d\tau$  equals the partial derivative, as the solid does not move ( $\vec{v} = 0$ ) and the specific heat  $c_p$  at constant pressure is the same as the specific heat  $c_v$  at constant volume ( $c_p = c_v = c$ ). Equation (2-1-1) then takes the form

$$c \rho \frac{\partial T}{\partial \tau} = \text{div} (\lambda \text{grad } T) + I_q \quad (2-1-2)$$

In a general case the thermal conductivity  $\lambda$ , specific heat  $c$ , and the heat source  $I_q$  depend on temperature. Generally  $\lambda$  and  $c\rho$  are assumed to be constant and equation (2-1-2) then changes into the Fourier classical heat-conduction equation

$$\frac{\partial T}{\partial \tau} = a \nabla^2 T + \frac{I_q}{c\rho} \quad (2-1-3)$$

where  $a$  is the thermal diffusivity ( $a = \lambda/c\rho$ ). For a solid body the Fourier heat-conduction equation may be written as

$$\bar{q} = - \bar{n}^1 \lambda \frac{\partial T}{\partial n} = - \bar{n}^1 \lambda \left( \frac{\partial T}{\partial u_v} \right) \frac{\partial u_v}{\partial n} = - a_v \text{grad } u_v \quad (2-1-4)$$

where  $u_v$  is the volume concentration of internal energy and  $a_v$  is the thermal diffusivity at constant volume, ( $a_v = \lambda/c_v\rho$ ), as

$$\left( \frac{\partial u_v}{\partial T} \right)_v = c_v \rho \quad (2-1-5)$$

Therefore, the heat flux density, according to equation (2-1-4), is directly proportional to the body's internal energy concentration gradient. The proportionality factor  $a_v$  is the internal energy diffusion coefficient.

At constant pressure the thermal diffusivity is defined by

$$a_p = \lambda / c_p \rho \quad (2-1-6)$$

where  $c_p$  is the specific heat at constant pressure

$$c_p = \frac{1}{\rho} \frac{\partial H_v}{\partial T} \quad (2-1-7)$$

in which  $H_v$  is the volume concentration of enthalpy.

In this case the Fourier heat-conduction equation may be written as

$$\vec{q} = -\bar{n}^1 \lambda \left( \frac{\partial T}{\partial H_v} \right) \frac{\partial H_v}{\partial n} = -a_p \text{grad } H_v \quad (2-1-8)$$

where the proportionality factor  $a_p$  between the heat flow and enthalpy volume concentration gradient is the enthalpy diffusion coefficient.

Thus, thermal diffusivity is the internal energy ( $a_v$ ) or enthalpy ( $a_p$ ) diffusion coefficient depending on the mode of interaction of the body with the surroundings ( $V = \text{const}$  or  $p = \text{const}$ ).

For anisotropic bodies the thermal conductivity is a tensor of the second order  $\lambda_{ij}$ .

In this case the Fourier heat-conduction law will be written as:

$$q_i = -\lambda_{ij} \frac{\partial T}{\partial x_j}, \quad i, j = 1, 2, 3 \quad (2-1-9)$$

and the differential equation of heat-conduction will take the form

$$c_p \rho \frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x_i} \left( \lambda_{ij} \frac{\partial T}{\partial x_j} \right) + I_q \quad (2-1-10)$$

A transition from rectangular coordinates  $O \ x_i$  ( $i = 1, 2, 3$ ) to rectangular coordinates  $Q \ \xi_i$  ( $i = 1, 2, 3$ ) directed along the principal axes of a crystal (anisotropic body) results in equation (2-1-10) in the form

$$c_p \rho \frac{\partial T}{\partial \tau} = \lambda_i \frac{\partial^2 T}{\partial \xi_i^2} + I_q, \quad i = 1, 2, 3 \quad (2-1-11)$$

where  $\lambda_i$  is the conductivity in the direction of principal axes.

Let us now introduce new coordinates  $\eta_i$  defined by the relation

$$\eta_i = \left( \frac{\Lambda}{\lambda_{(i)}} \right)^{\frac{1}{2}} \xi_{(i)}, \quad i = 1, 2, 3 \quad (2-1-12)$$

where  $\Lambda$  is a constant. Equation (2-1-10) now takes the form

$$c_p \rho \frac{\partial T}{\partial \tau} = \Lambda \frac{\partial^2 T}{\partial \eta_i \partial \eta_i}, \quad i = 1, 2, 3 \quad (2-1-13)$$

The thermal conductivity of fluids may depend on the deformation gradient  $G$  (see Sec. 1-9); in this case the heat flux equation will be written as

$$\bar{q} = -\lambda(T, G) \text{ grad } T \quad (2-1-14)$$

The heat-conduction equation then reads

$$c\rho \frac{\partial T}{\partial \tau} = \text{div} (\lambda \text{ grad } T) = \text{tr} (\lambda_s \text{ grad}^2 T) + \text{div} (\tilde{\lambda} \text{ grad } T) \quad (2-1-15)$$

where  $\lambda_s$  is the symmetric part of the tensor

$$\lambda_s = \frac{1}{2} (\lambda + \tilde{\lambda}) \quad (2-1-16)$$

If the density  $\rho$  and deformation gradient  $G$  are constant, and heat capacity  $c(T)$  and thermal conductivity  $\lambda(T)$  are functions of temperature only, then equation (2-1-15) may be rewritten as

$$c\rho \frac{\partial T}{\partial \tau} \text{tr} (\lambda_s \text{ grad}^2 T) + \text{grad } T \cdot \frac{\partial \lambda_s}{\partial T} \text{grad } T \quad (2-1-17)$$

## 2-2. INITIAL AND BOUNDARY CONDITIONS

The differential equation for heat conduction establishes a relationship between time and space changes of body temperature; this equation is a mathematical description of heat transfer within a body. In order to find the temperature field within the body at any time, i.e. to solve the differential equation, it is necessary to know the temperature distribution pattern within the body at the initial moment (initial condition), the geometry and the mode of interaction between the body surface and the surroundings (boundary conditions).

The initial condition is defined by the law of temperature distribution within the body at the initial moment, i.e. when

$$\tau = 0; \quad T_0 = f_0(x_1, x_2, \dots, x_n) = f_0(x_i) \quad (2-2-1)$$

where  $f_0(x_i)$  is a known function of coordinates  $x_1, x_2, \dots, x_n$ . In Cartesian coordinates  $f_0(x_i) = f(x, y, z)$ , because  $x_1 \equiv x; x_2 \equiv y; x_3 \equiv z$ .

In many problems it is assumed that at the initial moment the temperature distribution is even, i.e.

$$T_0 = \text{const} \quad (2-2-2)$$

Boundary conditions may be defined as follows:

1. The first kind boundary condition resides in preassigning surface temperature distribution at any time

$$T_s = \varphi_0(\tau, x_{is}) \quad (2-2-3)$$

where subscript  $s$  indicates the surface of a body.



In a particular case,  $T_s = T_c = \text{const}$ , i.e. the temperature at the surface remains constant throughout the heat transfer process (the first kind homogeneous limiting condition). This may be achieved by artificially maintaining the temperature at a constant level or under specific heat transfer conditions (see boundary conditions of the third kind).

2. The second kind boundary condition consists in preassigning heat flux at any surface point of the body as a function of time, i.e.

$$\lambda_{js} \left( \frac{\partial T}{\partial n_j} \right)_s = q_{js}(\tau, x_{js}) \quad (2-2-4)$$

where  $j = 1, 2, 3 \dots$ ;  $k$  is the number of continuous boundary surfaces; for example, for a parallelepiped  $k = 6$ .

The simplest case of a second-kind boundary condition (a homogeneous boundary condition of the second kind) consists in keeping the heat flux density constant:

$$-\lambda_{js} \left( \frac{\partial T}{\partial n_j} \right)_s = q_c = \text{const} \quad (2-2-5)$$

Such a case of heat transfer occurs when bodies are heated in high-temperature furnaces, in which heat transfer takes place basically by radiation (as per the Stefan-Boltzmann law) and when the temperature of the body is much less than the temperature of radiating surfaces.

3. The third kind boundary conditions generally characterize the law of convective heat transfer between the surface of a body and its surroundings for a constant heat flux (steady state temperature field). In this case, the amount of heat transferred per unit time from a unit area into the surrounding medium with temperature  $T_c$  in the process of cooling ( $T_s > T_c$ ) is directly proportional to the difference in temperature between the surface and the surrounding medium, i.e.

$$q_{js} = \alpha_j(T_{js} - T_c); \quad j = 1, 2, \dots, k \quad (2-2-6)$$

where  $\alpha_j$  is the proportionality factor designated as heat-transfer coefficient,  $\text{W/m}^2 \cdot ^\circ\text{C}$ .

For a heating process a similar relation may be written by interchanging  $T_{js}$  and  $T_c$ . The heat transfer coefficient is numerically equal to the amount of heat released (or received) by a unit surface area of the body in a unit time when the difference in temperature between the surface and the surroundings is  $1^\circ\text{C}$ .

Relation (2-2-6) may be obtained from the Fourier heat conduction law with the assumption that when the surface of a body is flown around by liquid or gas, the transfer of heat from liquid to the body close to its surface takes place as per the Fourier law.

$$\bar{q}_{js} = \lambda_f \left( \frac{\partial T_f}{\partial n_j} \right) \bar{\mathbf{i}}_{nj} = \frac{\lambda_f(T_{js} - T_c) \bar{\mathbf{i}}_{nj}}{\delta_j} = \alpha_j(T_{js} - T_c) (-\bar{\mathbf{i}}_{nj}) \quad (2-2-7)$$

where  $\lambda_f$  is the fluid thermal conductivity;  $\delta_j$  is the conventional boundary layer thickness;  $\vec{1}_{n_j}$  is the unit vector normal to the isothermal surface;  $\alpha_j = \lambda_f/\delta_j$ .

The heat flux vector  $\vec{q}_{js}$  is thus directed along the normal  $n_j$  to the isothermal surface and its scalar quantity equals  $q_{js}$ .

The conventional boundary layer thickness  $\delta_j$  depends on the velocity of fluid and its physical properties. Therefore the heat transfer coefficient depends on the fluid velocity and its temperature and varies along the surface of the body in the direction of motion.

As an approximation, the heat transfer coefficient may be assumed constant, independent of temperature and uniform over the whole surface.

Strictly speaking, relation (2-2-6) holds true only at constant temperature  $T_{js}$ . But in most cases  $T_{js}$  varies along the surface of the body. This variation depends upon the properties of the body in flow. Therefore, for convective heat transfer, relation (2-2-6) may be taken as the boundary condition for the first approximation.

The third kind boundary conditions may be used in the study of problems of radiation heating or cooling. From the Stefan-Boltzmann law the radiant heat flow between two surfaces equals

$$q_{js} = \sigma_j^*(T_{js}^4 - T_a^4) \quad (2-2-8)$$

where  $\sigma_j^*$  is the dimensionless emission coefficient;  $T_a$  is the absolute temperature of the heat-receiving body.

For a small temperature difference  $T_{js} - T_a$ , relation (2-2-8) may be approximately written as

$$q_{js} = \alpha_j(T) (T_{js} - T_a) \quad (2-2-9)$$

where  $\alpha_j(T)$  is the radiation heat-transfer coefficient, measured in the same units as the convective heat-transfer coefficient

$$\alpha_j(T) = (T_{js} + T_a) (T_{js}^2 + T_a^2) \sigma_j^* = \sigma_j^* k(T) \quad (2-2-10)$$

In Fig. 2-1  $K(T)$  is plotted versus  $T_{js}$  and  $T_a$  ( $K = 10^{-8}k$ ).

Relation (2-2-9) is the expression of Newton's law on cooling and heating processes; here  $T_a$  is the heat-receiving surface temperature. Although relation (2-2-9) is similar to expression (2-2-6) for convective heat transfer at constant heat flow, its physical significance is quite different. The radiation heat transfer coefficient  $\alpha_j(T)$  depends on temperature (Fig. 2-1) as well as on the properties of surfaces that are involved in heat transfer. If temperature  $T_{js}$  varies slightly, the coefficient  $\alpha_j(T)$  may be assumed to be almost constant.

If the ambient temperature  $T_c$  and the temperature  $T_a$  of the heat-receiving body are the same, and the radiation absorption coefficient of the medium is very small, then  $T_c$  may be substituted for  $T_a$  in relation (2-2-9). Here, a small fraction of heat, transferred from the solid by convection,

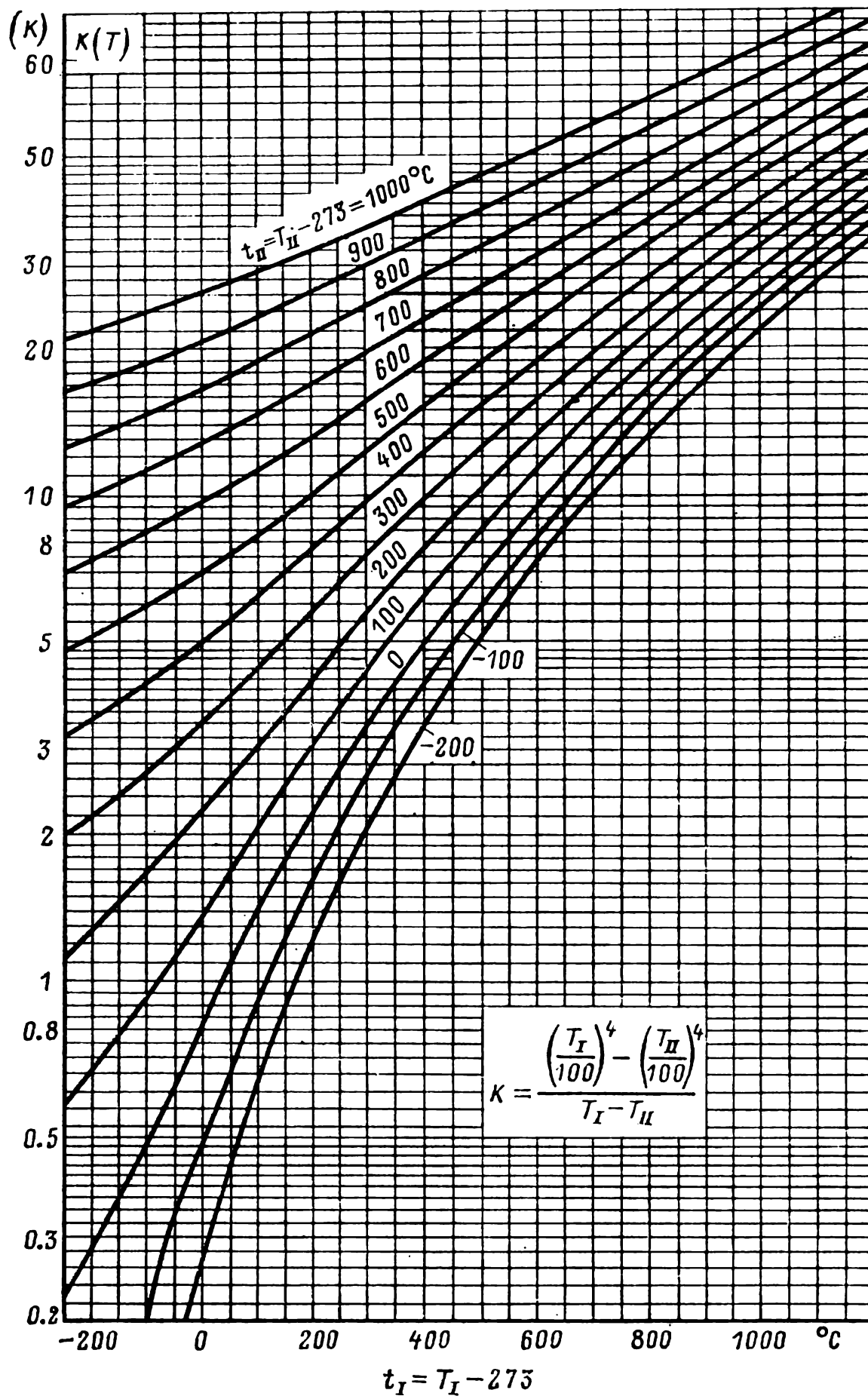


Fig. 2-1. Plot of coefficient  $K(T)$  versus temperatures  $T_I$  and  $T_{II}$

may be assumed equal to  $\alpha_{jk}\Delta T$  where  $\alpha_{jk}$  is the convective heat transfer coefficient. In this case, in the relation

$$q_{js} = \alpha_j(T_{js} - T_c) \quad (2-2-11)$$

the coefficient  $\alpha_j$  is the total heat transfer coefficient

$$\alpha_j = \alpha_{jk} + \alpha_j(T) \quad (2-2-12)$$

The heat transfer mechanism, governed by relation (2-2-11), will further be referred to as heat transfer according to Newton's law.

According to the law of conservation of energy the amount of heat  $q_{js}$  released by the solid surface is equal to the amount of heat supplied per unit time from the bulk of the solid to unit surface area by heat conduction, i.e.

$$\lambda_{js} \left( \frac{\partial T}{\partial n_j} \right)_s + \alpha_j T_{js} = \alpha_j T_c + f(x_{si}, \tau) \quad (2-2-13)$$

where, for generality,  $T_c$  is assumed to be variable and the heat transfer coefficient  $\alpha_j(T)$  is approximately constant [ $\alpha_j(T) = \alpha_j = \text{const}$ ].

In case of a uniform boundary condition, we have

$$\lambda_{js} \left( \frac{\partial T}{\partial n_j} \right)_s + \alpha_j [T_{js} - T_c] = 0, \quad (2-2-14)$$

The boundary condition of the first kind may be obtained from the boundary condition of the third kind as a particular case. If the ratio  $\alpha/\lambda$  tends to infinity, i.e. the heat transfer coefficient is large ( $\alpha \rightarrow \infty$ ) or the heat conduction coefficient ( $\lambda_j \rightarrow 0$ ) is small, then

$$T_{js} - T_c = \lim_{\substack{\alpha_j \rightarrow \infty \\ \lambda_{js}}} \left[ \frac{\lambda_{js}}{\alpha_j} \left( \frac{\partial T}{\partial n_j} \right)_s \right] = 0 \quad (2-2-15)$$

hence

$$T_{js} = T_c$$

i.e. the surface temperature of the heat-emitting body is equal to the ambient temperature. Similarly, when  $\alpha_j \rightarrow 0$  a particular case of a boundary condition of the second kind (i.e. the heat flow through the solid surface is equal to zero) is obtained from relation (2-2-14).

4. The boundary condition of the fourth kind conforms to heat transfer between a body and the surroundings (convective heat transfer between a body and a fluid) or heat transfer of solids in contact when their temperatures are equal. When a solid body is flown around by liquid (or gas), the transfer of heat from the liquid (gas) to the surface of the body in the immediate vicinity (a laminar boun-

dary layer or a laminar sublayer) takes place by conduction (molecular heat transfer), i.e. heat transfer conforming to the boundary condition of the fourth kind takes place.

$$T_{js}(x_{is}, \tau) = [T_c(x_i, \tau)]_{js} \quad (2-2-16)$$

Besides equal temperatures, heat fluxes are also equal

$$-\lambda_{cj} \left( \frac{\partial T_c}{\partial n_j} \right)_n = -\lambda_{js} \left( \frac{\partial T}{\partial n_j} \right)_s \quad (2-2-17)$$

Upon changing the state of aggregation (in the problems concerning melting, freezing, etc.), condition (2-2-17) is substituted by

$$\lambda_{1j} \left( \frac{\partial T_1}{\partial n_j} \right)_s = -\lambda_{2j} \left( \frac{\partial T_2}{\partial n_j} \right)_s + \psi_j(\tau) \quad (2-2-18)$$

where  $\psi_j(\tau)$  is a function that characterizes the phase conversion heat flux.

A differential equation together with initial and boundary conditions completely defines the problem, i.e. knowing the geometrical form of a body and the initial and boundary conditions, a complete solution of the differential equation can be obtained, and, therefore, the function representing temperature distribution at any time can be obtained. Thus, the function

$$T(x_i, \tau) = \varphi(x_i, \tau) \quad (2-2-19)$$

should be obtained upon solving the equation.

The function  $\varphi(x_i, \tau)$  should satisfy the differential equation (when the function is substituted for  $T$  in the heat-conduction differential equation the latter should become an identity) and also the initial and boundary conditions.

According to the unique solution theorem, if a function  $T(x_i, \tau)$  satisfies the heat-conduction differential equation and the initial and boundary conditions, then this function is the only solution of the problem.

### 2-3. HEAT CONSUMPTION CALCULATION METHODS

When heated or cooled, solids receive or release a definite amount of heat. Three methods are available to calculate the amount of heat consumed in a heat transfer process.

1. To a surface element  $ds$  in time  $d\tau$  the heat quantity

$$-\lambda \left( \frac{\partial T}{\partial n} \right) ds d\tau \quad (2-3-1)$$

is supplied.

To determine the amount of heat  $\Delta Q$  received by the solid in the time interval  $\Delta\tau = \tau_2 - \tau_1$ , it is necessary to integrate relation (2-3-1) over the whole surface  $S_j$  and the time interval

$$\Delta Q = - \int_{\tau_1}^{\tau_2} \int_S \sum_{j=1}^k \lambda_{js} \left( \frac{\partial T}{\partial n_j} \right)_s dS d\tau; \quad j = 1, 2, 3, \dots, k \quad (2-3-2)$$

Generally, the temperature and temperature gradient are uniform along the surface and formula (2-3-2) is then simplified to

$$\Delta Q = Q_2 - Q_1 = - \lambda_j \bar{S}_j \int_{\tau_1}^{\tau_2} \left( \frac{\partial T}{\partial n_j} \right)_s d\tau \quad (2-3-3)$$

2. In time  $\Delta\tau = \tau_2 - \tau_1$  the volume element  $dV$  is heated from  $T_1$  to  $T_2$ , it receives the amount of heat equal to

$$c\rho(T_2 - T_1) dV \quad (2-3-4)$$

The total amount of heat  $\Delta Q$  spent on heating the solid in time  $\Delta\tau$  may be found after integration over the whole volume, i.e.

$$\Delta Q = Q_2 - Q_1 = c\rho \int_{(V)} (T_2 - T_1) dV = c\rho V \frac{1}{V} \int_{(V)} (T_2 - T_1) dV \quad (2-3-5)$$

Let us now denote the mean (integral) temperature over the whole volume by  $\bar{T}$ , i.e.

$$\bar{T} = \frac{1}{V} \int_{(V)} T dV$$

then

$$\Delta Q = Q_2 - Q_1 = c\rho V(\bar{T}_2 - \bar{T}_1) \quad (2-3-6)$$

because  $\bar{T}_2 > \bar{T}_1$ .

The amount of heat  $Q - Q_0$  spent on heating in time  $\tau$  starting from the very beginning ( $\tau_1 = 0$ ) equals

$$Q - Q_0 = c\rho V(\bar{T} - \bar{T}_0) \quad (2-3-7)$$

where  $\bar{T}_0$  is the mean (integral) initial temperature.

If the initial temperature is uniform at all points ( $\bar{T}_0 = T_0 = \text{const}$ ), then the specific heat consumption equals

$$\Delta Q_V = c\rho(\bar{T} - T_0) \quad (2-3-8)$$

Therefore, the basic task in this case amounts to finding  $\bar{T}(\tau)$ .

3. The surface element  $dS$  in time  $d\tau$  receives from its surroundings an amount of heat equal to

$$\alpha_j(T_c - T_{js}) dS d\tau \quad (2-3-9)$$

The total amount of heat  $\Delta Q$  received by the whole surface is found by integration of equation (2-3-9) over the whole surface and time interval  $\Delta\tau$ .

If the solid surface temperature is uniform at all points and  $\alpha_j$  is independent of temperature, then

$$\Delta Q = Q_2 - Q_1 = \alpha_j S_j \int_{\tau_1}^{\tau_2} [T_c - T_{js}(\tau)] d\tau \quad (2-3-10)$$

In further solutions  $\bar{T}$  is to be found simultaneously with  $T$ . Knowing  $\bar{T}$ , the heat content  $Q$  of a body as well as the amount of heat supplied for heating ( $Q - Q_0$ ) or heat lost during cooling ( $Q_0 - Q$ ) can be determined.

In those cases when it is difficult to determine  $\bar{T}$ , the amount of heat consumed may be calculated by formula (2-3-3).

For one-dimensional problems when temperature is a function of time and one coordinate  $\bar{T}(\tau)$ , the average over volume temperature is equal to

$$\bar{T}(\tau) = \frac{\Gamma + 1}{R^{\Gamma+1}} \int_0^R x_1^{\Gamma} T(x_1, \tau) dx \quad (2-3-11)$$

where  $\Gamma$  is a constant. For a plate  $\Gamma = 0$  ( $x_1 \equiv x$ ), for a cylinder  $\Gamma = 1$  ( $x_1 \equiv r$ ), and for a sphere  $\Gamma = 2$  ( $x_1 \equiv r$ ).

## 2-4. METHODS OF SOLVING HEAT-CONDUCTION PROBLEMS

Heat transfer by conduction or mass transfer by diffusion is described by parabolic equations of the type

$$c\rho \frac{\partial T}{\partial \tau} = \nabla(\lambda \nabla T) + I_q$$

for a steady state  $\frac{\partial T}{\partial \tau} = 0$ .

An analytical theory for solving these equations containing variable parameters that depend upon temperature, position and time has still not been developed. The available solutions are dedicated only to some particular problems. In the present section the methods of solving heat conduction problems will be considered when thermal properties are constant or depend on time and space.

*(a) Method of Separation of Variables  
for One-Dimensional  
Heat-Conduction Problems (the Fourier Method)*

This method, applicable only for bodies of finite dimensions, resides in the fact that the solution is found as a set of particular solutions satisfying the homogeneous boundary conditions. Let us now consider the equation

$$c(\tau) \rho(\tau) \frac{\partial T}{\partial \tau} = \nabla[\lambda(x_i) \nabla T] \quad (2-4-1)$$

at homogeneous boundary conditions. The particular solution will be represented by a product of functions, one of them depends on time alone, the other — on the coordinates, i.e.

$$T = A\theta(\tau) v(x_i) \quad (2-4-2)$$

Substitution of (2-4-2) into (2-4-1) yields

$$c(\tau) \rho(\tau) \theta'(\tau) = -\mu^2 \theta(\tau) \quad (2-4-3)$$

$$\nabla[\lambda(x_i) \nabla v(x_i)] = -\mu^2 v(x_i) \quad (2-4-4)$$

The solution of (2-4-3) is readily found as

$$\theta(\tau) = \exp \left[ -\mu^2 \int_0^\tau c^{-1}(\tau_1) \rho^{-1}(\tau_1) d\tau_1 \right] \quad (2-4-5)$$

The general solution of (2-4-4) is not available (solutions are available only for some particular cases).

At constant thermal properties ( $c = \text{const}$ ;  $\rho = \text{const}$ ;  $\lambda = \text{const}$ ) equation (2-4-4) becomes the Helmholtz equation. Equation (2-4-4) has non-trivial solutions for definite values  $\mu = \mu_n$  called eigenvalues; its solution is expressed as:

$$T = \sum_{n=1}^{\infty} A_n v_n(x_i) e^{-\mu_n^2 \tau} \quad (2-4-6)$$

where  $v_n(x_i) = v(\mu_n, x_i)$  are the eigenfunctions; the constants  $A_n$  are found from the initial condition: at  $\tau = 0$   $T = f(x_i)$ , i.e.

$$A_n = \frac{1}{N} \int_{(R)} v_n(x_i) f(x_i) dx_i$$

where index  $R$  refers to a finite range;  $N$  is the norm of the characteristic function  $v_n(x_i)$  equal to

$$N = \int_{(R)} v_n^2(x_i) dx_i \quad (2-4-7)$$

Integration is carried out with respect to all the characteristic coordinates.



Thus the solution will look like

$$T(x_i, \tau) = \sum_{n=1}^{\infty} \frac{1}{N} v_n(x_i) e^{-\alpha v_n^2 \tau} \int_{(R)} v_n(x_i) f(x_i) dx_i \quad (2-4-8)$$

It has been shown that there are 11 orthogonal systems\* in which the Helmholtz equation is divided into ordinary differential equations (Table 2-1).

Table 2-1

**Orthogonal Coordinates which Allow Separation of Variables in the Helmholtz Equation**

System of Coordinates	Functions appearing in the solutions of the Helmholtz Equation
Rectangular	Exponential, trigonometric, hyperbolic
Circular cylindrical	Exponential, Bessel, trigonometric
Elliptical cylindrical	Matier, trigonometric
Parabolic cylindrical	Weber, trigonometric
Spherical	Legendre, power, trigonometric
Prolate spheroid	Legendre, trigonometric
Oblate spheroid	Legendre, trigonometric
Parabolic	Bessel, trigonometric
Conical	Lame, power
Ellipsoidal	Lame
Paraboloidal	Ber

The summation in (2-4-8) is made over the whole discontinuous spectrum of eigenvalues  $v_n$ . For three-dimensional problems in a finite summation range, the equation (2-4-8) is a triple sum.

Sometimes eigenfunctions may be obtained such that the norms are equal to unity. Then, having determined the normalized eigenfunctions in the form

$$K(v_n, x_i) = \frac{1}{\sqrt{N}} v_n(x_i) \quad (2-4-7')$$

solution (2-4-8) is written as

$$T(x_i, \tau) = \sum_{n=1}^{\infty} K(v_n, x_i) e^{-\alpha v_n^2 \tau} \int_{(R)} K(v_n, x_i) f(x_i) dx_i \quad (2-4-9)$$

The details concerning the Fourier method may be found in [2-1], [2-2], [2-3], [2-18], [2-19].

\* Orthogonality of the functions  $v_n$  is determined from the properties of eigenfunctions in the Sturm Liouville problem (see further).

The disadvantages of the separation of variables method are as follows: (1) it is inapplicable to semi-infinite and infinite bodies\*; (2) it cannot be directly applied to inhomogeneous boundary conditions which must be first reduced to homogeneous boundary conditions (this is not always easily achieved); (3) considerable difficulties associated with the solution of boundary-value problems with the boundary conditions of the fourth kind.

*(b) The Source Function (Green Functions) Method*

The source function (Green function) method enables the solution of boundary-value problems to be found at inhomogeneous boundary conditions both for finite and infinite bodies. To do this the Green function is determined as a transfer potential at point  $P(x, y, z)$  at time  $\tau$ , defined by the action of instantaneous point source of unit power placed at point  $P_1(x_1, y_1, z_1)$  when  $\tau = \tau_1$ . Besides, the initial transfer potential at the initial time and surface conditions are homogeneous. Let us solve the equation

$$L[T] \equiv c(\tau) \rho(\tau) \frac{\partial T}{\partial \tau} - \nabla^2 [\lambda(x, y, z) \nabla^2 T] = W(x, y, z, \tau) \quad (2-4-10)$$

for inhomogeneous boundary and initial conditions. In this case, the Green function is found from the solution of equation

$$L[G] = \delta(x - x_1, y - y_1, z - z_1, \tau - \tau_1) \quad (2-4-11)$$

where  $\delta$  is the delta-function (for homogeneous boundary conditions).

With the Green function obtained, the explicit solution of equation (2-4-10) for inhomogeneous boundary conditions may be found. For example, for boundary conditions of the first kind the solution of equation (2-4-10) with conditions (2-2-1) through (2-2-3) takes the form [2-4], [2-26].

$$\begin{aligned} T = & \int_{(V)} G(P, P_1, \tau - \tau_1)|_{\tau_1=0} f(P_1) dV_1 + a \int_0^{\tau} d\tau_1 \int_{(S)} \varphi \frac{\partial G}{\partial n} dS_1 \\ & + a \int_0^{\tau} d\tau_1 \int_{(V)} \frac{W}{c\rho} G(P, P_1, \tau - \tau_1) dV_1 \end{aligned} \quad (2-4-12)$$

Similar formulas are available for boundary conditions (2-2-4) and (2-2-6) and also for mixed boundary conditions.

Several methods of plotting the Green functions are available for equations of the type (2-4-10). Let us consider the most simple and convenient method, from our view-point, in which the property of  $\delta$ -functions is used to plot Green's function [2-5], [2-6], [2-21], [2-22].

\* Except the case when the eigenfunction spectrum within infinite or semi-infinite intervals is discontinuous.

1. If  $f(x, y, z, \tau)$  is a continuous function in the range  $D$  ( $c_1 < x < b_1$ ,  $c_2 < y < b_2$ ,  $c_3 < z < b_3$ ,  $0 < \tau < \infty$ ), then

$$\int_0^\infty d\tau_1 \int_{c_1}^{b_1} \int_{c_2}^{b_2} \int_{c_3}^{b_3} f(x_1, y_1, z_1, \tau_1) \delta(x - x_1, y - y_1, z - z_1, \tau - \tau_1) \times dx_1 dy_1 dz_1 = f \quad (2-4-13)$$

Particularly

$$\int_0^\infty d\tau_1 \int_{c_1}^{b_1} \int_{c_2}^{b_2} \int_{c_3}^{b_3} \delta(x - x_1, y - y_1, z - z_1, \tau - \tau_1) dx_1 dy_1 dz_1 = 1 \quad (2-4-14)$$

2. Let us use the splitting property

$$\begin{aligned} & \delta(x - x_1, y - y_1, z - z_1, \tau - \tau_1) \\ &= \delta(x - x_1) \delta(y - y_1) \delta(z - z_1) \delta(\tau - \tau_1) \end{aligned} \quad (2-4-15)$$

3. Let us use series expansion of the  $\delta$ -function and represent it as integrals.

If  $\{v_k(x)\}$  is a closed orthonormal set in the finite range  $(c_1, b_1)$ , and  $\{\bar{v}_k(x)\}$  is a complex set associated with it, then

$$\delta(x - x_1) = \sum_k v_k(x) \bar{v}_k(x_1); \quad c_1 < x; \quad x_1 < b_1 \quad (2-4-16)$$

In case of infinite and semi-infinite ranges, formula (2-4-16) is generalized as

$$\delta(x - x_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i\alpha(x - x_1)] d\alpha; \quad -\infty < x; \quad x_1 < \infty \quad (2-4-17)$$

$$\delta(x - x_1) = \frac{1}{2\pi} \int_0^{\infty} \{\exp[i\alpha(x - x_1)] - \exp[i\alpha(x + x_1)]\} d\alpha;$$

$$0 \leq x; \quad x_1 < \infty \quad (2-4-18)$$

Now we shall discuss the method of plotting Green's function on a particular example. Let us consider the boundary-value problem

$$L[T] = \frac{\partial T}{\partial \tau} - a \frac{\partial^2 T}{\partial x^2} = W(x, \tau) \quad (2-4-19)$$

$$T(c, \tau) = \varphi_1(\tau); \quad T(b, \tau) = \varphi_2(\tau) \quad (2-4-20)$$

$$T(x, 0) = f(x) \quad (2-4-21)$$

If Green's function is known, then the solution of the boundary-value problem (2-4-19) through (2-4-21) may be written as

$$T(x, \tau) = \int_c^b G(x, x_1, \tau - \tau_1) \Big|_{\tau_1=0} f(x_1) dx_1 + a \int_0^\tau \left[ \varphi_1(\tau_1) \frac{\partial G}{\partial x_1} \Big|_{x_1=0} - \varphi_2(\tau_1) \frac{\partial G}{\partial x_1} \Big|_{x_1=0} \right] d\tau_1 + a \int_0^\tau d\tau_1 \int_0^b \frac{W}{c\rho} G(x, x_1, \tau - \tau_1) dx_1 \quad (2-4-22)$$

Green's function is found from the equation

$$L[G] = \delta(x - x_1, \tau - \tau_1) \quad (2-4-23)$$

hence, with consideration for (2-4-15) we get

$$L[G] = \delta(x - x_1) \delta(\tau - \tau_1) \quad (2-4-24)$$

Taking into account (2-4-16) and (2-4-17), we obtain

$$\delta(x - x_1) = \sum_k \psi_k(x) \bar{\psi}_k(x_1) \quad (2-4-25)$$

$$\delta(\tau - \tau_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i\alpha(\tau - \tau_1)] d\alpha \quad (2-4-26)$$

where  $\psi_k(x)$  is the solution of the corresponding Sturm-Liouville problem (eigenfunctions).

Substitution of (2-4-25) and (2-4-26) into (2-4-24) yields

$$\begin{aligned} G(x, x_1, \tau, \tau_1) &= L^{-1}[\delta(x - x_1) \delta(\tau - \tau_1)] \\ &= -\frac{1}{2\pi} \sum_k \psi_k(x_1) \int_{-\infty}^{\infty} \exp(-i\alpha\tau_1) L^{-1}[\psi_k(x) \exp(i\alpha\tau)] d\alpha \end{aligned} \quad (2-4-27)$$

Now it may be easily demonstrated that

$$L^{-1}[\psi_k(x) \exp(i\alpha\tau)] = \frac{\psi_k(x) \exp[i\alpha\tau]}{i\alpha - av_k^2} \quad (2-4-28)$$

Substitution of (2-4-28) into (2-4-27) and integration of the latter results in the final expression for Green's function

$$G(x, x_1, \tau, \tau_1) = \sum_k \exp[-av_k^2(\tau - \tau_1)] \psi_k(x) \psi_k(x_1) \quad (2-4-29)$$

The Green-function method is very useful as it is universal and may be applied to the solution of problems in the general form in finite and

infinite ranges, with inhomogeneous boundary and initial conditions and for inhomogeneous equations. The disadvantage of the method is as follows. The plotting of Green's function calls for certain ingenuity and sometimes presents difficulties.

*(c) Heat Potential Method*

The following integrals

$$U_1(P, \tau) = \int_0^\tau d\tau_1 \int_S \dots \int \chi_1(P_1, \tau_1) \Phi_1(P, P_1, \tau - \tau_1) dS_1 \quad (2-4-30)$$

$$U_2(P, \tau) = \int_0^\tau d\tau_1 \int_S \dots \int \chi_2(P_1, \tau_1) \frac{\partial}{\partial n} \Phi_1(P, P_1, \tau - \tau_1) dS_1 \quad (2-4-31)$$

are called heat potentials of single and double layer, respectively [2-4], [2-7], [2-8], [2-20].

In one-dimensional and two-dimensional cases for Cartesian coordinates, we get, respectively

$$U_1(x, \tau) = \frac{\sqrt{a}}{2\sqrt{\pi}} \int_0^\tau \frac{\chi_1(\tau_1)}{\sqrt{\tau - \tau_1}} \exp\left[-\frac{(\xi - x)^2}{4a(\tau - \tau_1)}\right] d\tau_1 \quad (2-4-32)$$

$$U_2(x, \tau) = \frac{1}{4\sqrt{\pi a}} \int_0^\tau \frac{\chi_2(\tau_1)}{\sqrt{(\tau - \tau_1)^3}} (x - \xi) \exp\left[-\frac{(\xi - x)^2}{4a(\tau - \tau_1)}\right] d\tau_1 \quad (2-4-33)$$

$$U_1(x, y, \tau) = \frac{a}{4\pi} \int_0^\tau d\tau_1 \int_S \frac{\chi_1(P_1, \tau_1)}{\tau - \tau_1} \exp\left[-\frac{r^2}{4a(\tau - \tau_1)}\right] dS_1 \quad (2-4-34)$$

$$\begin{aligned} U_2(x, y, \tau) &= \frac{1}{2\pi} \int_0^\tau d\tau_1 \int_S \frac{\chi_2(P_1, \tau_1)}{\tau - \tau_1} \frac{\partial}{\partial n} \exp\left[-\frac{r^2}{4a(\tau - \tau_1)}\right] dS_1 \\ &= \frac{1}{4a\pi} \int_0^\tau d\tau_1 \int_S \frac{\chi_2(P_1, \tau_1)}{(\tau - \tau_1)^2} \exp\left[-\frac{r^2}{4a(\tau - \tau_1)}\right] r \cos(\bar{r}, \bar{n}) dS_1 \end{aligned} \quad (2-4-35)$$

Heat potentials outside the points of surface  $S$ , over which integration is done, are the solutions of a homogeneous heat-conduction equation and satisfy homogeneous initial conditions; it follows from the fact that

they are constructed using fundamental solutions of the heat-conduction equation (source and dipole). When approaching the points of surface  $S$ , the heat potential of double layer and the normal derivative of heat potential of the layer of charge are discontinuous, which may be written analytically as

$$U_2^+(P^*, \tau) = 0.5 \kappa_2(P^*, \tau) + U_2(P^*, \tau) \quad (2-4-36)$$

$$U_2^-(P^*, \tau) = -0.5 \kappa_2(P^*, \tau) + U_2(P^*, \tau) \quad (2-4-37)$$

$$\frac{\partial U_1^+(P^*, \tau)}{\partial n} = -0.5 \kappa_1(P^*, \tau) + \frac{\partial U_1(P^*, \tau)}{\partial n} \quad (2-4-38)$$

$$\frac{\partial U_1^-(P^*, \tau)}{\partial n} = 0.5 \kappa_1(P^*, \tau) + \frac{\partial U_1(P^*, \tau)}{\partial n} \quad (2-4-39)$$

where  $U_i^+$ ,  $U_i^-$  ( $i = 1, 2$ ) are the limiting values of heat potentials when approached along the normal from the inside and outside, respectively, to point  $P^*$  of lateral surface  $S$ .

Let us consider homogeneous equation (2-4-6) for a zero initial condition. Its solution for boundary conditions of the first kind (2-2-3) is sought in the form of a double-layer potential

$$T(x, y, z, \tau) = \frac{1}{16 \sqrt{a^3 \pi^3}} \int_0^\tau d\tau_1 \int_S \frac{\chi_2(P_1, \tau_1)}{\sqrt{(\tau - \tau_1)^3}} \times \exp \left[ -\frac{r^2}{4a(\tau - \tau_1)} \right] r \cos(\bar{r}, \bar{n}) dS_1 \quad (2-4-40)$$

for boundary conditions of the second and third kind in the form of a single-layer potential

$$T(x, y, z, \tau) = \frac{1}{8 \sqrt{a \pi^3}} \int_0^\tau d\tau_1 \int_S \frac{\chi_1(P_1, \tau_1)}{\sqrt{(\tau - \tau_1)^3}} \exp \left[ -\frac{r^2}{4a(\tau - \tau_1)} \right] dS_1 \quad (2-4-41)$$

where

$$r^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2.$$

It should be mentioned that the densities  $\chi_1, \chi_2$  of heat potentials are unknown functions, which are determined from integral equations obtained by substituting equations (2-4-40) and (2-4-41), respectively, into the boundary conditions (2-2-3) and (2-2-4) with due regard for (2-4-36) through (2-4-39). It should also be noted that there are a number of problems which can be solved only by the heat potential method. These are: problems with moving boundaries, variable transfer coefficients, etc.

One of the advantages of the method is the possibility of reducing the solution of a parabolic differential equation to an integral equation which is more convenient for numerical calculations.

The disadvantage of the method is that it is a complicated and time consuming method, besides it cannot be applied directly to problems with inhomogeneous initial conditions (which must be first transformed to homogeneous ones); however, this is easy to overcome by using the Poisson integral.

#### (d) Integral Transformations

The earlier described classical methods, used for solving boundary-value problems, have a number of disadvantages. They call for certain ingenuity; they give solutions which are of little use for numerical calculations, etc. The integral transformation methods have a number of advantages over the classical methods. They are universal, provide solutions in forms convenient for calculations (for example, for large and small independent variables); the use of function tables speeds up and simplifies the process of finding solutions, etc. Besides the evident advantages, integral transformations have a common significant disadvantage: they are applicable only to linear equations.

The expression

$$f_1(P, x_2, \dots, x_n) = \int_a^b K(P, x_1) f(x_1, x_2, \dots, x_n) dx_1 \quad (2-4-42)$$

is known as an integral transform of the function  $f(x_1, x_2, \dots, x_n)$  with respect to variable  $x_1$ . Here,  $f_1(P, x_2, \dots, x_n)$  is the transform of the original function  $f(x_1, x_2, \dots, x_n)$  and  $K(P, x_1)$  is the kernel of the integral transform. It may be mentioned that together with transforms of the type (2-4-42), we shall consider the so-called multiple integral transforms, i.e. transforms with respect to several variables. Both finite and infinite ranges of integral transforms will be considered. Use of integral transform (2-4-42) reduces the number of independent variables by one and the resulting equations are generally more simple than the initial ones. The inversion is carried out by the inversion formula

$$f(x_1, x_2, \dots, x_n) = \int_c^d R(P, x_1) f_1(P, x_2, \dots, x_n) dP \quad (2-4-43)$$

#### (e) Integral Transforms in Infinite Ranges

Let us consider some of the most widely used integral transforms. The Laplace integral transform is given by the relation

$$f_1(P) = \int_0^{\infty} f(\tau) \exp[-P\tau] d\tau = \int_0^{\infty} f(\tau) \exp[-\sigma\tau] \exp[-i\alpha\tau] d\tau$$

$$(P = \sigma \pm i\alpha) \quad (2-4-44)$$

The original  $f(\tau)$ , which is a function of real variable  $\tau$ , meets the following conditions:

1.  $f(\tau)$  is a piecewise continuous function with its derivatives of a rather high order;
2.  $f(\tau)$  grows slower than the exponential function. This means that such constants  $M > 0$  and  $\sigma_0 \geq 0$  exist that

$$|f(\tau)| < M \exp(\sigma_0 \tau) \quad (2-4-45)$$

$$f(\tau) = 0 \text{ at } \tau < 0 \quad (2-4-46)$$

After the transformed solution has been obtained, inversion is carried out by the formula

$$f(\tau) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} f_1(P) \exp [P\tau] dP \quad (2-4-47)$$

The integration in (2-4-47) is carried out in a complex plane along the straight line  $\operatorname{Re} P = \sigma > \sigma_1 > \sigma_0$ .

Inversion of the transform may be very simple if the transform coincides with one of those listed in the table of transforms. Then, to find  $f(\tau)$ , formula (2-4-47) may be substituted by the following inversion formula

$$f(\tau) = \lim_{n \rightarrow \infty} \left\{ \frac{(-1)^n}{n!} \left( \frac{n}{\tau} \right)^{n+1} f_I^{(n)} \left( \frac{n}{\tau} \right) \right\}$$

This formula enables one to obtain the original function only by differentiation operations and limit transition. Alongside with transform (2-4-43), in some cases the so-called Laplace bilateral transform is discussed [2-9], [2-23], [2-24]

$$f(P) = \int_{-\infty}^{\infty} f(\tau) \exp [-P\tau] d\tau \quad (2-4-44')$$

Assuming  $P = \sigma + i\alpha$  and denoting

$$f(\tau) \exp [-\sigma\tau] = F(\tau); \quad \frac{1}{\sqrt{2\pi}} f_I(\sigma + i\alpha) = F_I(\alpha) \quad (2-4-48)$$

we get

$$F(\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) \exp [i\alpha\tau] d\alpha \quad (2-4-49)$$

$$F_I(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\tau) \exp [-i\alpha\tau] d\tau \quad (2-4-50)$$



Expressions (2-4-49) and (2-4-50) are the Fourier transform and Fourier inverted transform, respectively. It may be mentioned that the integral in (2-4-49) exists if the function  $F(\alpha)$  is absolutely integrable on the real axis

$$\int_{-\infty}^{\infty} |F(\alpha)| d\alpha < \infty \quad (2-4-51)$$

Condition (2-4-51) significantly confines the validity range of the Fourier integral transform (2-4-50). It is inapplicable even for  $F(\tau) = \text{const}$ ,  $\sin \tau$ ,  $\cos \tau$ , etc., that is for the functions most common in heat engineering calculations. The Laplace transform is free of this disadvantage, since the availability of an additional factor  $\exp[-\sigma\tau]$  in (2-4-44) extends the set of original functions to those which satisfy condition (2-4-45), i.e. for a majority of important functions met with in practice. If the function  $F(\tau)$  is given in the semi-infinite range  $(0, \infty)$  or  $(-\infty, 0)$ , it may be extended in the even or odd manner, respectively, to the range  $(-\infty, 0)$  or  $(0, \infty)$ . In the former case (2-4-50) and (2-4-49) are written as

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F(\tau) \cos \alpha\tau d\tau \quad (2-4-52)$$

$$F(\tau) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(\alpha) \cos \alpha\tau d\alpha \quad (2-4-53)$$

In the latter case we obtain

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F(\tau) \sin \alpha\tau d\tau \quad (2-4-54)$$

$$F(\tau) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(\alpha) \sin \alpha\tau d\alpha \quad (2-4-55)$$

Expressions (2-4-52) and (2-4-54) are the Fourier cosine and sine transforms, respectively, [2-10]–[2-12]. It may be mentioned, in particular, that for these expressions the limiting relation of the type (2-4-51) should be fulfilled, which significantly narrows the range of their application

$$\int_0^{\infty} |F(\tau)| d\tau < \infty \quad (2-4-56)$$

To widen the range of application of these transformations it is convenient to use instead of (2-4-52) through (2-4-55) the so-called Fourier generalized cosine and sine transforms.

$$F_C(\alpha) = \sqrt{\frac{2}{\pi}} \lim_{\sigma \rightarrow 0} \int_0^{\infty} \exp[-\sigma\tau] F(\tau) \cos \alpha\tau d\tau \quad (2-4-57)$$

$$F_S(\alpha) = \sqrt{\frac{2}{\pi}} \lim_{\sigma \rightarrow 0} \int_0^{\infty} \exp[-\sigma\tau] F(\tau) \sin \alpha\tau d\tau \quad (2-4-58)$$

$$F(\tau) = \sqrt{\frac{2}{\pi}} \lim_{\sigma \rightarrow 0} \int_0^{\infty} \exp[\sigma\tau] F_C(\alpha) \cos \alpha\tau d\alpha \quad (2-4-59)$$

$$F(\tau) = \sqrt{\frac{2}{\pi}} \lim_{\sigma \rightarrow 0} \int_0^{\infty} \exp[\sigma\tau] F_S(\alpha) \sin \alpha\tau d\alpha \quad (2-4-60)$$

The use of formulas (2-4-57) and (2-4-58) enables the inversion to be found for a wide range of functions, such as  $F(\tau) = \text{const}$ ,  $\cos \tau$ ,  $\sin \tau$ ,  $\tau$ , etc., i.e. for those cases when ordinary transforms are unsuitable.

Substitution of variables  $P$  by  $-P$  and  $\tau$  by  $t = e^\tau$  and functions  $f(\log_e t) = g(t)$ ,  $f_I(-P) = \bar{g}_I(P)$  results in

$$g_I(P) = \int_0^{\infty} g(t) t^{P-1} dt \quad (2-4-61)$$

$$g(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{g_I(P)}{t^P} dP \quad (2-4-62)$$

Expressions (2-4-61) and (2-4-62) give the Melline integral transform which is a modification of the Laplace integral transform. The properties of the Melline transform may be deduced from the corresponding properties of the Laplace transform [2-9], [2-13], [2-14].

The expression

$$f_I(\alpha) = \int_0^{\infty} r f(r) I_\nu(r\alpha) dr; \quad \nu > -0.5 \quad (2-4-63)$$

is known as the Hankel integral transform.

The inversion formula in this case takes the form

$$f(r) = \int_0^{\infty} f_I(\alpha) I_\nu(r\alpha) \alpha d\alpha \quad (2-4-64)$$

Formulas (2-4-63) and (2-4-64) may be obtained by the Fourier multiple (double) integral transform by changing over to polar coordinates.

There are a number of integral transforms which together with Hankel transforms are grouped with the so-called Bessel transforms, in the kernel of which Bessel functions appear. Among the latter are the Kantorovich-Lebedev transforms, Mayer transforms, etc. Unlike the earlier considered integral transforms, the range of validity of the latter ones for solving heat-conduction problems is much more narrow and they are of theoretical rather than practical value, because the problems for which they may be used are more easily solved by other methods.

### (f) Integral Transforms in Finite Ranges

The method of finite integral transformations is, in our opinion, the most useful one for solving inhomogeneous parabolic equations with inhomogeneous boundary conditions. The idea of this method was originally suggested by N. S. Koshlyakov [2-15]. But the theory of such integral transforms was developed in detail by G. A. Grinberg [2-16], who extended the method to step variations of the properties of material along the transformation coordinate.

A general solution of the parabolic heat-conduction equation was obtained by M. D. Mikhailov [2-17].

We shall consider unsteady-state heat conduction in a finite range ( $V_R$ ) with variable heat source  $Q(\vec{r}, \tau)$  and with heat radiation to the surrounding medium with zero temperature according to Newton's law.

In the range  $V_R$  the differential equation may be written as

$$C(\tau)\Gamma(\vec{r}) \frac{\partial T(\vec{r}, \tau)}{\partial \tau} = \text{div} [\lambda(\vec{r}) \text{grad } T(\vec{r}, \tau)] \quad (2-4-65)$$

$$+ [\varepsilon(\tau) \Gamma(\vec{r}) - \sigma(\vec{r})] T(\vec{r}, \tau) + Q(\vec{r}, \tau)$$

where  $C(\tau) = c\rho$  is the heat capacity per unit volume, which depends on time.

The initial condition takes the form

$$\text{at } \tau = 0 \quad T(\vec{r}, 0) = f_0(\vec{r}) \quad (2-4-66)$$

The boundary condition for surface  $S$  is written in a general form as

$$A(\vec{r}) \frac{\partial T(\vec{r}, \tau)}{\partial n} + B(\vec{r}) T(\vec{r}, \tau) = f(\vec{r}, \tau) \quad (2-4-67)$$

Here  $A(\vec{r})$ ,  $B(\vec{r})$  and  $f(\vec{r}, \tau)$  are the given piecewise regular functions. From condition (2-4-67) three kinds of boundary conditions are obtained. The first-kind boundary conditions:  $A(\vec{r}) = 0$ ,  $B(\vec{r}) = 1$ ; the second-kind boundary conditions:  $A(\vec{r}) = \lambda(\vec{r})$ ,  $B(\vec{r}) = 0$  and  $f(\vec{r}, \tau) = q(\vec{r}, \tau)$  and the third-kind boundary conditions:  $A(\vec{r}) = \lambda(\vec{r})/\alpha(\vec{r})$ ,  $B(\vec{r}) = 1$ ,  $f(\vec{r}, \tau) = T_f(\tau)$ , where  $T_f(\tau)$  is the temperature of the fluid that surrounds the solid body.

Eigenfunctions  $\psi(\vec{r})$  are found from the system of the Sturm-Liouville equations

in the range  $V_R$

$$\operatorname{div} [\lambda(\vec{r}) \operatorname{grad} \psi(\vec{r})] + [v^2 \Gamma(\vec{r}) - \sigma(\vec{r})] \psi(\vec{r}) = 0 \quad (2-4-68)$$

at the surface  $S$

$$\lambda(\vec{r}_s) \frac{\partial \psi(\vec{r}_s)}{\partial n} = \alpha(\vec{r}_s) \psi(\vec{r}_s) = 0 \quad (2-4-69)$$

where  $v_i$  are eigenvalues.

Eigenvalues  $v_i$  increase infinitely with the increase of the number  $i$ . As  $\sigma(\vec{r}) > 0$ , then  $v_i > 0$ . All eigenfunctions corresponding to one and the same eigennumber differ only by a constant factor. Let  $\psi_i(\vec{r})$  and  $\psi_j(\vec{r})$  be two eigenfunctions for different eigenvalues  $v_j$  and  $v_i$

$$v_i^2 \Gamma(\vec{r}) \psi_i(\vec{r}) = \sigma(\vec{r}) \psi_i(\vec{r}) - \operatorname{div} [\lambda(\vec{r}) \operatorname{grad} \psi_i(\vec{r})] \quad (2-4-70)$$

$$v_j^2 \Gamma(\vec{r}) \psi_j(\vec{r}) = \sigma(\vec{r}) \psi_j(\vec{r}) - \operatorname{div} [\lambda(\vec{r}) \operatorname{grad} \psi_j(\vec{r})] \quad (2-4-71)$$

provided that

$$A(\vec{r}_s) + \frac{\partial \psi_i(\vec{r}_s)}{\partial n} + B(\vec{r}_s) \psi_i(\vec{r}_s) = 0 \quad (2-4-72)$$

$$A(\vec{r}_s) + \frac{\partial \psi_j(\vec{r}_s)}{\partial n} + B(\vec{r}_s) \psi_j(\vec{r}_s) = 0 \quad (2-4-73)$$

Let us multiply equation (2-4-70) by  $\psi_j(\vec{r})$ . By subtracting from it equation (2-4-71), which was multiplied by  $\psi_i(\vec{r})$ , and integrating with regard to the Ostrogradsky-Gauss formula, we obtain

$$(\psi_i, \psi_j) = \int_V \Gamma(\vec{r}) \psi_i(\vec{r}) \psi_j(\vec{r}) dV = \frac{1}{v_i^2 - v_j^2} \int_S \lambda(\vec{r}) \begin{vmatrix} \psi_i(\vec{r}) & \psi_{i,n}(\vec{r}) \\ \psi_j(\vec{r}) & \psi_{j,n}(\vec{r}) \end{vmatrix} dS \quad (2-4-74)$$

where comma after the subscript indicates the derivative with respect to normal  $n$ , i.e.

$$\psi_{i,n} \equiv \partial \psi_i / \partial n, \quad \psi_{j,n} \equiv \partial \psi_j / \partial n$$

From boundary conditions (2-4-72) and (2-4-73) it follows that at  $i \neq j$ , the values  $(\psi_i, \psi_j) = 0$ , i.e. eigenfunctions corresponding to different eigenvalues, are mutually orthogonal with the weight  $\Gamma(\vec{r})$ .

If in relation (2-4-74), we assume that  $v_i = v_j$  then, we obtain an uncertainty. This uncertainty may be disclosed by the l'Hospital rule. Then the norm  $N \equiv [\psi_i, \psi_j]$  is

$$N = \int_V \Gamma(\vec{r}) \psi_i^2(\vec{r}) dV = \frac{1}{2v_i} \int_S \lambda(\vec{r}) \left| \begin{array}{cc} [\psi_{i,v}(\vec{r})]_{v=v_i} & [\psi_{i,n,v}(\vec{r})]_{v=v_i} \\ \psi_i(\vec{r}) & \psi_{i,n}(\vec{r}) \end{array} \right| dS \quad (2-4-75)$$

where

$$\psi_{i,n,v} \equiv \partial^2 \psi_i / \partial n \partial v; \quad \psi_{i,v} \equiv \partial \psi_i / \partial v.$$

If the eigenfunction is a kernel of the integral transform

$$\tilde{T}(v_i, \tau) = \int_V \psi_i(\vec{r}) T(\vec{r}, \tau) dV \quad (2-4-76)$$

then multiplication of (2-4-65) by  $\psi_i(\vec{r})$ , subtraction from it of (2-4-72), multiplied earlier by  $T(\vec{r}, \tau)$ , and subsequent integration over the volume yields

$$C(\tau) \frac{d\tilde{T}(v_i, \tau)}{d\tau} + [v_i^2 - \varepsilon(\tau)] \tilde{T}(v_i, \tau) = M(\tau) \quad (2-4-77)$$

where

$$\begin{aligned} M(\tau) = & \int_S \lambda(\vec{r}_s) \left| \begin{array}{cc} \psi_i(\vec{r}_s) & \psi_{i,n}(\vec{r}_s) \\ T(\vec{r}_s, \tau) & T_{,n}(\vec{r}_s, \tau) \end{array} \right| dS \\ & + \int_V \psi_i(\vec{r}) Q(\vec{r}, \tau) dV \end{aligned} \quad (2-4-78)$$

From (2-4-69) and (2-4-72) we find  $A(\vec{r}_s)$  and  $B(\vec{r}_s)$ . Summation of the results gives

$$\left| \begin{array}{cc} \psi_i(\vec{r}_s) & \psi_{i,n}(\vec{r}_s) \\ T(\vec{r}_s, \tau) & T_{,n}(\vec{r}_s, \tau) \end{array} \right| = f(\vec{r}_s, \tau) \frac{\psi_i(\vec{r}_s) - \psi_{i,n}(\vec{r}_s)}{A(\vec{r}_s) + B(\vec{r}_s)} \quad (2-4-79)$$

Substitution of (2-4-79) into (2-4-78) results in a linear equation of the first order which can be easily solved.

Then, using the inversion formula

$$T(\vec{r}, \tau) = \sum_{i=1}^{\infty} N^{-1} \psi_i(\vec{r}) \tilde{T}(v_i, \tau) \quad (2-4-80)$$

we get the solution

$$T(\vec{r}, \tau) = \sum_{i=1}^{\infty} N^{-1} \psi_i(\vec{r}) \exp \left( \int_0^{\tau} \frac{\varepsilon(r) - v_i^2}{C(\tau)} d\tau \right) \left\{ \int_V \Gamma(\vec{r}) \psi_i(\vec{r}) f_0(\vec{r}) dV \right. \\ \left. + \int_0^{\tau} \frac{1}{C(\tau)} \exp \left( \frac{v_i^2 - \varepsilon(\tau)}{C(\tau)} d\tau \right) M(\tau) d\tau \right\} \quad (2-4-81)$$

If  $\sigma(\vec{r}) = 0$  and  $B(\vec{r}_s) = 0$  (boundary conditions of the second kind),  $\mu_0 = 0$  and  $\psi_0 = \text{const}$  are also the eigenvalues and eigenfunction of the Sturm-Liouville problem. Then the solution takes the form

$$T^*(\vec{r}, \tau) = \left[ \int_V \Gamma(\vec{r}) dV \right]^{-1} \exp \left( \frac{\varepsilon(\tau)}{C(\tau)} d\tau \right) \left\{ \int_V \Gamma(\vec{r}) f_0(r) dV \right. \\ \left. + \int_0^{\tau} \exp \left( - \int_0^{\tau} \frac{\varepsilon(\tau)}{C(\tau)} d\tau \right) \frac{1}{C(\tau)} \left[ \int_S \lambda(\vec{r}_s) \frac{f(\vec{r}_s, \tau)}{A(\vec{r}_s)} dS \right. \right. \\ \left. \left. + \int_V Q(\vec{r}, \tau) dV \right] d\tau \right\} + \sum_{i=1}^{\infty} N^{-1} \psi_i(\vec{r}) \exp \left( \int_0^{\tau} \frac{\varepsilon(\tau) - v_i^2}{C(\tau)} d\tau \right) \left\{ \int_V \Gamma(\vec{r}) \psi_i(\vec{r}) \right. \\ \left. \times f_0(\vec{r}) dV + \int_0^{\tau} \exp \left( \int_0^{\tau} \frac{v_i^2 - \varepsilon(\tau)}{C(\tau)} d\tau \right) \frac{1}{C(\tau)} M(\tau) d\tau \right\} \quad (2-4-82)$$

To simplify the analysis, we take  $C(\tau) = 1$ ,  $\varepsilon(\tau) = 0$ . Then for better convergence of series (2-4-81) and (2-4-82), the time integrals involved are integrated by parts. This results in

$$T(\vec{r}, \tau) = T_0(\vec{r}, \tau) + \sum_{i=1}^{\infty} N^{-1} \psi_i(\vec{r}) e^{-v_i^2 \tau} \left\{ [\psi_i, f_0] \right. \\ \left. - \frac{1}{v_i^2} M(0) - \frac{1}{v_i^2} \int_0^{\tau} e^{v_i^2 \tau} \frac{\partial M(\tau)}{\partial \tau} d\tau \right\} \quad (2-4-83)$$

where

$$\left. \begin{aligned} T_0(\vec{r}, \tau) &= \sum_{i=1}^{\infty} N^{-1} \psi_i(\vec{r}) v_i^{-2} M(\tau) \\ [\psi_i, f_0] &= \int_V \Gamma(\vec{r}) \psi_i(\vec{r}) f_0(\vec{r}) dV \end{aligned} \right\} \quad (2-4-84)$$

Solution (2-4-83) was named by Ölger [2-24] the quasi-stationary solution of order zero.

A similar quasi-stationary solution of order zero for boundary conditions of the second kind is of the form

$$T^*(\vec{r}, \tau) = T_0^*(\vec{r}, \tau) + \left[ \int_V \Gamma(\vec{r}) dV \right]^{-1} \left\{ \int_V \Gamma(\vec{r}) f_0(\vec{r}) dV + \right. \\ \left. + \int_0^\tau \left[ \int_S \lambda(\vec{r}_s) \frac{f(\vec{r}_s, \tau)}{A(\vec{r}_s)} dS + \int_V \sigma(\vec{r}, \tau) dV \right] d\tau + \sum_{i=1}^\infty N^{-1} \psi_i(\vec{r}) e^{-v_i^2 \tau} \right. \\ \left. \times \left\{ [\psi_i, f_0] - \frac{M(0)}{v_i^2} - \frac{1}{v_i^2} \int_0^\tau e^{v_i^2 \tau} \frac{\partial M(\tau)}{\partial \tau} d\tau \right\} \right\} \quad (2-4-85)$$

where

$$T_0^*(\vec{r}, \tau) = \sum_{i=1}^\infty \frac{1}{v_i^2 N} \psi_i(\vec{r}) M(\tau) \quad (2-4-86)$$

Even better convergence of the series, appearing in the solution, is achieved by  $m$ -fold integration by parts of the time integrals. Thus

$$T(\vec{r}, \tau) = \sum_{n=0}^m T_n(\vec{r}, \tau) + \sum_{i=1}^\infty N^{-1} \psi_i(\vec{r}) e^{-v_i^2 \tau} \left\{ [\psi_i, f_0] - \sum_{n=0}^m (-1)^n \frac{1}{v_i^{2(n+1)}} \frac{\partial^n M(0)}{\partial \tau^n} + \frac{(-1)^{m+1}}{v_i^{2(m+1)}} \int_0^\tau e^{v_i^2 \tau} \frac{\partial^{m+1} M(\tau)}{\partial \tau^{m+1}} d\tau \right\} \quad (2-4-87)$$

where

$$T_n(\vec{r}, \tau) = \sum_{i=1}^\infty N^{-1} \psi_i(\vec{r}) \frac{(-1)^n}{v_i^{2(n+1)}} \frac{\partial^n M(\tau)}{\partial \tau^n} \quad (2-4-88)$$

Equation (2-4-87) is called by Ölger the quasi-stationary solution of the  $n$ -th order [2-24].

For convenience it is expeditious to introduce normalized eigenfunctions in transforms and inversion formulas according to the relation

$$K(\beta_n, \vec{r}) = \frac{1}{\sqrt{N}} \psi_n(\vec{r}) \quad (2-4-89)$$

Then integral transforms and inversion formulas become

$$T_1(\beta_n, \tau) = \int_V K(\beta_n, \vec{r}) T(\vec{r}, \tau) dV \quad (2-4-90)$$

$$T(\vec{r}, \tau) = \sum_{n=1}^\infty K(\beta_n, \vec{r}) T_1(\beta_n, \tau) \quad (2-4-91)$$

In this case the solution of the heat-conduction equation

$$\frac{\partial T(\vec{r}, \tau)}{\partial \tau} = a \nabla^2 T(\vec{r}, \tau) + \frac{Q(\vec{r}, \tau)}{c\rho} \quad (2-4-92)$$

at the initial and boundary conditions

$$\left. \begin{aligned} \lambda_{s_i} \frac{\partial T(\vec{r}_{s_i}, \tau)}{\partial n_i} + \alpha_i T(\vec{r}_{s_i}, \tau) &= f_i(\vec{r}_{s_i}, \tau) \\ T(\vec{r}, 0) &= f_0(\vec{r}) \end{aligned} \right\} \quad (2-4-93)$$

takes the form

$$\begin{aligned} T(\vec{r}, \tau) &= \sum_{n=1}^{\infty} e^{-a\beta_n^2 \tau} K(\beta_n, \vec{r}) \left\{ f_{01}(\beta_n) \right. \\ &\quad \left. + \int_0^{\tau} e^{a\beta_n^2 \tau} C(\beta_n, \vec{r}) d\tau \right\} \end{aligned} \quad (2-4-94)$$

where

$$C(\beta_n, \tau) = \frac{a}{\lambda} Q_1(\beta_n, \tau) + a \sum_{i=1}^k \int_{s_i} \frac{1}{\lambda_{s_i}} K(\beta_n, \vec{r}_{s_i}) f_i(\vec{r}_{s_i}) dS_i \quad (2-4-95)$$

$$f_{01}(\beta_n) = \int_V K(\beta_n, \vec{r}) f_0(\vec{r}) dV \quad (2-4-96)$$

When changing over to boundary conditions of the first kind, the quantity  $\frac{1}{\lambda_s} K(\beta_n, \vec{r})$  is substituted by  $\frac{1}{\alpha_s} \frac{\partial K(\beta_n, \vec{r})}{\partial n_s}$ . For better convergence the quasi-stationary solution of order zero is used

$$\begin{aligned} T(\vec{r}, \tau) &= \sum_{j=0}^k T_{0j}(\vec{r}, \tau) + \sum_{n=1}^{\infty} e^{-a\beta_n^2 \tau} K(\beta_n, \vec{r}) \left\{ f_{01}(\beta_n) \right. \\ &\quad \left. - \sum_{j=0}^k [T_{0j}(\beta_n, 0)]_I \right\} - \sum_{j=0}^k \left\{ \sum_{n=1}^{\infty} e^{-a\beta_n^2 \tau} K(\beta_n, \vec{r}) \int_0^{\tau} e^{a\beta_n^2 \tau} \{[\dot{T}_{0j}(\beta_n, \tau)]_I\} d\tau \right\} \end{aligned} \quad (2-4-97)$$

where

$$f_{01}(\beta_n) = \int_V K(\beta_n, \vec{r}) f_0(\vec{r}) dV$$

$$[T_{0j}(\beta_n, 0)]_I = \int_V K(\beta_n, \vec{r}) T_{0j}(\vec{r}, 0) dV$$



The dot over the quantity indicates the derivative with respect to time. When surface integration is easier to be done than volume integration, solution (2-4-97) may be written as

$$\begin{aligned}
 T(\vec{r}, \tau) = & \sum_{j=0}^k T_{0j}(\vec{r}, \tau) + \sum_{n=1}^{\infty} e^{-a\beta_n^2 \tau} K(\beta_n, \vec{r}) \\
 & \times \left\{ f_{01}(\beta_n) - \frac{1}{\beta_n^2} \left[ \frac{Q_1(\beta_n, 0)}{\lambda} + \sum_{i=1}^k \int_{S_i} K(\beta_n, r_{si}) \right. \right. \\
 & \times f_i(r_{si}, \tau) dS_i \left. \right] - \frac{1}{\beta_n^2} \int_0^{\tau} e^{a\beta_n^2 \tau} \left[ \frac{\dot{Q}(\beta_n, \tau)}{\lambda} \right. \\
 & \left. \left. + \sum_{i=1}^k \int_{S_i} \frac{1}{\lambda_{si}} K(\beta_n, r_{si}) \dot{f}_i(r_{si}, \tau) dS_i \right] d\tau \right\} \quad (2-4-98)
 \end{aligned}$$

The temperature field superposition principle allows an inhomogeneous problem to be divided into a series of simpler problems. A general solution is found by summation of auxiliary problems. The procedure is especially recommended for heat-source problems.

For one-dimensional problems M. D. Mikhailov suggested the Fourier-Hankel generalized integral transform which is a combination of the Fourier and Hankel transforms for a solid cylinder. This transform is of the form

$$f(\mu) = \int_0^1 \xi^{\Gamma} W_{\Gamma}(\mu, \xi) f(\xi) d\xi \quad (2-4-99)$$

where  $f(\xi)$  is a function which satisfies the Dirichlet conditions for  $\xi$  ranging from 0 to 1,  $\mu$  is the root of characteristic equations \*

$$W_{\Gamma}(\mu) = 0; \quad V_{\Gamma}(\mu) = 0 \quad (2-4-100)$$

$$\frac{W_{\Gamma}(\mu)}{V_{\Gamma}(\mu)} = \frac{\mu}{\text{Bi}} \quad (2-4-101)$$

where  $W_{\Gamma}$  and  $V_{\Gamma}$  are the following generalized functions

$$\begin{aligned}
 W_{\Gamma}(\xi) = & 1 - \frac{\xi^2}{2(\Gamma + 1)} + \frac{\xi^4}{2 \cdot 4(\Gamma + 1)(\Gamma + 3)} \\
 & - \frac{\xi^6}{2 \cdot 4 \cdot 6(\Gamma + 1)(\Gamma + 3)(\Gamma + 5)} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k \xi^{2k}}{(2k)!!(\Gamma + 2k - 1)!!} \quad (2-4-102)
 \end{aligned}$$

\* The relation between the characteristic numbers  $\nu_n$  and  $\mu_n$  is expressed as  $\mu_n = \nu_n/l_i$  where  $l_i$  are characteristic dimensions of the body. For one-dimensional problems  $l_i = l_1$ .

$$V_R(\xi) = \frac{\xi}{\Gamma + 1} - \frac{\xi^3}{2(\Gamma + 1)(\Gamma + 3)} + \frac{\xi^5}{2 \cdot 4(\Gamma + 1)(\Gamma + 3)(\Gamma + 5)} - \dots = \sum_{k=0}^{\infty} \frac{(-1)^k \xi^{2k+1}}{(2k)!!(\Gamma + 2k + 1)!!} \quad (2-4-103)$$

The functions  $W_R(\xi)$  and  $V_R(\xi)$  are uniformly convergent in the range of  $\xi$  from 0 to 1. Assuming  $\Gamma = 0$ , we obtain

$$W_0(\xi) = \cos \xi; \quad V_0(\xi) = \sin \xi \quad (2-4-104)$$

i.e. the Fourier cosine transform which is applicable to heat-conduction problems for a plate.

For  $\Gamma = 1$

$$W_1(\xi) = J_0(\xi), \quad V_1(\xi) = J_1(\xi) \quad (2-4-105)$$

we obtain the Hankel finite transform. This may be used in solving problems for an infinite solid cylinder.

If  $\Gamma = 2$

$$W_2(\xi) = \frac{\sin \xi}{\xi}; \quad V_2(\xi) = \frac{\sin \xi - \xi \cos \xi}{\xi^2} \quad (2-4-106)$$

we obtain the Fourier generalized sine transform applicable to the solution of heat-conduction problems for a sphere.

Characteristic equations (2-4-100) are consistent with boundary conditions of the first and second kinds, and equation (2-4-101) corresponds to the third-kind boundary conditions.

The finite integral transform has a physical justification. Any integral transform with respect to space coordinates is, physically, a certain averaging of the physical quantity under study. It is quite natural that the averaging should be consistent not only with the nature of the process and the shape of the body (the type of the differential equation), but also with boundary conditions. In this case the solution for the transform will be of interest by itself, because physically such a transform is a transition from actual values of the functions considered (differential equations, conditions of uniqueness) to averaged values consistent with the specific problem.

The integral transform methods thus gain a significant edge over the classical ones as they allow us to find new relationships for physical processes based on the analysis of solutions for averaged values of the quantity under study (the analysis of the transformed solutions). Therefore, the analytical methods considered approach the similarity theory methods.

The integral transforms are particularly advantageous for the solution of the systems of partial differential equations. The procedure for the solution of the sets of equations, in this case, does not basically differ from

the procedure of solving individual equations and involves a number of successive operations. For example, for one-dimensional heat-conduction problems, depending upon coordinates and time, it is necessary to do the following:

1. to select a suitable integral transform or a set of integral transforms from an analysis of the differential equation and boundary conditions;
2. to multiply the differential equation and boundary conditions by the chosen transform kernel and to integrate the result over the necessary interval with respect to the variable to be eliminated; thus, instead of a set of partial differential equations for the original function, we get a set of ordinary differential equations for the transforms which account for the initial (when the Laplace transformation is used) or boundary (with the Fourier transformation) conditions;
3. to solve an ordinary differential equation for the transformed functions. If it is difficult to solve the equation, then it should be suitably transformed with respect to the second independent variable. The resultant algebraic equation is readily soluble. The functions transformed twice are then inversed. The result will be the solution of the differential equation;
4. to elaborate the expressions of arbitrary constants appearing in the solutions of equations. To do this, boundary conditions of the considered problem are used;
5. using the known relations between the transform and the original function or inversion formulas, it is necessary to find inversions of the transforms, and therefore, the final solution of the problem.

## 2-5 STEADY-STATE TEMPERATURE FIELD

For a steady state ( $\partial T/\partial \tau = 0$ ) with no internal heat sources ( $I_q = 0$ ) the differential heat-conduction equation (2-1-3) becomes

$$\operatorname{div}(\lambda \nabla T) = 0 \quad (2-5-1)$$

Thus in the steady state the heat-conduction transfer depends on the temperature gradient and thermal conductivity  $\lambda$ . The thermal conductivity for different materials ranges from 0.0074 kcal/(m · h · °C) (carbon tetrachloride at 100°C) to 358 kcal/(m · h · °C) (silver at 0°C). The thermal conductivity depends on the chemical composition, physical structure and the state of the material.

Heat conduction in gases and vapours is essentially controlled by molecular transfer of kinetic energy. Therefore, it is quite natural that the coefficients  $\lambda$  for gases and vapours have small values.

In liquids the heat-conduction transfer is similar to propagation of longitudinal oscillations (sound propagation). Thermal conductivities for liquids are, therefore, greater than those for gases. The molecular structure of crystal bodies favours heat transfer.

In metals, the heat-conduction transfer to a large extent is controlled by energy transfer via free electrons. Differences in thermal conductivities of various inhomogeneous materials may be attributed to porosity effect. For granular materials anisotropy, which manifests in the form of non-uniform thermal conductivity in various directions, is a typical disturbance of material homogeneity. The thermal conductivity depends on temperature; for metals it decreases linearly as the temperature increases.

The thermal conductivity for gases increases with temperature, and is practically independent of pressure except for very high (above 2000 kgf/cm<sup>2</sup>) and very low (below 10 mm Hg) pressures. For gaseous mixtures the thermal conductivity may only be found experimentally, as the additivity law does not hold for  $\lambda$ . Thermal conductivity for liquid ranges between 0.08 and 0.6 kcal/(m·h·°C). As the temperature rises, the thermal conductivity of most liquids decreases. Water and glicerine are exceptions. The thermal conductivity of non-metallic materials ranges within 0.02-2.5 kcal/(m·h·°C) and as the temperature increases, it rises almost linearly. Materials with a low thermal conductivity ( $\lambda \leq 0.2$  kcal)/(m·h·°C) are generally called heat-insulating materials.

For practical calculations, the thermal conductivity versus temperature curve for gases may be assumed to be linear

$$\beta_T = \frac{d\lambda}{dT} = \text{const} \quad (2-5-2)$$

The thermal conductivity in a definite temperature interval  $\Delta T = T_2 - T_1$  may be assumed to be constant and equal to the arithmetic mean of  $\lambda$  at  $T_2$  and  $T_1$ .

Much research has been devoted to the solution of steady-state heat-conduction problems. These problems have been systematized in [2-27 and 2-28]. In the following subsections some methods of solving such problems and some particular problems (as illustrations) will be considered.

### (a) One-Dimensional Problems

For a one-dimensional temperature field when temperature depends only on one coordinate, the solution of differential equation (2-5-1), for a constant thermal conductivity ( $\lambda = \text{const}$ ), takes the form:

(1) for a flat wall (an infinite plate)

$$T = B_1 x + B_2 \quad (2-5-3)$$

(2) for a hollow infinite cylinder ( $l \gg R$ )

$$T = B_1(\log_e r + B_2) \quad (2-5-4)$$

(3) for a hollow sphere

$$T = B_1 \frac{1}{r} + B_2 \quad (2-5-5)$$

Constants  $B_1$  and  $B_2$  are determined from the boundary conditions.

The case of an infinite plate will be considered in greater detail. For the boundary conditions of the first kind (when  $x = 0$   $T = T_1$ , when  $x = L$   $T = T_2$ ), the solution of equation (2-5-3) will be written as

$$T = T_1 - \frac{(T_1 - T_2)x}{L} \quad (2-5-6)$$

where  $L$  is the wall thickness (Fig. 2-2). According to the Fourier law the heat flux  $q$  is

$$q = -\lambda \frac{dT}{dx} = \frac{\lambda}{L} (T_1 - T_2) \quad (2-5-7)$$

Relation (2-5-7) is similar to the Ohm law for electric current (current intensity is a ratio of the potential difference to the resistance of a given

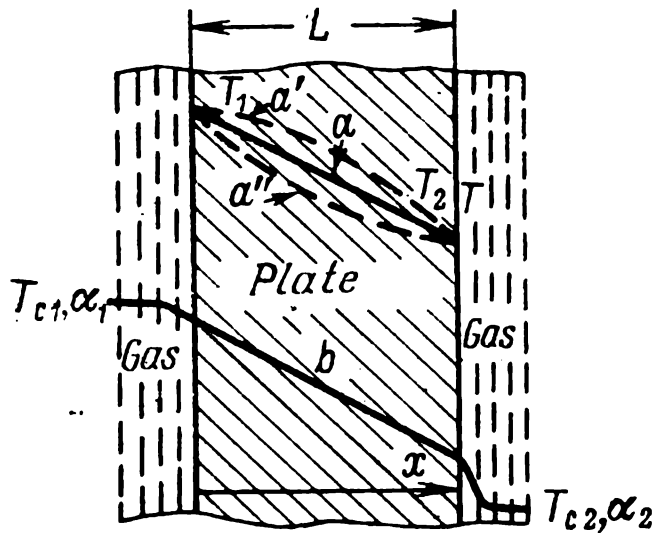


Fig. 2-2. Temperature distribution in an infinite plate in a gas flow in the steady state

$a$  — boundary conditions of the first kind ( $a'$ , at  $d\lambda/dT > 0$ ;  $a''$ , at  $d\lambda/dT < 0$ );  $b$  — boundary condition of the third kind

conductor section), therefore, by analogy, the quantity  $L/\lambda$  is called the heat or thermal resistance of a plate and its reciprocal  $\lambda/L$  is called the thermal conductivity of the plate.

For the boundary condition of the third kind (Fig. 2-2) when  $x = 0$

$$\lambda \frac{\partial T}{\partial x} + \alpha_1(T_{c1} - T_{x=0}) = 0 \quad (2-5-8)$$

when  $x = L$

$$-\lambda \frac{\partial T}{\partial x} + \alpha_2(T_{x=L} - T_{c2}) = 0 \quad (2-5-9)$$

where  $T_{c1}$  and  $T_{c2}$  are the gas temperatures on the left-hand and right-hand sides of the plate, respectively;  $\alpha_1$  and  $\alpha_2$  are the heat-transfer coefficients at the plate surfaces.

The constants  $B_1$  and  $B_2$  are determined by

$$B_1 = - \frac{T_{c1} - T_{c2}}{\lambda \left( \frac{1}{\alpha_1} + \frac{L}{\lambda} + \frac{1}{\alpha_2} \right)} ; \quad B_2 = T_{c1} + \frac{\lambda}{\alpha_1} B_1 \quad (2-5-10)$$

and the temperature distribution is expressed as:

$$T = T_{c1} - \frac{T_{c1} - T_{c2}}{\frac{1}{\alpha_1} + \frac{L}{\lambda} + \frac{1}{\alpha_2}} \left[ \frac{x}{\lambda} + \frac{1}{\alpha_1} \right] \quad (2-5-11)$$

The heat flux is found from formula (2-5-11)

$$q = - \lambda \frac{dT}{dx} = \frac{1}{\frac{1}{\alpha_1} + \frac{L}{\lambda} + \frac{1}{\alpha_2}} (T_{c1} - T_{c2}) \quad (2-5-12)$$

In this case the thermal resistance

$$R_q = \frac{1}{\alpha_1} + \frac{L}{\lambda} + \frac{1}{\alpha_2} \quad (2-5-13)$$

If the heat-transfer coefficients  $\alpha_1$  and  $\alpha_2$  are large ( $\alpha_1 \rightarrow \infty$ ,  $\alpha_2 \rightarrow \infty$ ), then the thermal resistance

$$R_q = L/\lambda \quad (2-5-14)$$

For a multi-layer wall composed of homogeneous plates of thickness  $L_i$  ( $i = 1, 2, 3, \dots, n$ ) having corresponding thermal conductivities  $\lambda_i$ , the thermal resistance is

$$R_q = \frac{1}{\alpha_1} + \sum_{i=1}^n \frac{L_i}{\lambda_i} + \frac{1}{\alpha_2} \quad (2-5-15)$$

Here, a perfect heat contact is assumed between individual plates which constitute the wall ( $T_i = T_{i+1}$ ). If the thermal conductivity is a variable quantity ( $d\lambda/dT \neq 0$ ), then the temperature distribution across the plate will not be linear. As the thermal conductivity increases with temperature ( $d\lambda/dT > 0$ ), the temperature distribution  $T(x)$  is a curve with an upward hump (see Fig. 2-2), and vice versa, when  $d\lambda/dT < 0$  the hump of the curve  $T(x)$  faces the abscissa.

A similar method allows the formula for cylindrical and spherical shells to be found. The temperature distribution in parallelepiped-like shells, such as the walls of an ordinary room or an oven, cannot be described by a one-dimensional temperature field. These problems can hardly be solved analytically. Using the electrical analogy method, Langmuir succeeded in obtaining simple empirical formulas for determining hourly heat-flow rate  $Q$ , kcal/h, through parallelepiped-like shells [2-29].

Given the constant power  $I_q$ , kcal/(m<sup>3</sup>·h) of the source, the solution of a one-dimensional steady-state temperature field reduces to the solution

of the differential equation

$$\lambda \left( \frac{d^2 T}{d\xi^2} + \frac{\Gamma}{\xi} \frac{dT}{d\xi} \right) + I_q = 0 \quad (2-5-16)$$

where  $\Gamma$  is a constant (for an infinite plate  $\Gamma = 0$ , for a cylinder  $\Gamma = 1$ , for a sphere  $\Gamma = 2$ );  $\xi$  is a coordinate chosen to suit the body shape. Here, thermal conductivity is assumed constant.

For an infinite plate, when heat transfer between its opposite faces and the surrounding gas ( $T_c = \text{const}$ ) is uniform  $\alpha_1 = \alpha_2 = \alpha = \text{const}$ , the solution of equation (2-5-16) in a dimensionless form takes the shape

$$\frac{T - T_c}{T_c} = \frac{1}{2} \text{Po} \left( 1 - \frac{x^2}{R^2} + \frac{2}{\text{Bi}} \right) \quad (2-5-17)$$

wherein  $R$  is the half-thickness of the plate ( $L = 2R$ );  $\text{Bi}$  is the Biot number ( $\text{Bi} = \alpha R / \lambda$ ); the Pomerantsev number

$$\text{Po} = I_q R^2 / \lambda T_c \quad (2-5-18)$$

is the ratio between the amount of heat  $I_q R^2$ , liberated by the source per unit time in volume  $R^*$  and the amount of heat transferred by conduction

$\left( \lambda \frac{T_c}{R} \right)$  through a layer of thickness  $R$ , when the temperature drop in it equals  $T_c$ . Thus, the temperature distribution across the plate is governed by a parabolic law.

For solid infinite cylinder and sphere the solution may be written as:

$$\frac{T - T_c}{T_c} = \frac{1}{N} \text{Po} \left( 1 - \frac{r^2}{R^2} + \frac{2}{\text{Bi}} \right) \quad (2-5-19)$$

where  $N$  is a constant (for a cylinder  $N = 4$ , for a sphere  $N = 6$ ).

Let us consider how a porous plate cools when cold liquid or gas is passed through it.

If the void fraction  $\Pi$  (the ratio of the pore volume to the body volume) is equal to the surface void fraction (the ratio of the pore surface area to the body surface area), then a unit area for a liquid flow within the plate  $A'_f$  is equal to the porosity ( $A'_f = \Pi$ ). The unit area of the skeleton of the plate (the skeleton area per unit area of the plate surface)  $A'_T$  equals  $(1 - A'_f)$ , i.e.  $A'_T = 1 - \Pi$ . The mass-flow rate  $j_f$  kg/(m<sup>2</sup>·h) is equal to the product of the fluid density by the fluid velocity  $v$  ( $j_f = \rho_f v$ ). The fluid velocity within the plate will be  $\Pi$  times greater, that is, the mass-flow rate within the plate will equal  $j_f / \Pi$ . For small values of  $j_f$ , defined by  $\text{Re} = \frac{j_f d_l}{\Pi \eta_f} < 1$ , the fluid temperature is equal to that of the body skeleton.

\* This is the volume of a parallelepiped whose base area equals unity and height,  $R$  (in metres).

The problem is formulated as follows: one face of the plate is heated up to a temperature  $T_2 = \text{const}$ , the opposite face is in contact with the fluid. Heat is transferred by conduction (boundary condition of the fourth kind). This problem was solved by Weinbaum and Wheller [2-34]. The differential equations for heat transfer in the fluid and plate are of the form

$$\lambda_f \frac{d^2 T}{dx^2} - c_{pf} \rho_f v \frac{dT}{dx} = 0 \quad (-\infty < x < 0) \quad (2-5-20)$$

$$\lambda_T(1 - \Pi) \frac{d^2 T}{dx^2} - c_{pf} \rho_f v \frac{dT}{dx} = 0 \quad (0 < x < L) \quad (2-5-21)$$

where  $c_{pf}$  is the specific heat of the fluid at constant pressure.

In the second term there is no factor dependent on  $\Pi$ , which may be attributed to the fact that fluid velocity within the plate is  $\bar{v}/\Pi$  and the unit surface area for fluid motion equals  $A'_f = \Pi$ . Here, it is assumed that heat within the body is transferred through the skeleton only. Therefore, the first term of equation (2-5-21) contains a factor  $(1 - \Pi)$  equal to the unit area of the skeleton. This assumption is justified when the thermal conductivity of a blown through gas is small in comparison to that of the body. However, for porous heat insulating materials, when liquid is passed through them, this assumption will not comply with the actual process as heat transfer by conduction through pores filled with liquid cannot be neglected. In this case,  $\lambda_T(1 - \Pi)$  should be replaced by the equivalent thermal conductivity  $\lambda_e$ , equal to  $\lambda_T + \Pi(\lambda_f - \lambda_T)$ .

The boundary conditions for the system of equations (2-5-20)-(2-5-21) are written as (Fig. 2-3).

$$\left. \begin{array}{l} \text{for } x = L \quad T = T_2 = \text{const} \\ \text{for } x = 0 \quad (T)_{+x \rightarrow 0} = (T)_{-x \rightarrow 0} \end{array} \right\} \quad (2-5-22)$$

$$\text{for } \lambda_f \left( \frac{dT}{dx} \right)_{-x \rightarrow 0} = \lambda_T(1 - \Pi) \left( \frac{dT}{dx} \right)_{+x \rightarrow 0} \quad (2-5-23)$$

$$\text{for } x = -\infty \quad T \rightarrow T_c = \text{const} \quad (2-5-24)$$

The solution of equation for temperature distribution in a plate is of the form

$$\frac{T - T_c}{T_2 - T_c} = \exp \left[ - \left( 1 - \frac{x}{L} \right) \text{Kn} \right] \quad (2-5-25)$$

where Kn is a dimensionless number for cooling porous bodies

$$\text{Kn} = \frac{j_f c_{pf} L_f}{\lambda_T(1 - \Pi)} \quad (2-5-26)$$



The physical significance of the Kn number is that it characterizes the ratio of the amount of heat transferred by convection within a porous plate to that transferred by conduction under similar temperature heads.

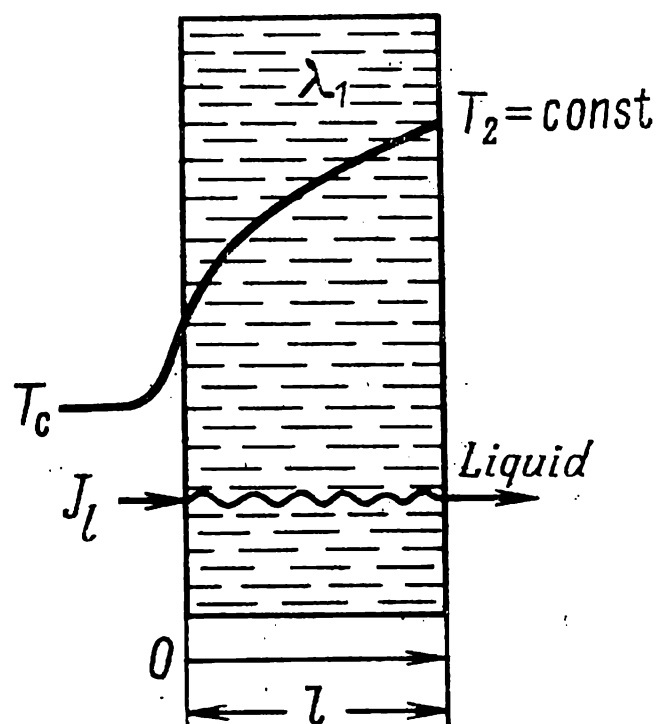


Fig. 2-3. Transpiration cooling of a flat plate ( $l \equiv L$ )

The Kn number may be expressed in terms of the Peclet number

$$\text{Kn} = \frac{c_{pf}\rho_f v L}{\lambda_T(1 - \Pi)} = \frac{vL}{a_f} \frac{\lambda_f}{\lambda_T(1 - \Pi)} = \text{Pe} \frac{\lambda_f}{(1 - \Pi)\lambda_T} \quad (2-5-27)$$

In Fig. 2-4 the curves showing temperature distribution across the plate are given for values of Kn numbers ranging from 0 to 10. It is seen from

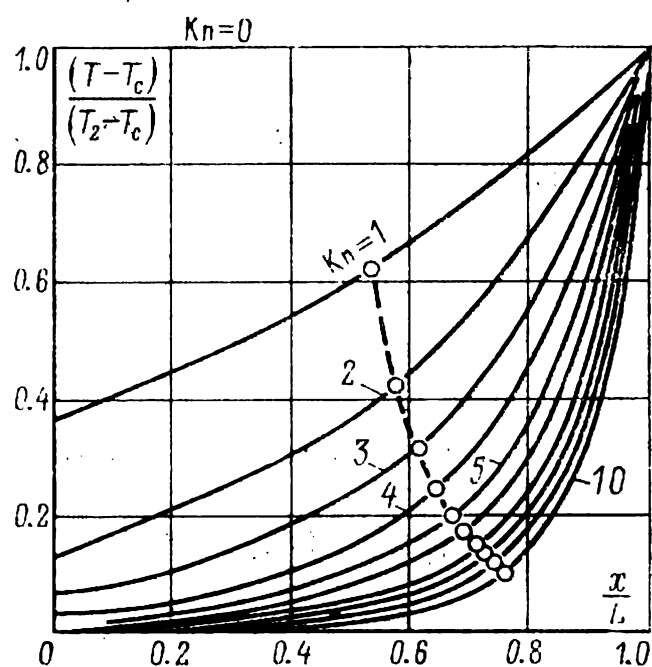


Fig. 2-4. Temperature distribution and mean temperature in a porous plate versus Kn number

Figure 2-4 that with a decrease in the Kn number, the temperature distribution tends to follow the linear law. The dashed line shows mean temper-

ature  $\bar{T}$  versus  $Kn$ . It has been plotted by using the following formula:

$$\frac{\bar{T} - T_c}{T_2 - T_c} = \frac{1}{(T_2 - T_c)L} \int_0^L (T - T_c) dx = \frac{1}{Kn} (1 - e^{-Kn}) \quad (2-5-28)$$

The transpiration cooling technique is at present widely used in modern engineering practice. Effective heat removal at small temperature gradients within a body is one of the main advantages of the method.

### (b) Two-Dimensional Problem

To conclude the section, we shall consider the problem of finding the temperature field in a wedge for which the heat-conduction equation is of the form ( $0 < r < \infty$ ,  $0 < \varphi < \alpha$ )

$$r^2 \frac{\partial^2 T}{\partial r^2} + r \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial \varphi^2} + W(r, \varphi) = 0 \quad (2-5-29)$$

for boundary conditions

$$T(r, 0) = \varphi_1(r); \quad T(r, \alpha) = \varphi_2(r) \quad (2-5-30)$$

It is assumed that functions  $\varphi_1(r)$  and  $\varphi_2(r)$  are such that the integral

$$\int_0^\infty \varphi_i(r) r^{P-1} dr; \quad i_1 = 1, 2 \quad (2-5-31)$$

exists. The solution of equation (2-5-29) for boundary conditions (2-5-30) may be obtained by the Melline transformation

$$T_1 \equiv \bar{T}(p) = \int_0^\infty T r^{P-1} dr \quad (2-5-32)$$

We assume that  $T$ ,  $r \partial T / \partial r$  are uniformly finite when  $r \rightarrow 0$  and at infinity they behave like  $r^{-\delta}$  ( $\delta > 0$ ).

Then applying (2-5-32) to (2-5-29) and (2-5-30), we obtain

$$\frac{d^2 \bar{T}}{d\varphi^2} + P^2 \bar{T} - \bar{W}(P, \varphi) = 0 \quad (2-5-33)$$

$$\bar{T}(r, 0) = \bar{\varphi}_1(P); \quad \bar{T}(r, \alpha) = \bar{\varphi}_2(P) \quad (2-5-34)$$

By integrating (2-5-33) with consideration for (2-5-34), we obtain

$$\begin{aligned} \bar{T}(P, \varphi) = & \bar{\varphi}_1(P) \frac{\sin P(\alpha - \varphi)}{\sin P\alpha} + \bar{\varphi}_2(P) \frac{\sin P\varphi}{\sin P\alpha} \\ & - \int_0^\alpha G(P, \varphi, \varphi_1) \bar{W}(P, \varphi_1) d\varphi_1 \end{aligned} \quad (2-5-35)$$

where

$$G(P, \varphi, \varphi_1) = \begin{cases} \frac{\sin P\varphi \sin P(\alpha - \varphi_1)}{P \sin P\alpha}, & \varphi \leq \varphi_1 \\ \frac{\sin P\varphi_1 \sin P(\alpha - \varphi)}{P \sin P\alpha}, & \varphi > \varphi_1 \end{cases} \quad (2-5-36)$$

The transforms are inverted by the formula

$$T(r, \varphi) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} T(P, \varphi) r^{-P} dP \quad (2-5-37)$$

In the inversion the line  $\operatorname{Re} P = \sigma$  is supplemented until the contour closes, and on the assumption that  $\bar{T}$  is a meromorphous function and at sufficiently large values of  $R$  the integral over the arc  $|P| = R$  tends to zero, we may use the residue theorem.

Let us consider inversion in more detail for the particular case  $W = 0$  and

$$T(r, 0) = \begin{cases} A_0 + B_0 r, & 0 < r \leq b \\ 0, & r > b \end{cases} \quad (2-5-38)$$

$$T(r, \alpha) = \begin{cases} A_1 + B_1 r, & 0 < r \leq b \\ 0, & r > b \end{cases} \quad (2-5-39)$$

Formula (2-5-35) will then take the form

$$\begin{aligned} \bar{T}(P, \varphi) = & \left( A_0 \frac{b^P}{P} + B_0 \frac{b^{P+1}}{P+1} \right) \frac{\sin P(\alpha - \varphi)}{\sin P\alpha} \\ & + \left( A_1 \frac{b^P}{P} + B_1 \frac{b^{P+1}}{P+1} \right) \frac{\sin P\varphi}{\sin P\alpha} \end{aligned} \quad (2-5-40)$$

It may be easily seen that  $\bar{T}$  is a meromorphous function and integral (2-5-37) over the arc  $|P| = R$  tends to zero at sufficiently large values of  $R$ .

Then substituting (2-5-40) into (2-5-37) and using the residue theorem (the integration contour is supplemented with an arc of a circle of an infinitely large radius, which is located on the left of  $\text{Re } P = \sigma$  when  $r < b$  and to the right of the line when  $r > b$ ) we obtain a final solution of the boundary-value problem (2-5-29)-(2-5-30), (2-5-38)-(2-5-39)

$$T(r, \varphi) = \frac{A_0(\alpha - \varphi)}{\alpha} + \frac{A_1\varphi}{\alpha} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{r}{b}\right)^{n\pi/\alpha} \\ \times \left[ A_1 \sin \frac{n\pi\varphi}{\alpha} + A_0 \sin \frac{n\pi(\varphi - \alpha)}{\alpha} \right] + r \left[ B_0 \frac{\sin(\alpha - \varphi)}{\sin \alpha} + B_1 \frac{\sin \varphi}{\sin \alpha} \right] \\ + \frac{1}{b} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\alpha - n\pi} \left(\frac{r}{b}\right)^{n\pi/\alpha} \left[ B_0 \sin \frac{n\pi}{\alpha}(\alpha - \varphi) + B_1 \sin \frac{n\pi\varphi}{\alpha} \right] \quad (0 < r < b) \quad (2-5-41)$$

$$T(r, \varphi) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{b}{r}\right)^{n\pi/\alpha} \left[ A_1 \sin \frac{n\pi\varphi}{\alpha} + A_0 \sin \frac{n\pi(\varphi - \alpha)}{\alpha} \right] \\ + \frac{1}{b} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\alpha + n\pi} \left(\frac{b}{r}\right)^{n\pi/\alpha} \left[ B_0 \sin \frac{n\pi(\alpha - \varphi)}{\alpha} + B_1 \sin \frac{n\pi\varphi}{\alpha} \right] \quad (2-5-42)$$

(when  $b < r < \infty$ ).

## 2-6. SOLUTION OF STEADY-STATE PROBLEMS BY THE CONFORMAL MAPPING TECHNIQUE

This technique is used to solve a two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (2-6-1)$$

which describes the processes of steady-state transfer of heat, mass, etc. The conformal mapping technique is advantageous to other classical methods, because it gives the solution of equation (2-6-1) not only for very simple plane geometries (a plate, a cylinder, etc.), but also for almost any plane configuration. This advantage is very important as very complicated plane configurations, encountered in practice, cannot be represented by simple geometries.

As is known, any analytical function of a complex variable satisfies the Laplace equation (2-6-1), therefore, the conformal mapping technique consists in using a certain analytical function to transform the given complicated geometry into a simple geometry (say, a half-plane) for which the solution is readily obtainable. In doing so the Laplace equation (2-6-1) and boundary conditions retain their original form. Therefore, if in the resultant simple geometry a suitable analytical function is found consistent with the given boundary conditions, the problem is considered solved.

1. Let us now introduce the concept of a complex transfer potential similar to the concept adopted in fluid mechanics. If we denote

$$q = -\lambda \operatorname{grad} u \quad (2-6-2)$$

then from the heat-conduction equation it follows that

$$\operatorname{div} q = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = 0 \quad (2-6-3)$$

Moreover, since

$$\operatorname{rot} \operatorname{grad} u = 0 \quad (2-6-4)$$

then from (2-6-2) when  $\lambda = \text{const}$ , we get

$$\frac{\partial q_y}{\partial x} - \frac{\partial q_x}{\partial y} = 0 \quad (2-6-5)$$

The complex potential of the heat flux is a function of the complex variable

$$\psi(z) = V(x, y) + iu(x, y) \quad (2-6-6)$$

where  $V(x, y)$  is a function named by analogy to fluid mechanics terminology a stream function and related to  $u(x, y)$  by the Cauchy-Riemann condition

$$\frac{\partial V}{\partial x} = \frac{\partial u}{\partial y} = -\frac{1}{\lambda} q_y; \quad \frac{\partial V}{\partial y} = -\frac{\partial u}{\partial x} = \frac{1}{\lambda} q_x \quad (2-6-7)$$

Thus, the unknown function  $u(x, y)$  is defined by

$$u(x, y) = \operatorname{Im} [\psi(z)] \quad (2-6-8)$$

It should be emphasized that the conformal mapping technique is applicable only to steady-state transfer problems and cannot be used for the solution of unsteady-state problems.

2. It is extremely difficult to find a function transforming the given region  $\mathfrak{B}$  into a half-plane  $\mathfrak{z}$  (in the subsequent text the upper half-plane will be considered as an elementary region). Here, polygonal regions will be considered which are mapped on the upper plane by the Kristoffel-Schwartz integral [2-14], [2-30]

$$W = C \int_0^z (z - a_1)^{\alpha_1 - 1} (z - a_2)^{\alpha_2 - 1} \dots (z - a_n)^{\alpha_n - 1} dz + C_1 \quad (2-6-9)$$

where  $a_i$  are points on the real axis of the half-plane  $z$ , to which the points of the polygon transform during conformal mapping;  $\alpha_i$  are the polygon angles expressed in fractions of  $\pi$ ;  $C, C_1$  are constants. It should be noted that in (2-6-9) the factors corresponding to the points lying at infinitely large distances drop out. According to the theorem of conformal mapping existence [2-30], three points on a real axis may be taken arbitrarily and

the remaining points as well as constants  $C$  and  $C_1$  are found from the problem conditions.

**Example 1.** Find the steady-state temperature distribution at the wall close to the building corner when the temperatures of the outer and inner surfaces differ and remain constant (Fig. 2-5a) [2-14], [2-31]. Mapping

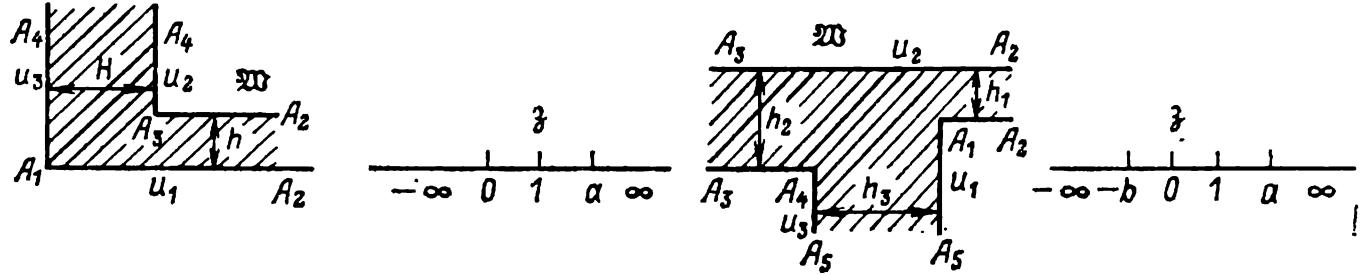


Fig. 2-5. Schematic for the calculation of temperature field  
(a) corner of a building; (b) pentagonal wall

of tetragonal  $A_1 A_2 A_3 A_4 A_1$  with apices  $A_1 = 0$ ,  $A_2 = \infty$ ,  $A_3 = H + ih$ ,  $A_4 = \infty$  and respective apex angles  $\alpha_1 = 1/2$ ,  $\alpha_2 = 0$ ,  $\alpha_3 = 3/2$ ,  $\alpha_4 = 0$  will be performed using the Kristoffel-Schwartz integral. For convenience the data are listed in a table in which three out of the four points on the real axis are given arbitrarily.

$k$	$A_k$	$\alpha_k$	$a_k$
1	0	0.5	0
2	$\infty$	0	1
3	$H + ih$	1.5	$a$
4	$\infty$	0	$\infty$

Then integral (2-6-9) becomes

$$W = C \int_0^z \frac{\sqrt{z-a}}{\sqrt{z}(z-1)} dz + C_1 \quad (2-6-10)$$

To find the constants  $C$ ,  $C_1$  and  $a$ , three equations are necessary, of which two are obtained from the correspondence conditions at the constraints  $a_1 \rightarrow A_1$ ,  $a_3 \rightarrow A_3$ . Since

$$W = 2C \left[ \sqrt{a-1} \tan^{-1} \sqrt{\frac{z(a-1)}{z-a}} + \tanh^{-1} \sqrt{\frac{z}{z-a}} \right] + C_1 \quad (2-6-11)$$

then

$$C_1 = 0;$$

$$H + ih = C\pi [\sqrt{a-1} - i] \quad (2-6-12)$$

To find the third equation we shall make use of the following consideration: when passing from ray  $A_1 A_2$  to ray  $A_2 A_3$  in the plane  $\mathfrak{W}$  the function  $W$  is incremented.

$$\Delta W = ih + 0(\rho) \quad (2-6-13)$$

where  $0(\rho)$  is a complex function which is infinitesimal when  $\rho \rightarrow 0$ . This corresponds to circumvention of a semicircle of an infinitesimal radius with the center at point 0 in the plane  $z$ . Substitution of  $z = 1 + \rho e^{i\varphi}$  into (2-6-10) and calculation of the integral over the semicircle when  $\rho \rightarrow 0$  give

$$\Delta W = -C\pi\sqrt{a-1} \quad (2-6-14)$$

Comparison of equation (2-6-13) with (2-6-14) results in

$$C = -\frac{ih}{\pi\sqrt{(a-1)}} \quad (2-6-15)$$

After the constants have been found from equations (2-6-12) and (2-6-15) a final expression for the function mapping the half-plane to the tetragon  $A_1A_2A_3A_4A_1$  may be obtained

$$W = \frac{2i}{\pi} \left[ h \tan^{-1} \frac{h}{H} \sqrt{\frac{z}{z-a}} + H \tanh^{-1} \sqrt{\frac{z}{z-a}} \right] \quad (2-6-16)$$

It may be easily seen from formula (2-6-16) that the internal boundary of "tetragon"  $A_2A_3A_4$  becomes the negative part of the real axis, and the external boundary  $A_4A_1A_2$  — the positive one. Now we select an analytical function in the upper half-plane, the imaginary part of which in the interval  $(-\infty, 0)$  has a value  $u_3$ , in the interval  $(0, 1)$  —  $u_1$ , in the interval  $(1, \infty)$  —  $u_2$ . Such a function may be found in the form [2-30]

$$u = C_1 \arg(-z) + C_2 \arg(z-1) + C_3 \quad (2-6-17)$$

where

$$\arg y = \begin{cases} 0, & y > 0 \\ \pi, & y < 0 \end{cases} \quad (2-6-18)$$

The constants in equation (2-6-17) are found from the boundary conditions with allowance made for (2-6-18)

$$C_1 = \frac{u_1 - u_3}{\pi}; \quad C_2 = \frac{u_1 - u_2}{\pi}; \quad C_3 = u_3 - u_1 + u_2$$

Then

$$u = \frac{u_1 - u_3}{\pi} \arg(-z) + \frac{u_1 - u_2}{\pi} \arg(z-1) + u_3 - u_1 + u_2 \quad (2-6-19)$$

Now an analytical function with an imaginary part of the form (2-5-19) may easily be selected, for example the following function could be chosen

$$\psi(z) = \frac{u_1 - u_3}{\pi} \log_e(-z) + \frac{u_1 - u_2}{\pi} \log_e(z-1) + u_3 - u_1 + u_2$$

The final solution of the problem will then take the form

$$u = \operatorname{Im} \left[ \frac{u_1 - u_3}{\pi} \log_e (-z) + \frac{u_1 - u_2}{\pi} \log_e (z - 1) \right] + u_3 - u_1 + u_2 \quad (2-6-20)$$

where  $z$  is expressed in terms of  $W$  by formula (2-6-16).

Since  $z$  is not expressed explicitly in terms of  $W$ , it should first be assigned a value using formula (2-6-20), and then the appropriate value of  $W$  may be found from (2-6-16). Thus, the transfer potential in the given plane is obtained.

**Example 2.** Find a steady-state temperature field in a plane bounded by "pentagon"  $A_1 A_2 A_3 A_4 A_5 A_1$  with the following boundary conditions:  $u = u_1$  over the broken line  $A_5 A_1 A_2$ ;  $u = u_2$  over the broken line  $A_2 A_3$ ;  $u = u_3$  over the broken line  $A_3 A_4 A_5$  (Fig. 2-5b).

As in the previous example, let us represent conformly the upper half-plane in configuration  $A_1 A_2 A_3 A_4 A_5 A_1$  using the Kristoffel-Schwartz integral. The data as before are summarized in the table.

$k$	$A_k$	$\alpha_k$	$a_k$
1	0	1.5	0
2	$\infty$	0	1
3	$\infty$	0	$a$
4	$-h_3 - i(h_2 - h_1)$	1.5	$\infty$
5	$\infty$	0	$-b$

In the table points  $a_k$  are also given, three of which are taken arbitrarily, namely,  $a_1 = 0$ ,  $a_2 = 1$ ,  $a_4 = \infty$ ; the remaining points  $a_3 = a$ ,  $a_5 = -b$  on the real axis are unknown and are to be found later. In this case, integral (2-6-9) is of the form

$$W = C \int_0^z \frac{\sqrt{z_1} dz_1}{(z_1 - 1)(z_1 - a)(z_1 + b)} \quad (2-6-21)$$

To find the unknown constants  $C$ ,  $a$  and  $b$ , the following should be taken into account: transition from ray  $A_1 A_2$  to ray  $A_2 A_3$  in the plane  $\mathfrak{B}$  results in an increment of the function  $W$

$$\Delta W = ih_1 + 0(\rho) \quad (2-6-22)$$

where  $0(\rho)$  is a complex function which is infinitesimal when  $\rho \rightarrow 0$ . This conforms to circumvention of a semicircle of an infinitesimal radius with its center at point 1 in plane  $\mathfrak{z}$ . Substitution of  $z = 1 + \rho e^{i\varphi}$  into (2-6-21) and calculation of the integral over the semicircle at  $\rho \rightarrow 0$  give

$$\Delta W = \frac{C\pi i}{(1 - a)(1 + b)} \quad (2-6-23)$$



Comparison of equation (2-6-22) with (2-6-23) results in

$$h_1 = \frac{C\pi}{(1-a)(1+b)} \quad (2-6-24)$$

Similarly, transition in the plane  $\mathfrak{W}$  from the rays  $A_2 A_3$  and  $A_4 A_5$  to the rays  $A_3 A_4$  and  $A_5 A_1$ , respectively, (which conforms to circumvention of a semicircle of an infinitesimal radius about points  $a$  and  $-b$ , respectively, in the plane  $\mathfrak{z}$ ) results in

$$h_3 = \frac{C\pi\sqrt{a}}{(a+b)(a-1)} \quad (2-6-25)$$

$$h_3 = \frac{C\pi\sqrt{b}}{(a+b)(b+1)} \quad (2-6-26)$$

The constants are found from equations (2-6-24)-(2-6-26). Calculation of integral (2-6-21) yields

$$W(z) = \frac{2C}{a+b} \left[ \frac{\sqrt{a}}{2(a-1)} \log_e \frac{\sqrt{z} - \sqrt{a}}{\sqrt{z} + \sqrt{a}} - \frac{\sqrt{b}}{1+b} \tan^{-1} \frac{\sqrt{z}}{\sqrt{b}} \right. \\ \left. + \frac{a+b}{2(1-a)(1+b)} \log_e \frac{\sqrt{z} - 1}{\sqrt{z} + 1} \right] \quad (2-6-27)$$

where constants  $C$ ,  $a$  and  $b$  are found from (2-6-24)-(2-6-26).

When mapping (2-6-27), the broken lines  $A_5 A_1 A_2$ ,  $A_2 A_3$ ,  $A_3 A_4 A_5$ , bounding the "pentagon"  $A_1 A_2 A_3 A_4 A_5 A_1$  change into sections of the real axis of the plane, namely,  $[-b, 1]$ ,  $[1, a]$ ,  $[-\infty, -b]$ ,  $[a, \infty]$ .

Thus, the considered problem amounts to finding an analytical function in the upper half-plane, the imaginary part of which at the sections of the real axis attains the values  $u_1$ ,  $u_2$  and  $u_3$ , respectively. Such a function is found in a similar manner, as was done in the previous example, in the form

$$u = \frac{1}{\pi} \operatorname{Im} [(u_3 - u_1) \log_e (z + b) + (u_1 - u_2) \log_e (z - 1) \\ + (u_2 - u_3) \log_e (z - a)] + u_3$$

**Example 3.** Find the temperature distribution in a plane defined by rhomb  $A_1 A_2 A_3 A_4$  (Fig. 2-6) whose adjacent sides  $A_4 A_1$ ,  $A_1 A_2$  and  $A_2 A_3$ ,  $A_3 A_4$  have the given temperatures  $u_1$  and  $u_2$ , respectively [2-31].

The Kristoffel-Schwartz integral is used to map  $A_1 A_2 A_3 A_4$  on the upper half-plane. Conformity of the points is given as

$$\mathfrak{W}[A_1 = 0, \quad A_2 = d, \quad A_3 = d(1 + e^{i\pi\alpha}), \quad A_4 = de^{i\pi\alpha}] \rightarrow \mathfrak{z} \quad (0, 1, \infty, b)$$

In this case the Kristoffel-Schwartz integral takes the form

$$W = C \int_0^z z_1^{\alpha-1} (z_1 - 1)^{-\alpha} (z_1 - b)^{-\alpha} dz_1 + C_1 \quad (2-6-28)$$

As in the previous examples, we arbitrarily choose three points in the plane  $z$ :  $a_1 = 0$ ,  $a_2 = 1$ ,  $a_3 = \infty$ . The fourth point  $a_4 = b$  is found from the symmetry principle. As the configuration  $A_1 A_2 A_3 A_4$  is symmetrical

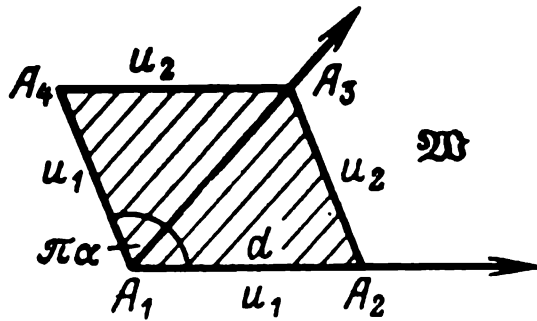


Fig. 2-6. Schematic for the calculation of temperature field of a rhomboidal flat wall

relative to diagonal  $A_1 A_3$ , and the point  $A_3$  is mapped to  $i\infty$  in the plane  $z$ , then from the symmetry principle, triangle  $A_1 A_2 A_3$  is mapped to the right-hand side of the upper half-plane and triangle  $A_3 A_4 A_1$  to its left-hand side. It may now be inferred that  $b = -1$ . Constants  $C$  and  $C_1$  are found from the conformity conditions of points  $A_1 = 0$  and  $A_2 = d$  in the plane  $\mathfrak{B}$  to points  $a_1 = 0$  and  $a_2 = 1$  in the plane  $z$

$$C_1 = 0 \quad (2-6-29)$$

$$C = 2 \left( \int_0^1 (1 - z_1)^{\alpha/2-1} z_1^{-\alpha} dz_1 \right)^{-1} = \frac{2d}{B \left( 1 - \alpha; \frac{\alpha}{2} \right)} \quad (2-6-30)$$

The analytical function mapping the upper half-plane to rhomb  $A_1 A_2 A_3 A_4$ , with consideration for (2-6-29) and (2-6-30), is written as

$$W(z) = \frac{2d}{B \left( 1 - \alpha; \frac{\alpha}{2} \right)} \int_0^z z_1^{\alpha-1} (1 - z_1^2)^{-\alpha} dz_1 \quad (2-6-31)$$

The boundary conditions after mapping to the plane  $z$  are written as  
 $u_1 = u_2, (-\infty < x < -1); u = u_1,$   
 $(-1 < x < 1); u = u_2 (1 < x < \infty)$

Now we shall find in the upper half-plane an analytical function satisfying these conditions. This function may easily be found as

$$u = C_1 \arg(z + 1) + C_2 \arg(z - 1) + C_3 \quad (2-6-32)$$

To find the constants, use will be made of the conditions

$$C_1\pi + C_2\pi + C_3 = u_2$$

$$C_2\pi + C_3 = u_1$$

$$C_3 = u_2$$

hence

$$C_3 = u_2, \quad C_1 = \frac{u_1 - u_2}{\pi}, \quad C_2 = \frac{u_2 - u_1}{\pi}$$

or in the final form

$$u = \frac{1}{\pi} \operatorname{Im} [(u_2 - u_1) \log_e (z + 1) + (u_1 - u_2) \log_e (z - 1)] + u_2 \quad (2-6-33)$$

The procedures described in the previous section may be used to solve several problems of steady-state heat conduction.

## 2-7. ONE-DIMENSIONAL UNSTEADY-STATE FIELD (PLATE, SPHERE, CYLINDER)

Solutions of problems on an unsteady-temperature field are treated in some fundamental works [2-18] and [2-22]. Here, we shall present some of the available generalized solutions. First of all we shall consider one-dimensional problems for the most simple configurations, such as an infinite plate ( $R \ll l$ ), an infinite cylinder ( $R \ll l$ ) and a sphere. Here  $R$  is the radius or one half of the plate thickness and  $l$  is the length.

There are two kinds of unsteady states, namely, aperiodic and periodic. In the aperiodic state temperature at any point of the body is a function of time. In the periodic unsteady state, temperature at any point of a body is a periodic function of time. The periodic variation may be either regular or irregular, but it must be cyclic. Regular periodic variation is characterized by a harmonic sinusoidal or cosinusoidal function, and irregular periodic changes are characterized by any cyclic function.

Heat-conduction differential equation may, for the present case, be written as

$$\frac{\partial T(x, \tau)}{\partial \tau} = a \left( \frac{\partial^2 T(x, \tau)}{\partial x^2} + \frac{\Gamma}{x} \frac{\partial T(x, \tau)}{\partial x} \right) + \frac{I_q(x, \tau)}{c\rho} \quad (2-7-1)$$

where  $\Gamma$  is a constant; for a plate  $\Gamma = 0$  ( $x \equiv x$ ), for a cylinder  $\Gamma = 1$  ( $x \equiv r$ ) and for a sphere  $\Gamma = 2$  ( $x \equiv r$ ).

In general, a heat source is a function of temperature. In some particular cases the equation may be written as

$$I_q(x, \tau) = Q(x, \tau) + b[T(x, \tau) - T_0] \quad (2-7-2)$$

where  $b$  is a certain time-dependent coefficient;  $T_0$  is the body temperature at the initial time, i.e.

$$T(x, 0) = T_0$$

Solutions will be presented for symmetrical problems, i.e.

$$\frac{\partial T(0, \tau)}{\partial x} = 0 \quad (2-7-3)$$

The average volumetric temperature  $\bar{T}(\tau)$  necessary for calculation of heat consumption is defined by

$$\bar{T}(\tau) = \frac{\Gamma + 1}{R^{\Gamma+1}} \int_0^R x^{\Gamma} T(x, \tau) dx \quad (2-7-4)$$

We shall use the notations

$$\left. \begin{aligned} \xi &= \frac{x}{R}; \quad \text{Fo} = \frac{a\tau}{R^2}; \quad \theta(\xi, \text{Fo}) = \frac{T(x, \tau) - T_0}{T_c - T_0} \\ A(\text{Fo}) &= \frac{b(\tau)R^2}{\lambda}; \quad \text{Po}(\xi, \text{Fo}) = \frac{Q(x, \tau)R^2}{\lambda(T_c - T_0)} \end{aligned} \right\} \quad (2-7-5)$$

where Fo is the Fourier number and Po is the Pomerantsev number.

The differential equation then takes the form

$$\frac{\partial \theta(\xi, \text{Fo})}{\partial \text{Fo}} = \frac{\partial^2 \theta(\xi, \text{Fo})}{\partial \xi^2} + \frac{\Gamma}{\xi} \frac{\partial \theta(\xi, \text{Fo})}{\partial \xi} + A(\text{Fo})\theta(\xi, \text{Fo}) + \text{Po}(\xi, \text{Fo}) \quad (2-7-6)$$

This equation will be solved for boundary conditions of the 1st, 2nd and 3d kinds and for an arbitrary temperature distribution at the initial moment.

To this end the following similarity criteria will be introduced

$$\left. \begin{aligned} \text{Ki}(\text{Fo}) &= \frac{q_n(\tau)R}{\lambda(T_c - T_0)}; \quad \text{Bi} = \frac{\alpha R}{\lambda} \\ \theta_c(\text{Fo}) &= \frac{T_c(\tau) - T_0}{T_c - T_0} \end{aligned} \right\} \quad (2-7-7)$$

where  $q_n(\tau)$  is the specific heat flux at the surface of the body;  $T_c(\tau)$  is the ambient temperature; Ki (Fo) and Bi are the Kirpichev and Biot numbers, respectively.

The initial condition is written as

$$\theta(\xi, 0) = f(\xi) \quad (2-7-8)$$

The boundary conditions are as follows:  
of the first kind

$$\theta(1, Fo) = \varphi(Fo) \quad (2-7-9)$$

of the second kind

$$\frac{\partial \theta(1, Fo)}{\partial \xi} = Ki(Fo) \quad (2-7-10)$$

of the third kind

$$\frac{1}{Bi} \frac{\partial \theta(1, Fo)}{\partial \xi} + \theta(1, Fo) = \theta_c(Fo) \quad (2-7-11)$$

The symmetry conditions are

$$\frac{\partial \theta(0, Fo)}{\partial \xi} = 0; \quad \theta(0, Fo) \neq \infty \quad (2-7-12)$$

The last inequality indicates that temperature at the center of the body cannot be infinite. The average dimensionless temperature

$$\bar{\theta}(Fo) = (\Gamma + 1) \int_0^1 \xi \Gamma_\theta(\xi, Fo) d\xi \quad (2-7-13)$$

To generalize solutions for a plate, cylinder and sphere, use will be made of the generalized finite integral transformation developed by M.D. Mikhailov. We shall introduce functions  $W_r(\xi)$  and  $V_r(\xi)$ , defined by relations (2-4-102) and (2-4-103). Equation (2-7-6) will be solved for boundary conditions (2-7-9) with the help of transformation (2-4-99). The average temperature is determined from formula (2-7-13).

For the boundary condition of the first kind the solutions are of the form [2-32]

$$\begin{aligned} \theta(\xi, Fo) = & \sum_{n=1}^{\infty} \frac{2W_r(\mu_n, \xi)}{V_r^2(\mu_n)} \exp \left[ \int_0^{Fo} A(Fo) dFo - \mu_n^2 Fo \right] \\ & \times \left\{ \int_0^1 \xi^\Gamma W_r(\mu_n, \xi) f(\xi) d\xi + \mu_n V_r(\mu_n) \int_0^{Fo} \varphi(Fo^*) \right. \\ & \times \exp \left[ \mu_n^2 Fo^* - \int_0^{Fo^*} A(Fo) dFo \right] dFo^* + \int_0^1 \int_0^{Fo} \xi^\Gamma W_r(\mu_n, \xi) Po(\xi, Fo^*) \\ & \left. \times \exp \left[ \mu_n^2 Fo^* - \int_0^{Fo^*} A(Fo) dFo \right] d\xi dFo^* \right\} \end{aligned} \quad (2-7-14)$$

The average dimensionless temperature equals

$$\begin{aligned}
 \bar{\theta}(\text{Fo}) = & \sum_{n=1}^{\infty} \frac{2(\Gamma + 1)}{\mu_n V_r(\mu_n)} \exp \left[ \int_0^{\text{Fo}} A(\text{Fo}) d\text{Fo} - \mu_n^2 \text{Fo} \right] \\
 & \times \left\{ \int_0^1 \xi^{\Gamma} W_r(\mu_n, \xi) f(\xi) d\xi + \mu_n V_r(\mu_n) \int_0^{\text{Fo}} \varphi(\text{Fo}^*) \right. \\
 & \times \exp \left[ \mu_n^2 \text{Fo}^* - \int_0^{\text{Fo}^*} A(\text{Fo}) d\text{Fo} \right] d\text{Fo}^* + \int_0^1 \int_0^{\text{Fo}} \xi^{\Gamma} W_r(\mu_n, \xi) \text{Po}(\xi, \text{Fo}^*) \\
 & \left. \times \exp \left[ \mu_n^2 \text{Fo} - \int_0^{\text{Fo}^*} A(\text{Fo}) d\text{Fo} \right] d\xi d\text{Fo}^* \right\} \quad (2-7-15)
 \end{aligned}$$

where  $\mu_n$  are the roots of the characteristic equation

$$W_r(\mu) = 0$$

If corresponding expressions for  $W_r(\xi)$  and  $V_r(\xi)$  taken from (2-4-94)-(2-4-96) are substituted into (2-7-14) and (2-7-15), then we obtain solutions for a plate, cylinder and sphere.

Let us now consider some particular cases

$$\text{Po}(\xi, \text{Fo}) = 0; \quad \varphi(\text{Fo}) = 0; \quad A = \text{const}$$

Solutions of (2-7-14) and (2-7-15) will assume the following forms

$$\theta(\xi, \text{Fo}) = \sum_{n=1}^{\infty} \frac{2W_r(\mu_n, \xi)}{V_r^2(\mu_n)} \exp[(A - \mu_n^2)\text{Fo}] \int_0^1 \xi^{\Gamma} W_r(\mu_n, \xi) f(\xi) d\xi \quad (2-7-16)$$

$$\bar{\theta}(\text{Fo}) = \sum_{n=1}^{\infty} \frac{2(\Gamma + 1)}{\mu_n V_r(\mu_n)} \exp[(A - \mu_n^2)\text{Fo}] \int_0^{\xi} \xi^{\Gamma} W_r(\mu_n, \xi) f(\xi) d\xi \quad (2-7-17)$$

If  $A = 0$  and  $f(\xi) = 1$  (a uniform initial temperature distribution), then from (2-7-16) and (2-7-17) we obtain

$$\theta(\xi, \text{Fo}) = \sum_{n=1}^{\infty} \frac{2W_r(\mu_n, \xi)}{\mu_n V_r(\mu_n)} e^{-\mu_n^2 \text{Fo}} \quad (2-7-18)$$

$$\bar{\theta}(\text{Fo}) = \sum_{n=1}^{\infty} \frac{2(\Gamma + 1)}{\mu_n^2} \exp(-\mu_n^2 \text{Fo}) \quad (2-7-19)$$

For small values of time or, more precisely, for small Fourier numbers, the solutions of (2-7-18) and (2-7-19) may be written in a Laplace form for a plate

$$\theta = \sum_{n=1}^{\infty} (-1)^{n+1} \left[ \operatorname{erfc} \frac{(2n-1) - \xi}{2\sqrt{\text{Fo}}} + \operatorname{erfc} \frac{(2n-1) + \xi}{2\sqrt{\text{Fo}}} \right] \quad (2-7-20)$$

$$\bar{\theta} = 1 - 2\sqrt{\frac{\text{Fo}}{\pi}} + 4\sqrt{\text{Fo}} \sum_{n=1}^{\infty} (-1)^{n+1} \operatorname{erfc} \frac{n}{\sqrt{\text{Fo}}} \quad (2-7-21)$$

for a sphere

$$\theta = 1 - \frac{1}{\xi} \sum_{n=1}^{\infty} \left[ \operatorname{erfc} \frac{(2n-1) - \xi}{2\sqrt{\text{Fo}}} - \operatorname{erfc} \frac{(2n-1) + \xi}{2\sqrt{\text{Fo}}} \right] \quad (2-7-22)$$

$$\bar{\theta} = 1 + 3\text{Fo} - 6\sqrt{\text{Fo}} \left[ \frac{1}{\sqrt{\pi}} + 2 \sum_{n=1}^{\infty} \operatorname{ierfc} \frac{n}{\sqrt{\text{Fo}}} \right] \quad (2-7-22')$$

For a cylinder only an approximate solution may be obtained

$$\begin{aligned} \theta \approx 1 - \left\{ \sqrt{\frac{1}{\xi}} \operatorname{erfc} \frac{1 - \xi}{2\sqrt{\text{Fo}}} + \frac{1}{4} \left( \frac{1}{\xi} - 1 \right) \sqrt{\frac{1}{\xi}} \text{Fo} \operatorname{ierfc} \frac{1 - \xi}{2\sqrt{\text{Fo}}} \right. \\ \left. + \frac{1}{32} \sqrt{\frac{1}{\xi}} \left[ 9 \left( \frac{1}{\xi} \right)^2 - 7 - 2 \left( \frac{1}{\xi} \right) \right] \text{Fo} i^2 \operatorname{erfc} \frac{1 - \xi}{2\sqrt{\text{Fo}}} + \dots \right\} \quad (2-7-23) \end{aligned}$$

For a parabolic initial temperature distribution  $f(\xi) = (1 - \xi^2)$ , the solutions will be of the form

$$\theta(\xi, \text{Fo}) = \sum_{n=1}^{\infty} \frac{4(\Gamma + 1)W_r(\mu_n, \xi)}{\mu_n^3 V_r(\mu_n)} \exp(-\mu_n^2 \text{Fo}) \quad (2-7-24)$$

$$\bar{\theta}(\text{Fo}) = \sum_{n=1}^{\infty} \frac{4(\Gamma + 1)^2}{\mu_n^4} \exp(-\mu_n^2 \text{Fo}) \quad (2-7-24')$$

A particular case of interest is when  $f(\xi) = W_r(\mu_1, \xi)$ . As demonstrated in [2-32]

$$\int_0^1 \xi^{\Gamma} W_r(\mu_n, \xi) W_r(\mu_1, \xi) d\xi = \begin{cases} 0; & (\mu_n \neq \mu_1) \\ \frac{V_r^2(\mu_n)}{2}; & (\mu_n = \mu_1) \end{cases} \quad (2-7-25)$$

Then the solutions of (2-7-16) and (2-7-17) will have the form (for  $\text{Po}(\xi, \text{Fo}) = 0$ ,  $\varphi(\text{Fo}) = 0$  and  $A = 0$ )

$$\theta(\xi, \text{Fo}) = W_r(\mu_1, \xi) \exp(-\mu_1^2 \text{Fo}) \quad (2-7-26)$$

$$\bar{\theta}(\text{Fo}) = \frac{\Gamma + 1}{\mu_1} V_r(\mu_1) \exp(-\mu_1^2 \text{Fo}) \quad (2-7-27)$$

The solutions of (2-7-26) and (2-7-27) are of interest as they describe the unsteady process of heating or cooling by a simple exponent. The given heating conditions are ordinarily called a regular thermal regime. Solutions of (2-7-26) and (2-7-27) imply that at a certain given initial temperature distribution regular conditions of heat transfer will be observed from the very beginning of the process. This conclusion was first made by the present author in 1941 [2-33].

For a heating process when  $Po(\xi, Fo) = 0$ ,  $\theta(\xi, 0) = f(\xi) = 0$ ,  $A = 0$  the solutions take the forms

$$\theta(\xi, Fo) = \sum_{n=1}^{\infty} \frac{2\mu_n W_r(\mu_n, \xi)}{V_r(\mu_n)} \exp(-\mu_n^2 Fo) \times \int_0^{Fo} \varphi(Fo^*) \exp(\mu_n^2 Fo^*) dFo^* \quad (2-7-28)$$

$$\bar{\theta}(\xi, Fo) = 2(\Gamma + 1) \sum_{n=1}^{\infty} \exp(-\mu_n^2 Fo) \int_0^{Fo} \varphi(Fo^*) \exp(\mu_n^2 Fo^*) dFo^* \quad (2-7-29)$$

Let us now consider a case when the surface temperature of a body is a linear time function  $\theta(1, Fo) = \varphi(Fo) = Pd Fo$ , where  $Pd$  is the Predvoditelev number [ $Pd = bR^2/a (T_c - T_0)$ ]. The solutions then become

$$\frac{\theta(\xi, Fo)}{Pd} = Fo - \frac{1 - \xi^2}{2(\Gamma + 1)} + \sum_{n=1}^{\infty} \frac{2W_r(\mu_n, \xi)}{\mu_n^3 V_r(\mu_n)} \exp(-\mu_n^2 Fo) \quad (2-7-30)$$

$$\frac{\bar{\theta}(Fo)}{Pd} = Fo - \frac{1}{(\Gamma + 1)(\Gamma + 3)} + \sum_{n=1}^{\infty} \frac{2(\Gamma + 1)}{\mu_n^4} \exp(-\mu_n^2 Fo) \quad (2-7-31)$$

If the surface temperature of the body follows the exponential law  $\theta(1, Fo) = \varphi(Fo) = 1 - \exp(-Pd Fo)$ , then the solutions will take the form

$$\theta(\xi, Fo) = 1 - \frac{W_r(\xi \sqrt{Pd})}{W_r(\sqrt{Pd})} \exp(-Pd Fo) \quad (2-7-32)$$

$$- \sum_{n=1}^{\infty} \frac{2W_r(\mu_n, \xi)}{\mu_n V_r(\mu_n) \left[ 1 - \frac{\mu_n^2}{Pd} \right]} \exp(-\mu_n^2 Fo)$$

$$\bar{\theta}(Fo) = 1 - \frac{(\Gamma + 1)W_r(\sqrt{Pd})}{\sqrt{Pd} W_r(\sqrt{Pd})} \exp(-Pd Fo)$$

$$- \sum_{n=1}^{\infty} \frac{2(\Gamma + 1)}{\mu_n^2 \left( 1 - \frac{\mu_n^2}{Pd} \right)} \exp(-\mu_n^2 Fo) \quad (2-7-33)$$



For the case  $\theta(\xi, 0) = 0$ ,  $\varphi(\text{Fo}) = 0$ ,  $A=0$  and  $\text{Po}(\xi, \text{Fo}) = \text{Po} = \text{const}$  the solutions are as follows

$$\frac{\theta(\xi, \text{Fo})}{\text{Po}} = \frac{1 - \xi^2}{2(\Gamma + 1)} - \sum_{n=1}^{\infty} \frac{2W_r(\mu_n, \xi)}{\mu_n^3 V_r(\mu_n)} \exp(-\mu_n^2 \text{Fo}) \quad (2-7-34)$$

$$\frac{\bar{\theta}(\text{Fo})}{\text{Po}} = \frac{1}{(\Gamma + 1)(\Gamma + 3)} - \sum_{n=1}^{\infty} \frac{2(\Gamma + 1)}{\mu_n^4} \exp(-\mu_n^2 \text{Fo}) \quad (2-7-35)$$

If a heat source is a parabolic time function, then the solutions become

$$\frac{\theta(\xi, \text{Fo})}{\text{Po}} = \frac{1 - \xi^2}{2(\Gamma + 1)} - \frac{1 - \xi^4}{2(\Gamma + 3)} - 2(\Gamma + 1) \sum_{n=1}^{\infty} \frac{2W_r(\mu_n, \xi)}{\mu_n^5 V_r(\mu_n)} \exp(-\mu_n^2 \text{Fo}) \quad (2-7-36)$$

$$\frac{\bar{\theta}(\text{Fo})}{\text{Po}} = \frac{4}{(\Gamma + 1)(\Gamma + 3)(\Gamma + 5)} - \sum_{n=1}^{\infty} \frac{4(\Gamma + 1)^2}{\mu_n^6} \exp(-\mu_n^2 \text{Fo}) \quad (2-7-37)$$

The heat source  $\text{Po}(\xi, \text{Fo}) = \text{Po} W_r(\mu_1, \xi)$ . In this particular case we obtain solutions

$$\frac{\theta(\xi, \text{Fo})}{\text{Po}} = W_r(\mu_1, \xi) [1 - \exp(-\mu_1^2 \text{Fo})] \quad (2-7-38)$$

$$\frac{\bar{\theta}(\text{Fo})}{\text{Po}} = \frac{\Gamma + 1}{\mu_1} V_m(\mu_1) [1 - \exp(-\mu_1^2 \text{Fo})] \quad (2-7-39)$$

If a heat source is an exponential function of time  $\text{Po}(\xi, \text{Fo}) = \text{Po} \exp(-\text{Pd} \text{Fo})$ , then the solutions are

$$\begin{aligned} \frac{\theta(\xi, \text{Fo})}{\text{Po}} = & \left[ \frac{W_r(\xi \sqrt{\text{Pd}})}{W_r(\sqrt{\text{Pd}})} - 1 \right] \frac{1}{\text{Pd}} \exp(-\text{Pd} \text{Fo}) \\ & - \sum_{n=1}^{\infty} \frac{2W_r(\mu_n, \xi)}{\mu_n(\mu_n^2 - \text{Pd})V_r(\mu_n)} \exp(-\mu_n^2 \text{Fo}) \end{aligned} \quad (2-7-40)$$

$$\begin{aligned} \frac{\bar{\theta}(\text{Fo})}{\text{Po}} = & \left[ \frac{(\Gamma + 1)V_r(\sqrt{\text{Pd}})}{\sqrt{\text{Pd}} W_r(\sqrt{\text{Pd}})} - 1 \right] \frac{1}{\text{Pd}} \exp(-\text{Pd} \text{Fo}) \\ & - \sum_{n=1}^{\infty} \frac{2(\Gamma + 1)}{\mu_n^2(\mu_n^2 - \text{Pd})} \exp(-\mu_n^2 \text{Fo}) \end{aligned} \quad (2-7-41)$$

The solutions of the differential equation (2-7-6) for the boundary condition (2-7-10) of the second kind are as follows [2-32]

$$\begin{aligned}
 \theta(\xi, Fo) = & (\Gamma + 1) \exp \left[ \int_0^{Fo} A(Fo^*) dFo^* \right] \left\{ \int_0^1 \xi^{\Gamma} f(\xi) d\xi \right. \\
 & + \int_0^{Fo} Ki(Fo^*) \exp \left[ - \int_0^{Fo^*} A(Fo^{**}) dFo^{**} \right] dFo^* \\
 & + \int_0^1 \int_0^{Fo} \xi^{\Gamma} Po(\xi, Fo^*) \exp \left[ - \int_0^{Fo^*} A(Fo^{**}) dFo^{**} \right] d\xi dFo^* \Big\} \\
 & + \sum_{n=1}^{\infty} \frac{2W_r(\mu_n, \xi)}{W_r^2(\mu_n)} \exp \left[ \int_0^{Fo} A(Fo^*) dFo^* - \mu_n^2 Fo \right] \\
 & \times \left\{ \int_0^1 \xi^{\Gamma} W_r(\mu_n, \xi) f(\xi) d\xi + W_r(\mu_n) \int_0^{Fo} Ki(Fo^*) \exp \left[ \mu_n^2 Fo^* \right. \right. \\
 & \left. \left. - \int_0^{Fo^*} A(Fo^{**}) dFo^{**} \right] dFo^* + \int_0^1 \int_0^{Fo} \xi^{\Gamma} W_r(\mu_n, \xi) Po(\xi, Fo^*) \exp \left[ \mu_n^2 Fo^* \right. \right. \\
 & \left. \left. - \int_0^{Fo^*} A(Fo^{**}) dFo^{**} \right] d\xi dFo^* \right\} \quad (2-7-42)
 \end{aligned}$$

$$\begin{aligned}
 \bar{\theta}(Fo) = & (\Gamma + 1) \exp \left[ \int_0^{Fo} A(Fo^{**}) dFo^{**} \right] \left\{ \int_0^1 \xi^{\Gamma} f(\xi) d\xi \right. \\
 & + \int_0^{Fo} Ki(Fo^*) \exp \left[ - \int_0^{Fo^*} A(Fo^{**}) dFo^{**} \right] dFo^* \\
 & + \int_0^1 \int_0^{Fo} \xi^{\Gamma} Po(\xi, Fo^*) \exp \left[ - \int_0^{Fo^*} A(Fo^{**}) dFo^{**} \right] d\xi dFo^* \Big\} \quad (2-7-43)
 \end{aligned}$$

Let us now consider a particular example

$$A = 0; \quad \text{Po}(\xi, \text{Fo}) = 0; \quad \theta(\xi, 0) = f(\xi) = 0; \quad \text{Ki}(\text{Fo}) = \text{Ki} = \text{const}$$

In this particular case we have

$$\begin{aligned} \frac{\theta(\xi, \text{Fo})}{\text{Ki}} &= (\Gamma + 1) \text{Fo} + \frac{1}{2} \left( \xi^2 + \frac{\Gamma + 1}{\Gamma + 3} \right) \\ &- \sum_{n=1}^{\infty} \frac{2W_r(\mu_n, \xi)}{\mu_n^2 W_r(\mu_n)} \exp(-\mu_n^2 \text{Fo}) \end{aligned} \quad (2-7-44)$$

For small Fourier numbers the solutions may be written in the Laplace form

for a plate

$$\theta = 2 \text{Ki} \sqrt{\text{Fo}} \sum_{n=1}^{\infty} \left[ \text{ierfc} \frac{(2n-1) - \xi}{2\sqrt{\text{Fo}}} + \text{ierfc} \frac{(2n-1) + \xi}{2\sqrt{\text{Fo}}} \right] \quad (2-7-45)$$

for a sphere when  $\xi > 0$

$$\theta = \frac{\text{Ki}}{\xi} \left[ \exp \left( \text{Fo} - \frac{1}{\xi} + 1 \right) \text{erfc} \left( \frac{1 - \xi}{2\sqrt{\text{Fo}}} - \sqrt{\text{Fo}} \right) - \text{erfc} \frac{1 - \xi}{2\sqrt{\text{Fo}}} \right] \quad (2-7-46)$$

for a cylinder when  $\xi > 0$  (an approximate solution)

$$\theta \approx \frac{\text{Ki} \sqrt{\text{Fo}}}{\sqrt{\xi}} \left[ \text{ierfc} \frac{1 - \xi}{2\sqrt{\text{Fo}}} + \frac{(1 + 3\xi)\text{Fo}}{2\xi \sqrt{\xi}} i^2 \text{erfc} \frac{1 - \xi}{2\sqrt{\text{Fo}}} + \dots \right] \quad (2-7-47)$$

The average temperature is

$$\frac{\bar{\theta}(\text{Fo})}{\text{Ki}} = (\Gamma + 1) \text{Fo}$$

The Kirpichev number varies with time following the exponential law  $\text{Ki}(\text{Fo}) = \text{Ki} \exp(-\text{Pd} \text{Fo})$  provided  $A = 0$ ;  $\text{Po}(\xi, \text{Fo}) = 0$

$$\begin{aligned} \frac{\theta(\xi, \text{Fo})}{\text{Ki}} &= \frac{\Gamma + 1}{\text{Pd}} - \frac{W_r(\xi \sqrt{\text{Pd}})}{\sqrt{\text{Pd}} V_r(\sqrt{\text{Pd}})} \exp(-\text{Pd} \text{Fo}) \\ &- \sum_{n=1}^{\infty} \frac{2W_r(\mu_n, \xi)}{(\mu_n^2 - \text{Pd}^2) W_r(\mu_n)} \exp(-\mu_n^2 \text{Fo}) \end{aligned} \quad (2-7-48)$$

$$\frac{\bar{\theta}(\text{Fo})}{\text{Ki}} = \frac{\Gamma + 1}{\text{Pd}} [1 - \exp(-\text{Pd} \text{Fo})] \quad (2-7-49)$$

The solutions for the boundary condition of the third kind may be written as [2-32]

$$\begin{aligned} \theta(\xi, \text{Fo}) = & \sum_{n=1}^{\infty} \frac{2W_r(\mu_n, \xi)}{V_r^2(\mu_n)} \frac{\text{Bi}^2}{\mu_n^2 + \text{Bi}^2 + (1 - \Gamma) \text{Bi}} \\ & \times \exp \left[ \int_0^{\text{Fo}} A(\text{Fo}^*) d\text{Fo}^* - \mu_n^2 \text{Fo} \right] \left\{ \int_0^1 \xi^{\Gamma} W_r(\mu_n, \xi) f(\xi) d\xi \right. \\ & + \mu_n^2 V_r(\mu_n) \int_0^{\text{Fo}} \theta_c(\text{Fo}^*) \exp \left[ \mu_n^2 \text{Fo}^* - \int_0^{\text{Fo}^*} A(\text{Fo}^*) d\text{Fo}^* \right] d\text{Fo}^* + \int_0^1 \int_0^{\text{Fo}} \xi^{\Gamma} \\ & \times W_r(\mu_n, \xi) \text{Po}(\xi, \text{Fo}^*) \exp \left[ \mu_n^2 \text{Fo}^* - \int_0^{\text{Fo}^*} A(\text{Fo}^*) d\text{Fo}^* \right] d\xi d\text{Fo}^* \left. \right\} \end{aligned} \quad (2-7-50)$$

$$\begin{aligned} \bar{\theta}(\text{Fo}) = & \sum_{n=1}^{\infty} \frac{2(\Gamma + 1)}{\mu_n V_r(\mu_n)} \frac{\text{Bi}^2}{\mu_n^2 + \text{Bi}^2 + (1 - \Gamma) \text{Bi}} \\ & \times \exp \left[ \int_0^{\text{Fo}} A(\text{Fo}^*) d\text{Fo}^* - \mu_n^2 \text{Fo} \right] \left\{ \int_0^1 \xi^{\Gamma} W_r(\mu_n, \xi) f(\xi) d\xi + \mu_n V_r(\mu_n) \int_0^{\text{Fo}} \theta_c(\text{Fo}^*) \right. \\ & \times \exp \left[ \mu_n^2 \text{Fo}^* - \int_0^{\text{Fo}^*} A(\text{Fo}^*) d\text{Fo}^* \right] d\text{Fo}^* + \int_0^1 \int_0^{\text{Fo}} \xi^{\Gamma} W_r(\mu_n, \xi) \\ & \times \text{Po}(\xi, \text{Fo}^*) \exp \left[ \mu_n^2 \text{Fo}^* - \int_0^{\text{Fo}^*} A(\text{Fo}^*) d\text{Fo}^* \right] d\xi d\text{Fo}^* \left. \right\} \end{aligned} \quad (2-7-51)$$

where  $\mu_n$  are the roots of the characteristic equation

$$\frac{W_r(\mu)}{V_r(\mu)} = \frac{\mu}{\text{Bi}} \quad (2-7-52)$$

Solutions (2-7-14) and (2-7-15) may be obtained from the solutions of (2-7-50) and (2-7-51) if it is assumed that  $\text{Bi} = \infty$ .

Let us consider some particular cases.

1.  $A(\text{Fo}) = 0$ ;  $\text{Po}(\xi, \text{Fo}) = 0$ ;  $f(\xi) = 0$ ;  $\theta_c(\text{Fo}) = 1$

$$\theta(\xi, \text{Fo}) = 1 - \sum_{n=1}^{\infty} \frac{2W_r(\mu_n, \xi)}{\mu_n V_r(\mu_n)} C_n \exp(-\mu_n^2 \text{Fo}) \quad (2-7-53)$$

$$\bar{\theta}(\text{Fo}) = 1 - \sum_{n=1}^{\infty} \frac{2(\Gamma + 1)}{\mu_n^2} C_n \exp(-\mu_n^2 \text{Fo}) \quad (2-7-54)$$

where  $C_n = \frac{\text{Bi}^2}{\mu_n^2 + \text{Bi}^2(1 - \Gamma)\text{Bi}}$  are constants depending on the Biot criterion Bi (see Tables 2-2 to 2-4).

For a plate ( $\Gamma = 0$ ) an approximate solution may be given which is valid for small Fourier numbers

$$\theta \approx \text{erfc} \frac{1 - \xi}{2\sqrt{\text{Fo}}} - \exp[\text{Bi}(1 - \xi) + \text{Bi}^2 \text{Fo}] \text{erfc} \left( \frac{1 - \xi}{2\sqrt{\text{Fo}}} + \text{Bi} \sqrt{\text{Fo}} \right) \\ + \text{erfc} \frac{1 + \xi}{2\sqrt{\text{Fo}}} - \exp[\text{Bi}(1 + \xi) + \text{Bi}^2 \text{Fo}] \text{erfc} \left( \frac{1 + \xi}{2\sqrt{\text{Fo}}} + \text{Bi} \sqrt{\text{Fo}} \right) \quad (2-7-55)$$

Table 2-2

Values of Coefficients  $C_n$  ( $n = 1, 2, \dots, 6$ ) for a Plate at Different Values of Bi

Bi	$C_1$	$C_2$	$C_3$	$C_4$	$C_5$	$C_6$
0.01	0.0050	0.0000				
0.02	0.0099	0.0000				
0.04	0.0197	0.0002	0.0000			
0.06	0.0294	0.0004	0.0001	0.0000		
0.08	0.0390	0.0006	0.0002	0.0001	0.0000	
0.10	0.0484	0.0010	0.0002	0.0001	0.0001	0.0000
0.15	0.0714	0.0022	0.0006	0.0002	0.0001	0.0001
0.20	0.0936	0.0038	0.0010	0.0004	0.0002	0.0002
0.30	0.1359	0.0083	0.0022	0.0010	0.0006	0.0004
0.40	0.1754	0.0143	0.0039	0.0018	0.0010	0.0006
0.50	0.2124	0.0216	0.0061	0.0028	0.0016	0.0010
0.60	0.2471	0.0300	0.0086	0.0040	0.0022	0.0014
0.70	0.2795	0.0395	0.0116	0.0054	0.0030	0.0020
0.80	0.3098	0.0499	0.0151	0.0070	0.0040	0.0026
0.90	0.3383	0.0610	0.0188	0.0088	0.0050	0.0032
1.0	0.3649	0.0728	0.0230	0.0108	0.0062	0.0040
1.5	0.4760	0.1381	0.0488	0.0236	0.0137	0.0089
2.0	0.5587	0.2075	0.0812	0.0405	0.0238	0.0156
3.0	0.6705	0.3395	0.1580	0.0845	0.0512	0.0340
4.0	0.7408	0.4509	0.2408	0.1376	0.0862	0.0583
5.0	0.7880	0.5403	0.3216	0.1955	0.1267	0.0873
6.0	0.8215	0.6111	0.3960	0.2547	0.1706	0.1200
7.0	0.8484	0.6673	0.4627	0.3127	0.2163	0.1552
8.0	0.8654	0.7122	0.5212	0.3680	0.2623	0.1920
9.0	0.8804	0.7484	0.5722	0.4196	0.3076	0.2295
10.0	0.8925	0.7780	0.6163	0.4672	0.3514	0.2671
15.0	0.9291	0.8668	0.7635	0.6466	0.5360	0.4407
20.0	0.9473	0.9087	0.8400	0.7540	0.6629	0.5754
30.0	0.9653	0.9466	0.9111	0.8625	0.8051	0.7431
40.0	0.9742	0.9632	0.9419	0.9116	0.8740	0.8312
50.0	0.9795	0.9722	0.9581	0.9376	0.9116	0.8811
60.0	0.9830	0.9779	0.9678	0.9531	0.9342	0.9116
80.0	0.9873	0.9844	0.9786	0.9700	0.9588	0.9454
100.0	0.9899	0.9880	0.9842	0.9786	0.9712	0.9622

Table 2-3

Values of Coefficients  $C_n$  ( $n = 1, 2, \dots, 6$ ) for a Sphere at Different Values of Bi

Bi	$C_1$	$C_2$	$C_3$	$C_4$	$C_5$	$C_6$
0.01	0.0050	0.0000				
0.02	0.0100	0.0000				
0.04	0.0198	0.0001	0.0000			
0.06	0.0296	0.0002	0.0001	0.0000		
0.08	0.0394	0.0003	0.0001	0.0000		
0.10	0.0490	0.0005	0.0002	0.0001	0.0000	
0.15	0.0728	0.0011	0.0004	0.0002	0.0001	0.0001
0.20	0.0960	0.0020	0.0007	0.0003	0.0002	0.0001
0.30	0.1411	0.0044	0.0015	0.0008	0.0004	0.0003
0.40	0.1842	0.0077	0.0026	0.0013	0.0008	0.0005
0.50	0.2255	0.0119	0.0041	0.0021	0.0012	0.0008
0.60	0.2650	0.0170	0.0059	0.0030	0.0018	0.0012
0.70	0.3026	0.0229	0.0080	0.0041	0.0025	0.0016
0.80	0.3385	0.0296	0.0105	0.0053	0.0032	0.0021
0.90	0.3727	0.0370	0.0132	0.0067	0.0040	0.0027
1.0	0.4053	0.0450	0.0162	0.0083	0.0050	0.0033
1.5	0.5457	0.0940	0.0355	0.0183	0.0112	0.0075
2.0	0.6540	0.1530	0.0609	0.0320	0.0196	0.0132
3.0	0.8008	0.2823	0.1258	0.0688	0.0429	0.0292
4.0	0.8874	0.4063	0.2017	0.1154	0.0735	0.0506
5.0	0.9396	0.5137	0.2811	0.1684	0.1098	0.0766
6.0	0.9719	0.6025	0.3585	0.2248	0.1504	0.1064
7.0	0.9923	0.6743	0.4308	0.2820	0.1936	0.1392
8.0	1.0055	0.7319	0.4963	0.3383	0.2382	0.1740
9.0	1.0142	0.7780	0.5547	0.3922	0.2830	0.2102
10.0	1.0199	0.8151	0.6062	0.4429	0.3271	0.2469
15.0	1.0292	0.9197	0.7799	0.6415	0.5211	0.4230
20.0	1.0285	0.9621	0.8682	0.7632	0.6599	0.5656
30.0	1.0236	0.9924	0.9442	0.8840	0.8168	0.7472
40.0	1.0195	1.0015	0.9729	0.9355	0.8913	0.8426
50.0	1.0165	1.0048	0.9860	0.9609	0.9303	0.8955
60.0	1.0142	1.0061	0.9928	0.9749	0.9527	0.9269
80.0	1.0111	1.0065	0.9990	0.9886	0.9755	0.9600
100.0	1.0091	1.0062	1.0013	0.9946	0.9860	0.9758

2. The ambient temperature is a linear time function

$$\frac{\theta(\xi, Fo)}{Pd} = Fo - \frac{1 + \frac{2}{Bi} - \xi^2}{2(\Gamma + 1)} + \sum_{n=1}^{\infty} \frac{2W_r(\mu_n, \xi)}{\mu_n^3 V_r(\mu_n)} C_n \exp(-\mu_n^2 Fo) \quad (2-7-56)$$

$$\frac{\bar{\theta}(Fo)}{Pd} = Fo - \frac{1 + \frac{\Gamma + 3}{Bi}}{(\Gamma + 1)(\Gamma + 3)} + \sum_{n=1}^{\infty} \frac{2(\Gamma + 1)}{\mu_n^4} C_n \exp(-\mu_n^2 Fo) \quad (2-7-57)$$

An analysis of the solution of (2-7-56) reveals that starting from a definite value  $Fo \geq Fo_1$  the sum may be neglected. The temperature at any point of the body will then be a linear time function, and the temperature distribution will be described by a parabola.

Table 2-4

Values of  $C_n$  ( $n = 1, 2, \dots, 6$ ) for a Cylinder at Different Values of Bi

Bi	$ C_1 $	$ C_2 $	$ C_3 $	$ C_4 $	$ C_5 $	$ C_6 $
0.01	0.0050	0.0000				
0.02	0.0100	0.0000				
0.04	0.0198	0.0001	0.0000			
0.06	0.0296	0.0002	0.0001	0.0000		
0.08	0.0392	0.0004	0.0001	0.0001	0.0000	
0.10	0.0488	0.0007	0.0002	0.0001	0.0000	
0.15	0.0722	0.0015	0.0004	0.0002	0.0001	0.0001
0.20	0.0951	0.0026	0.0008	0.0004	0.0002	0.0001
0.30	0.1390	0.0058	0.0018	0.0009	0.0005	0.0003
0.40	0.1808	0.0102	0.0032	0.0015	0.0009	0.0006
0.50	0.2202	0.0157	0.0050	0.0024	0.0014	0.0009
0.60	0.2576	0.0222	0.0071	0.0034	0.0020	0.0013
0.70	0.2930	0.0296	0.0096	0.0046	0.0027	0.0018
0.80	0.3265	0.0379	0.0124	0.0060	0.0036	0.0023
0.90	0.3581	0.0469	0.0155	0.0076	0.0045	0.0030
1.0	0.3880	0.0567	0.0192	0.0094	0.0055	0.0036
1.5	0.5146	0.1136	0.0413	0.0207	0.0123	0.0081
2.0	0.6099	0.1785	0.0700	0.0359	0.0216	0.0143
3.0	0.7377	0.3112	0.1408	0.0760	0.0468	0.0314
4.0	0.8146	0.4304	0.2205	0.1258	0.0795	0.0542
5.0	0.8633	0.5295	0.3011	0.1814	0.1179	0.0817
6.0	0.8956	0.6094	0.3776	0.2393	0.1601	0.1129
7.0	0.9179	0.6732	0.4474	0.2972	0.2046	0.1469
8.0	0.9339	0.7241	0.5097	0.3532	0.2500	0.1827
9.0	0.9457	0.7649	0.5645	0.4061	0.2951	0.2196
10.0	0.9546	0.7979	0.6123	0.4544	0.3392	0.2568
15.0	0.9780	0.8936	0.7724	0.6446	0.5289	0.4320
20.0	0.9871	0.9354	0.8545	0.7591	0.6618	0.5708
30.0	0.9940	0.9693	0.9277	0.8734	0.8112	0.7454
40.0	0.9966	0.9822	0.9573	0.9236	0.8828	0.8370
50.0	0.9978	0.9884	0.9720	0.9492	0.9210	0.8883
60.0	0.9984	0.9919	0.9803	0.9640	0.9434	0.9193
80.0	0.9991	0.9954	0.9887	0.9792	0.9671	0.9526
100.0	0.9994	0.9970	0.9927	0.9866	0.9786	0.9690

3. The ambient temperature is an exponential time function

$\theta_c(\text{Fo}) = 1 - \exp(\text{Pd Fo})$  when  $A(\text{Fo}) = 0$ ;  $\text{Po}(\xi, \text{Fo}) = 0$ ,  $f(\xi) = 0$

$$\theta(\xi, \text{Fo}) = 1 - \frac{W_r(\xi \sqrt{\text{Pd}})}{W_r(\sqrt{\text{Pd}}) - \frac{\sqrt{\text{Pd}}}{\text{Bi}} V_r(\sqrt{\text{Pd}})} \exp(-\text{Pd Fo}) - \sum_{n=1}^{\infty} \frac{2W_r(\mu_n, \xi)}{\mu_n V_r(\mu_n) \left[1 - \frac{\mu_n^2}{\text{Pd}}\right]} C_n \exp(-\mu_n^2 \text{Fo}) \quad (2-7-58)$$

$$\bar{\theta}(\text{Fo}) = 1 - \frac{(\Gamma + 1) \exp(-\text{Pd Fo})}{\frac{\sqrt{\text{Pd}} W_r(\sqrt{\text{Pd}})}{V_r(\sqrt{\text{Pd}})} - \frac{\text{Pd}}{\text{Bi}}} - \sum_{n=1}^{\infty} \frac{2(\Gamma + 1)}{\mu_n^2 \left(1 - \frac{\mu_n^2}{\text{Pd}}\right)} C_n \exp(-\mu_n^2 \text{Fo}) \quad (2-7-59)$$

$$C_n = \frac{\text{Bi}^2}{\mu_n^2 + \text{Bi}^2 + (1 - \Gamma)\text{Bi}} \quad (2-7-60)$$

The numerical values of coefficients  $C_n$  for various geometric bodies are listed in Tables 2-2 through 2-4.

Solution of unsteady-state homogeneous asymmetric problems for an infinite plate at boundary conditions of the third kind (Fig. 2-7) may be obtained using the integral transform

$$T_I(\mu_n, \tau) = \int_0^l K\left(\mu_n, \frac{x}{l}\right) T(x, \tau) dx \quad (2-7-61)$$

Inversion is made following the formula

$$T(x, \tau) = \sum_{n=1}^{\infty} K\left(\mu_n, \frac{x}{l}\right) T_I(\mu_n, \tau) \quad (2-7-62)$$

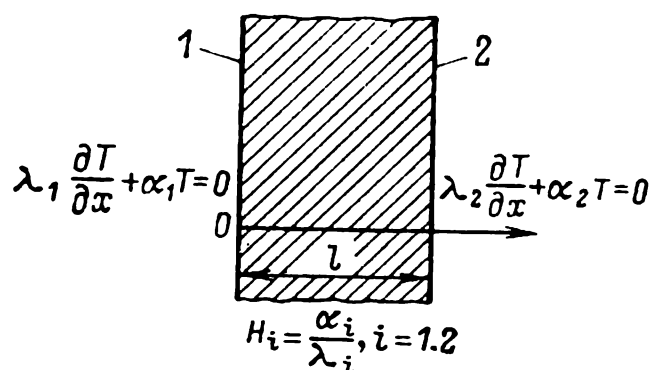


Fig. 2-7. Calculation of integral transform kernels (Table 2-5) for asymmetric heat transfer in an infinite plate



Kernels of Finite Integral Transform and Characteristic Equation for an Infinite Plate

Boundary conditions		Kernel $K(\mu_n, x/l)$	Characteristic equations for $\mu_n$
at $x = 0$	at $x = l$		
$H_1 = \infty$	$H_2 = \infty$	$\sqrt{\frac{2}{l}} \sin \mu_n x/l$	$\sin \mu = 0$
$H_1 = \infty$	$H_2 = 0$	$\sqrt{\frac{2}{l}} \sin \mu_n x/l$	$\cos \mu = 0$
$H_1 = 0$	$H_2 = \infty$	$\sqrt{\frac{2}{l}} \cos \mu_n x/l$	$\sin \mu = 0$
$H_1 = 0$	$H_2 = 0$	$\sqrt{\frac{2}{l}} \cos \mu_n x/l^*$	$\sin \mu = 0$
$H_1 = 0$	$H_2 = \text{const}$	$\sqrt{\frac{2}{l}} \left[ \frac{\mu_n^2 + H_2^2 l^2}{\mu_n^2 + H_2^2 l^2 + H_2 l} \right]^{1/2} \cos \mu_n x/l$	$\mu \tan \mu = H_2 l$
$H_1 = \text{const}$	$H_2 = \infty$	$\sqrt{\frac{2}{l}} \left[ \frac{\mu_n^2 + H_1^2 l^2}{\mu_n^2 + H_1^2 l^2 + H_1 l} \right]^{1/2} \sin \mu_n (1 - x/l)$	$\mu \cot \mu = -H_1 l$
$H_1 = \text{const}$	$H_2 = 0$	$\sqrt{\frac{2}{l}} \left[ \frac{\mu_n^2 + H_1^2 l^2}{\mu_n^2 + H_1^2 l^2 + H_1 l} \right]^{1/2} \cos \mu_n (1 - x/l)$	$\mu \tan \mu = H_1 l$
$H_1 = \text{const}$	$H_2 = \text{const}$	$\sqrt{\frac{2}{l}} \frac{\mu_n \cos \mu_n x/l + H_1 l \sin \mu_n x/l}{\left[ (\mu_n^2 + H_1^2 l^2) \left( 1 + \frac{H_2 l}{\mu_n^2 + H_2^2 l^2} + H_1 l \right) \right]^{1/2}}$	$\tan \mu = \frac{\mu(H_1 l + H_2 l)}{\mu^2 - H_1 H_2 l^2}$

Note: The first root  $\mu$  is zero.  $H_1 = \infty$  are boundary conditions of the 1-st kind,  $H_2 = 0$  are boundary conditions of the second kind,  $H_2 = \text{const}$  are boundary conditions of the third kind;  $\mu = \beta l$ .

Kernels  $K\left(\mu_n, \frac{x}{l}\right)$  for different boundary conditions are tabulated in Table 2-5.

To illustrate what has been mentioned above we shall consider a particular example.

The boundary conditions for an infinite plate are assumed as:

$$\lambda T_{,x} + \alpha_i T = f_i(\tau); \quad i = 1 \text{ at } x = 0; \quad i = 2 \text{ at } x = l$$

The solution for the transform is of the form

$$T_I(\beta_n, \tau) = e^{-a\beta_n^2 \tau} \left\{ f_0(\beta_n) + \int_0^\tau e^{a\beta_n^2 \tau} A(\beta_n, \tau) d\tau \right\} \quad (2-7-63)$$

where

$$A(\beta_n, \tau) = \frac{a}{\lambda_1} Q_I(\beta_n, \tau) + \left[ \frac{aK(\beta_n, x)}{\lambda} f_1(\tau) \right]_{x=0} + \left[ \frac{K(\beta_n, x)}{\lambda_2} \right]_{x=l} f_2(\tau)$$

$$Q_I(\beta_n, \tau) = \int_0^l K(\beta_n, x) Q(x, \tau) dx$$

where  $Q(x, \tau)$  is a heat source.

The solution for the original function

$$T(x, \tau) = \sum_{n=1}^{\infty} e^{-a\beta_n^2 \tau} K(\beta_n, x) \left\{ f_0(\beta_n) + \int_0^\tau e^{a\beta_n^2 \tau} A(\beta_n, \tau) d\tau \right\} \quad (2-7-64)$$

Let  $f_1(\tau) = f_2(\tau) = 0$  and  $f_0(x) = 0$ . Then the kernel of the integral transform from Table 2-5 (see the case  $H_1 = \text{const}$ ,  $H_2 = \text{const}$ ) is

$$K(\beta_n, x) = \frac{(\beta_n \cos \beta_n x + H_1 \sin \beta_n x) \sqrt{2}}{\{(\beta_n^2 + H_1^2)[l + H_2/(\beta_n^2 + H_2^2)] + H_1\}^{\frac{1}{2}}} \quad (2-7-65)$$

The solution of the problem is

$$T(x, \tau) = \sum_{n=1}^{\infty} e^{-a\beta_n^2 \tau} K(\beta_n, x) \int_0^\tau e^{a\beta_n^2 \tau} \frac{a}{\lambda_1} Q_I(\beta_n, \tau) d\tau \quad (2-7-66)$$

Let us assume that

$$Q(x, \tau) = Q_0 \delta(x - b) \delta(\tau - 0) \quad (2-7-67)$$

where  $Q_0$  is an instantaneous heat source located in plane  $x = b$ .

Then

$$T(x, \tau) = \frac{2Q_0}{c\rho l} \sum_{n=1}^{\infty} \frac{(\mu_n \cos \mu_n x/l + \text{Bi}_1 \sin \mu_n x/l)(\mu_n \cos \mu_n b/l + \text{Bi}_1 \sin \mu_n b/l)}{(\mu_n^2 + \text{Bi}_1^2)(1 + \text{Bi}_2)/(\mu_n^2 + \text{Bi}_2^2) + \text{Bi}_1} \times \exp(-a\mu_n^2 \tau/l^2) \quad (2-7-68)$$

If  $\alpha_1 = 0$ ,  $\partial T(0, \tau)/\partial x = 0$ ,  $\text{Bi}_1 = 0$ ,  $\text{Bi}_2 = \text{Bi}$ , then from (2-7-68) we get

$$T(x, \tau) = \frac{2Q_0}{c\rho l} \sum_{n=1}^{\infty} \frac{\mu_n}{\mu_n + \sin \mu_n \cos \mu_n} \cos \mu_n x/l \cos \mu_n b/l e^{-\mu_n^2 \text{Fo}} \quad (2-7-69)$$

because

$$1 + \text{Bi} (\mu_n^2 + \text{Bi}^2)^{-1} = \mu_n^{-1} (\mu_n + \sin \mu_n \cos \mu_n)$$

The remaining cases with constant heat sources or with uniform initial temperature distribution are solved in a similar way.

To conclude the section we shall consider very simple one-dimensional problems (plate, cylinder and sphere) generalized with the aid of the Bessel equations of a semi-integral order.

The heat-conduction equation (2-7-6) for the case  $A(\text{Fo}) = \text{Po}(\xi, \text{Fo}) = 0$  and for uniform initial temperature distribution may be solved using the Laplace transform

$$L[f(\text{Fo})] = f_L(s) = \int_0^{\infty} f(\text{Fo}) \exp(-s\text{Fo}) d\text{Fo} \quad (2-7-70)$$

Application of transform (2-7-70) to equation (2-7-6) yields

$$\theta_L''(\xi, s) + \frac{2\nu + 1}{\xi} \theta_L'(\xi, s) - s \theta_L(\xi, s) = 0 \quad (2-7-71)$$

where  $\nu = (\Gamma - 1)/2$ , i.e. for a plate  $\Gamma = 0$ ,  $\nu = -\frac{1}{2}$ ; for a cylinder

$\Gamma = 1$ ,  $\nu = 0$  and for a sphere  $\Gamma = 2$ ,  $\nu = 1/2$ .

Substitution of variables in equation (2-7-71)

$$\theta = u(\sqrt{s\xi})^{-\nu} \quad (2-7-72)$$

gives the modified Bessel equation of the  $\nu$ -th order

$$\xi^2 u'' + \xi u' - (\nu^2 + s\xi^2)u = 0 \quad (2-7-73)$$

Its solution is of the form

$$u = A_1 I_{\nu}(\sqrt{s\xi}) + A_2 K_{\nu}(\sqrt{s\xi}) \quad (2-7-74)$$

where  $I_\nu$  and  $K_\nu$  are the modified Bessel functions of the  $\nu$ -th order of the first and second kinds.

For symmetrical problems  $A_2 = 0$ , as for  $\xi \rightarrow 0$ ,  $K_\nu \rightarrow \infty$ , the solutions for transforms take the form

$$\theta_L(\xi, s) = A(\sqrt{s} \xi)^{-\nu} I_\nu(\sqrt{s} \xi) \quad (2-7-75)$$

For boundary conditions of the first kind (a constant surface temperature)

$$\theta_L(1, s) = \frac{\theta_c}{s} \quad (2-7-76)$$

The solution of (2-7-75) will take the form

$$\theta_L(\xi, s) = \frac{\theta_c I_\nu(\sqrt{s} \xi)}{\xi^\nu I_\nu(\sqrt{s})} = \frac{\Phi(s)}{\psi(s)} \quad (2-7-77)$$

Let us use the expansion theorem

$$\theta(\xi, \text{Fo}) = \lim_{s \rightarrow 0} \frac{\Phi(s)}{\psi'(s)} + \sum_{n=1}^{\infty} \frac{\Phi(s_n)}{\psi'(s_n)} e^{s_n \text{Fo}} \quad (2-7-78)$$

where  $s_n$  are the roots of the characteristic equation

$$I_\nu(\sqrt{s}) = 0 \quad (2-7-79)$$

The solution then takes the form

$$\frac{\theta(\xi, \text{Fo})}{\theta_c} = 1 - \sum_{n=1}^{\infty} \left( \frac{1}{\xi} \right)^\nu \frac{2}{\mu_n} \frac{J_\nu(\mu_n \xi)}{J_{\nu+1}(\mu_n)} \exp(-\mu_n^2 \text{Fo}) \quad (2-7-80)$$

where  $\mu_n = i \sqrt{s_n}$  and  $J_\nu$  is the first kind Bessel function of the  $\nu$ -th order

$$i^\nu I_\nu(x) = J_\nu(i, x)$$

The characteristic equation may be written as

$$J_\nu(\mu) = 0 \quad (2-7-81)$$

The solution of (2-7-80) is the generalized solution for a plate, cylinder and sphere.

For a plate  $\nu = -1/2$

$$J_{-\frac{1}{2}}(\mu_n, \xi) = \sqrt{\frac{2}{\mu_n \pi \xi}} \cos \mu_n \xi \quad (2-7-82)$$

Consequently, according to (2-7-81),  $\cos \mu_n = 0$  and  $\mu_n = (2n - 1) \frac{\pi}{2}$ .

Then from (2-7-80) we get

$$\frac{\theta(\xi, Fo)}{\theta_c} = 1 - \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{\mu_n} \cos \mu_n \xi e^{-\mu_n^2 Fo} \quad (2-7-83)$$

For a cylinder  $v = 0$

$$\frac{\theta(\xi, Fo)}{\theta_c} = 1 - \sum_{n=1}^{\infty} \frac{2}{\mu_n} \frac{J_0(\mu_n \xi)}{J_1(\mu_n)} e^{-\mu_n^2 Fo} \quad (2-7-84)$$

The solution for a sphere  $\left(v = \frac{1}{2}\right)$  is found from the relations

$$J_{\frac{1}{2}}(\mu_n \xi) = \left(\frac{2}{\pi \mu_n \xi}\right)^{\frac{1}{2}} \sin \mu_n \xi \quad (2-7-85)$$

$$J_{\frac{3}{2}}(\mu_n) = \left(\frac{2}{\pi \mu_n}\right)^{\frac{1}{2}} \left(\frac{\sin \mu_n}{\mu_n} - \cos \mu_n\right) \quad (2-7-86)$$

From (2-7-82) we find  $\sin \mu_n = 0$  and  $\mu_n = n\pi$ .  
Therefore,

$$\frac{\theta(\xi, Fo)}{\theta_c} = 1 - \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{\mu_n} \frac{\sin \mu_n \xi}{\xi} e^{-\mu_n^2 Fo} \quad (2-7-87)$$

For boundary conditions of the third kind

$$-\frac{\partial \theta_L(1, s)}{\partial \xi} + \text{Bi} \left[ \frac{\theta_c}{s} - \theta_L(1, s) \right] = 0 \quad (2-7-88)$$

The solution of equation (2-7-77) becomes

$$\frac{\theta_L(\xi, s)}{\theta_c} = \frac{1}{s \xi^v} \frac{I_v(\sqrt{s} \xi)}{\left[ I_v(\sqrt{s}) + \frac{1}{\text{Bi}} \sqrt{s} I_{v+1}(\sqrt{s}) \right]} \quad (2-7-89)$$

The solution for the original will be

$$\frac{\theta(\xi, \tau)}{\theta_c} = 1 - \sum_{n=1}^{\infty} \frac{2\text{Bi}^2}{\mu_n} \xi^{-v} \frac{J_v(\mu_n \xi)}{(\text{Bi}^2 + \mu_n^2 - 2v\text{Bi}) J_{v+1}(\mu_n)} \exp(-\mu_n^2 Fo) \quad (2-7-90)$$

where  $\mu_n = i\sqrt{s_n}$  and  $s_n$  is found from the characteristic equation which can be written as follows:

$$I_v(\sqrt{s}) = -\frac{\sqrt{s}}{\text{Bi}} I_{v+1}(\sqrt{s}) \quad (2-7-91)$$

$$\frac{J_v(\mu)}{J_{v+1}(\mu)} = \frac{\mu}{\text{Bi}} \quad (2-7-92)$$

Substitution of suitable values of  $\nu$  gives the solution for a plate, cylinder and sphere and the appropriate characteristic equations.

For boundary conditions of the second kind with a constant heat flux the transforms take the form

$$\frac{\partial \theta_L(1, s)}{\partial \xi} = \frac{Ki\theta_c}{s} = 0 \quad (2-7-93)$$

The solutions for the transform become

$$\theta_L(\xi, s) = \frac{Ki\theta_c J_\nu(i\sqrt{s}\xi)}{\xi^\nu s i \sqrt{s} J_{\nu+1}(i\sqrt{s})} \quad (2-7-94)$$

To determine the multiplicity of the root  $s = 0$ , the solution for the transform is rewritten as

$$\frac{\theta_L(\xi, s)}{\theta_c} = \frac{Ki J_\nu(i\sqrt{s}\xi)}{i\xi^\nu s^2 F(s)} = \frac{\Phi(s)}{s^2 \psi(s)} \quad (2-7-95)$$

where

$$F(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k + \nu + 2)k!} \left(\frac{i}{2}\right)^{2k+\nu+1} s^{\frac{2k+\nu}{2}}$$

We have the following roots:  $s = 0$  (a twofold root) and an infinite number of the roots  $i\sqrt{s_n} = \mu_n$  which are found from the characteristic equation

$$J_{\nu+1}(\mu) = 0 \quad (2-7-96)$$

Use of the expansion theorem and of the relation

$$\lim_{s \rightarrow 0} \left\{ \frac{d}{ds} \left[ \frac{\Phi(s)}{s^2 \psi(s)} e^{sFo} \right] \right\} = Ki \left[ Fo \cdot 2(\nu + 1) + \frac{1}{2} \xi^2 + \frac{\nu}{15} (\nu - 2) - \frac{1}{4} \right] \quad (2-7-97)$$

yields the solution for the original [2-37]

$$\begin{aligned} \frac{\theta(\xi, Fo)}{\theta_c} &= Ki \left[ 2(\nu + 1)Fo + \frac{1}{2} \xi^2 + \frac{1}{15} \nu(\nu - 2) - \frac{1}{4} \right] - \\ &- \frac{1}{\xi^\nu} \sum_{n=1}^{\infty} \frac{2}{\mu_n^2} \frac{J_\nu(\mu_n \xi)}{J'_{\nu+1}(\mu_n)} e^{-\mu_n^2 Fo} \end{aligned} \quad (2-7-98)$$

## 2-8. TEMPERATURE WAVES

Among unsteady temperature fields, the fields having periodic temperature variations of the body are of great interest.

Let us consider the simplest case, when the surface temperature of a semi-infinite body varies following a simple harmonic oscillation

$$\vartheta_n = T_n - \bar{T}_n = T_{mn} \cos \omega \tau \quad (2-8-1)$$

where  $\bar{T}_n$  is the mean surface temperature;  $T_{mn}$  is the amplitude of temperature variation at the surface (total range of temperature variation);  $\omega$  is the cyclic frequency of temperature oscillation ( $\omega = 2\pi\nu$ ,  $\nu$  is the number of oscillations per unit time).

At the beginning of the process the initial temperature distribution affects the temperature field of the body. After a definite period, a steady periodic state sets in which is characterized by the harmonic oscillations of temperature at any point of the body, the amplitude gradually decreases as the point moves away from the surface of the body (developed heat flow of a steady-state type).

The solution for a semi-infinite body takes the form

$$\theta = \frac{T - \bar{T}_n}{T_{mn}} = e^{-\sqrt{\frac{\omega}{2a}}x} \cos\left(\omega\tau - \sqrt{\frac{\omega}{2a}}x\right) \quad (2-8-2)$$

The decrease in amplitude with the depth may be seen from Fig. 2-8. Curves exhibiting temperature variations with time at different depths are shown in Fig. 2-8. The maxima and minima of temperature oscillations

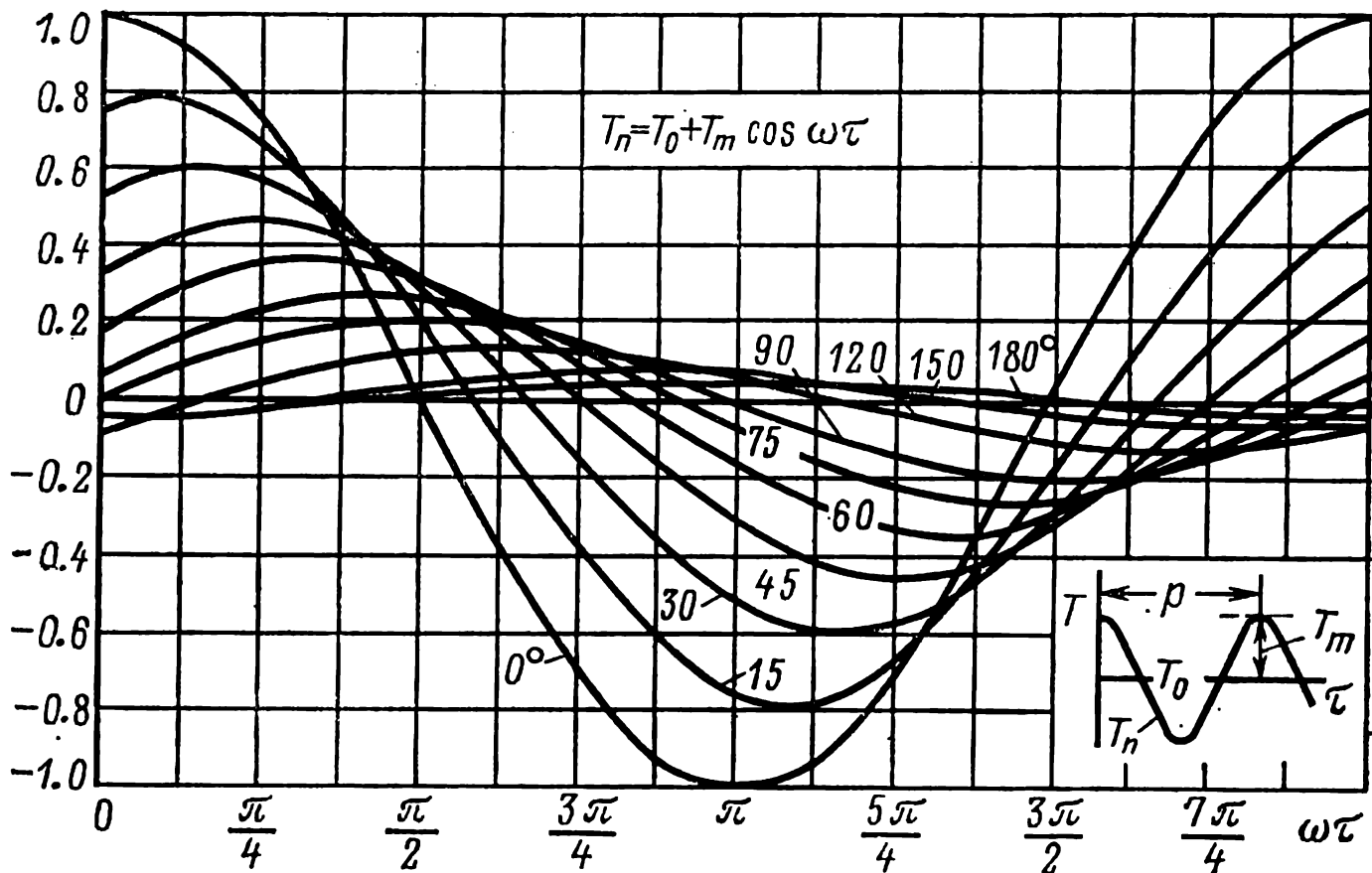


Fig. 2-8. Plot of temperature  $\theta$  versus time  $\omega\tau$  in a semi-infinite body at various depths  $x(\pi/aP)^{1/2}$  when surface temperature varies following the harmonic oscillation law

oscillations decrease with depth, and as the distance from the surface increases they occur with ever greater delay. The temperature at the surface attains its mean value  $\bar{T}_n$  when  $\cos \omega\tau = 0$ , i.e. at  $(\tau_1)_{x=0} = \frac{\pi}{2} \omega^{-1}$ .

At a depth  $x$  the temperature becomes equal to  $\bar{T}$  when

$$\cos\left(\omega\tau - \sqrt{\frac{\omega}{2a}}x\right) = 0, \text{ or at the moment}$$

$$(\tau_1)_x = \frac{\pi}{2}\omega^{-1} + \sqrt{\frac{1}{2a\omega}}x \quad (2-8-3)$$

which occurs after an interval of

$$\Delta\tau = \sqrt{\frac{1}{2a\omega}}x = \frac{1}{2}\sqrt{\frac{P}{\pi a}}x \quad (2-8-4)$$

where  $P$  is the oscillation period ( $\omega = 2\pi/P$ ).

The quantity  $\Delta\tau$  is the time delay of the temperature wave. For a given depth the time delay is directly proportional to the square root of the temperature oscillation period.

Let us denote the temperature wave propagation velocity by  $u$ , then

$$u = \frac{x}{\Delta\tau} = \sqrt{2a\omega} = 2\sqrt{\frac{\pi a}{P}} \quad (2-8-5)$$

The wave length  $l$  of periodic temperature equals the distance covered by the wave during one oscillation, i.e. in time equal to the oscillation period

$$l = uP = \pi\sqrt{\frac{8a}{\omega}} = 2\sqrt{\pi aP} \quad (2-8-6)$$

The heat flux per unit area of the solid surface, kcal/(m<sup>2</sup>·h)

$$q_n = -\lambda\left(\frac{\partial T}{\partial x}\right)_{x=0} = -\lambda T_{mn}\sqrt{\frac{\omega}{2a}}(\sin\omega\tau - \cos\omega\tau) \quad (2-8-7)$$

By using the known equality  $\cos(\omega\tau + \pi/4) = (\cos\omega\tau - \sin\omega\tau)/\sqrt{2}$ , for an instantaneous heat flux at the surface of the body we obtain the following expression

$$q_n = T_{mn}\sqrt{\lambda c\rho\omega}\cos(\omega\tau + \pi/4) \quad (2-8-8)$$

Integration of this equation gives the total heat flow on the surface area  $A$ (m<sup>2</sup>). In this case  $Q_n$  will be positive when  $\cos(\omega\tau + \pi/4) > 0$ , i.e. when the cosine argument varies between  $-\pi/2$  and  $+\pi/2$  which corresponds to the interval from  $\tau_1 = 3\pi/4\omega$  to  $\tau_2 = \pi/4\omega$ . Taking this into consideration,



we get

$$Q_n = A \int_{\tau_1}^{\tau_2} q_n d\tau = 2AT_{mc} \sqrt{\frac{\lambda c \rho}{\omega}} \quad (2-8-9)$$

The amount of heat  $Q_n$  consumed within the first half of the cycle is positive, during the rest of the cycle the same amount of heat is removed from the surface of the body.

If the ambient temperature varies according to the law of simple harmonic oscillation, then heat transfer between the surface of a semi-infinite body and the surroundings obeys Newton's law (boundary condition of the third kind)

$$-\lambda \frac{\partial T(0, \tau)}{\partial x} + \alpha [T_{mc} \cos \omega \tau - T(0, \tau)] = 0 \quad (2-8-10)$$

In this case the solution is of the form

$$\theta = \frac{T(x, \tau) - T_c}{T_{mc}} = \theta_A e^{-x \sqrt{\frac{\pi}{aP}}} \cos \left( \frac{2\pi}{P} \tau - x \sqrt{\frac{\pi}{aP}} - M \right) \quad (2-8-11)$$

where

$$\theta_A = \left( 1 + \frac{2\lambda}{\alpha} \sqrt{\frac{\pi}{aP}} + \frac{2\pi\lambda^2}{\alpha^2 aP} \right)^{-\frac{1}{2}}$$

is the maximum dimensionless amplitude of temperature oscillations at the surface of the body;  $T_{mc}$  is the mean ambient temperature;  $M$  is the phase shift of surface temperature oscillations with respect to ambient temperature oscillations

$$M = \tan^{-1} \left( \frac{1}{1 + \frac{\alpha}{\lambda} \sqrt{\frac{aP}{\pi}}} \right) \quad (2-8-12)$$

If the Biot number  $Bi = \infty$  ( $\alpha/\lambda = \infty$ ), the amplitude of the oscillation of relative temperature at the surface equals unity. The phase shift between the cosine curves  $\theta_c(\tau)$  and  $\theta_n(\tau)$  equals  $M$ ; when  $H = \infty$  this shift will be equal to  $\pi/2$ .

Let us write solution (2-8-11) in terms of generalized variables, and denote the local Fourier number as  $Fo_x = a\tau/x^2$ , the local Predvoditelev

number as  $Pd_x = \omega x^2/a$ , the Fourier number (periodicity number) as  $Fo_{xp} = aP/2\pi x^2$  and the generalized Biot number  $Bi^*$  as

$$Bi^* = \frac{\alpha}{\sqrt{\lambda c \rho \omega}} \quad (2-8-13)$$

then

$$\theta = \frac{T(x, \tau) - \bar{T}_n}{T_{mc}} = \theta_A \exp \left[ -\sqrt{\frac{1}{2}} Pd_x \right] \cos \left( \frac{Fo_x}{Fo_{xp}} - \sqrt{\frac{1}{2}} Pd_x + M \right) \quad (2-8-14)$$

where

$$\theta_A = \left[ 1 + \frac{\sqrt{2}}{Bi^*} + \left( \frac{1}{Bi^*} \right)^2 \right]^{-1/2} \quad (2-8-15)$$

$$M = \tan^{-1} \left( \frac{1}{1 + \sqrt{2} Bi^*} \right) \quad (2-8-16)$$

Now we shall explain the physical significance of the Biot number  $Bi^*$ . Instantaneous heat flux at the surface of a body equals

$$q(0, \tau) = -\lambda \left( \frac{\partial T}{\partial X} \right)_{x=0} = -\lambda T_{mn} \sqrt{\frac{\omega}{2a}} (\sin \omega \tau - \cos \omega \tau) \quad (2-8-1')$$

By using the relation

$$\cos \left( \omega \tau + \frac{\pi}{4} \right) = (\cos \omega \tau - \sin \omega \tau)(\sqrt{2})^{-1}$$

we may rewrite the equation for instantaneous heat flux at the solid surface of the body as

$$q(0, \tau) = \sqrt{\lambda c \gamma \omega} T_{mn} \cos \left( \omega \tau + \frac{\pi}{4} \right) = q_m \cos \left( \omega \tau + \frac{\pi}{4} \right) \quad (2-8)$$

where  $q_m$  is the maximum specific heat flux (or the amplitude of oscillation of the specific heat flux), equal to

$$q_m = T_{mn} \sqrt{\lambda c \rho \omega}$$

The ratio of heat flux oscillation amplitude to temperature

$$\frac{q_m}{T_{mn}} = \sqrt{\lambda c \rho \omega}$$

The value of  $\sqrt{\lambda c \rho \omega}$  is thus equal to the maximum instantaneous heat flux supplied to the solid surface when the amplitude of temperature oscillation at the surface of the wall equals unity ( $\theta_a T_{mc} = T_{mn} = 1^\circ\text{K}$ ).

The Biot number  $\text{Bi}^*$  thus is the ratio of the steady heat flux  $\alpha \Delta T$  with unit temperature head ( $\Delta T = 1$ ) to the maximum heat flux  $\sqrt{\lambda c \rho \omega}$  at a periodically steady state when the amplitude of temperature oscillation equals unity ( $T_{mn} = 1$ ).

In other words,  $\text{Bi}^*$  is the modified Biot number for a periodically steady state.

The solutions for an infinite plate, sphere and infinite cylinder may be written as

$$= \frac{1}{2} (N_i e^{i \text{Pd} \text{Fo}} + N_{-i} e^{-i \text{Pd} \text{Fo}}) \sum_{n=1}^{\infty} \frac{\mu_n^4}{\mu_n^4 + \text{Pd}^4} \frac{2W_r(\mu_n, \xi)}{\mu_n V_r(\mu_n)} C_n \exp(-\mu_n^2 \text{Fo}) \quad (2-8-21)$$

where  $\text{Pd}$  is the Predvoditelev number

$$= \frac{2n \nu R^2}{a} = \frac{\omega R^2}{a} \quad (2-8-22)$$

and  $N_{-i}$  are functions of the  $\text{Bi}$  and  $\text{Pd}$  numbers and of  $\xi$  [2-18]. Analysis reveals that coefficients  $N_i$  and  $N_{-i}$  satisfy the Dirichlet condition. Therefore they may be expanded in series in the range  $(-R \leq x \leq R)$  respect to fundamental functions  $W(\mu_n, \xi)$ .

After a few simple transformations, solution (2-8-21) may be written as

$$\theta(\xi, \text{Pd}) = \sum_{n=1}^{\infty} \frac{\mu_n^4}{\mu_n^4 + \text{Pd}^2} \frac{2W_r(\mu_n, \xi)}{\mu_n V_r(\mu_n)} C_n \left\{ \cos \text{Pd} \text{Fo} - \exp\left(-\frac{\text{Pd}}{\mu_n^2} \mu_n^2 \text{Fo}\right) \right\} = \frac{T(\xi, \text{Fo}, \text{Pd})}{T_{mc}} \quad (2-8-23)$$

Given enough time the quantity  $e^{-\mu_n^2 \text{Fo}}$  will be small, and then the periodic state will set in.

Mean temperature will be equal to

$$\bar{\theta} = \sum_{n=1}^{\infty} \frac{2(\Gamma + 1)}{\mu_n^2} \left( \frac{\mu_n^4}{\mu_n^4 + \text{Pd}^2} \right) C_n \left( \cos \text{Pd} \text{Fo} - \tan^{-1} \frac{\text{Pd}}{\mu_n^2} \right) \quad (2-8-24)$$

Hence, the mean temperature of the body will be a periodic function of time.

The specific heat flux rate for  $\Delta\tau = \tau_2 - \tau_1$  will be equal to

$$\Delta Q_v = c\rho[\bar{\theta}(\tau_2) - \bar{\theta}(\tau_1)]T_{mc} \quad (2-8-25)$$

that is

$$\Delta Q_v = T_{mc} c\rho \sum_{n=1}^{\infty} \frac{2(\Gamma + 1)}{\mu_n^2} C_n \frac{\mu_n^4}{\mu_n^4 + \text{Pd}^2} [\cos \text{Pd Fo}_2 - \cos \text{Pd Fo}_1] \quad (2-8-26)$$

Hence

$$\Delta Q_v = c\rho T_{mc} 2 \sum_{n=1}^{\infty} \frac{2(\Gamma + 1)}{\mu_n^2} C_n \frac{\mu_n^4}{\mu_n^4 + \text{Pd}^2} \sin \frac{\pi}{P} (\tau_1 - \tau_2) \sin \frac{\pi}{P} (\tau_1 + \tau_2) \quad (2-8-27)$$

Let us assume  $\tau_1 = 0$  (zero time), and  $\tau_2 = \frac{1}{2} P$ , i.e. let us consider the amount of heat spent in a half-period. In this case, both  $\sin \frac{\pi}{P} (\tau_1 + \tau_2)$  and  $\sin \frac{\pi}{P} (\tau_2 - \tau_1)$  will be equal to unity.

Let us introduce the following notation

$$K_\psi(\text{Bi}, \text{Pd}) = \sum_{n=1}^{\infty} \frac{\mu_n^4}{\mu_n^4 + \text{Pd}^2} \frac{2(\Gamma + 1)}{\mu_n^2} C_n \quad (2-8-28)$$

The heat spent in a half-period will then be expressed as

$$\Delta Q_v = 2c\rho T_{mc} K_\psi(\text{Bi}, \text{Pd}) \quad (2-8-29)$$

The quantity  $K_\psi$  is called the heat accumulation coefficient and is a function of the Pd and Bi numbers; when  $\text{Pd} \rightarrow 0$  (transition from periodic temperature variation to an aperiodic one)  $K_\psi \rightarrow 1$ , because

$$\sum_{n=1}^{\infty} \frac{2(\Gamma + 1) C_n}{\mu_n^2} = 1$$

The thermal amplitude coefficient of aperiodic mean temperature variation is denoted by

$$B_n = \frac{2(\Gamma + 1) C_n}{\mu_n^2} \quad (2-8-30)$$

In Tables 2-6 through 2-8 numerical values of  $B_n$  are presented as a function of Bi. When  $\text{Pd} \rightarrow \infty$ ,  $K_\psi = 0$ . For a thin plate when  $\text{Pd} \rightarrow 0$  and  $\text{Bi} \rightarrow 0$  the heat-accumulation coefficient equals unity. Therefore, the

Table 2-6

Values of  $B_n$  as a Function of Bi for an Infinite Plate

$$B_n = \frac{2 \text{Bi}^2}{\mu_n^2 (\mu_n^2 + \text{Bi}^2 + \text{Bi})}, \quad n = 1, 2, 3, \dots, 6$$

Bi	$B_1$	$B_2$	$B_3$	$B_4$	$B_5$	$B_6$
$\infty$	0.8106	0.0901	0.0324	0.0165	0.0100	0.0067
50.0	0.8260	0.0911	0.0323	0.0161	0.0095	0.0061
30.0	0.8355	0.0910	0.0315	0.0152	0.0086	0.0053
15.0	0.8565	0.0885	0.0279	0.0120	0.0060	0.0032
10.0	0.8743	0.0839	0.0236	0.0090	0.0040	0.0020
9.0	0.8796	0.0821	0.0222	0.0081	0.0035	0.0017
8.0	0.8858	0.0797	0.0205	0.0072	0.0030	0.0015
7.0	0.8932	0.0766	0.0185	0.0062	0.0025	0.0012
6.0	0.9021	0.0723	0.0162	0.0051	0.0020	0.0009
5.0	0.9131	0.0664	0.0135	0.0040	0.0015	0.0007
4.0	0.9264	0.0582	0.0104	0.0028	0.0010	0.0005
3.0	0.9430	0.0468	0.0070	0.0018	0.0006	0.0003
2.0	0.9635	0.0313	0.0038	0.0009	0.0003	0.0001
1.5	0.9748	0.0220	0.0023	0.0005	0.0002	0.0001
1.0	0.9861	0.0124	0.0011	0.0002	0.0001	
0.9	0.9882	0.0106	0.0009	0.0002	0.0001	
0.8	0.9902	0.0088	0.0007	0.0002		
0.7	0.9922	0.0070	0.0006	0.0001		
0.6	0.9940	0.0054	0.0004	0.0001		
0.5	0.9956	0.0040	0.0003	0.0001		
0.4	0.9971	0.0027	0.0002			
0.3	0.9983	0.0016	0.0001			
0.2	0.9992	0.0007				
0.1	0.9998	0.0002				

coefficient  $K_\psi$  numerically equals the ratio of heat accumulated by the wall of thickness  $2R$  to that accumulated by an infinitely thin wall ( $2R \rightarrow 0$ ) made of the same material under similar conditions of periodic heating.

Table 2-7

Values of  $B_n$  as a Function of Bi for a Sphere

$$B_n = \frac{6\text{Bi}^2}{\mu_n^2(\mu_n^2 + \text{Bi}^2 - \text{Bi})}, n = 1, 2, 3, \dots, 6$$

Bi	$B_1$	$B_2$	$B_3$	$B_4$	$B_5$	$B_6$
$\infty$	0.6079	0.1520	0.0675	0.0380	0.0243	0.0169
51.0	0.6427	0.1588	0.0693	0.0380	0.0236	0.0158
21.0	0.6886	0.1166	0.0652	0.0324	0.0180	0.0108
10.0	0.7607	0.1496	0.0485	0.0196	0.0091	0.0047
9.0	0.7737	0.1453	0.0450	0.0175	0.0079	0.0040
8.0	0.7889	0.1396	0.0408	0.0152	0.0067	0.0033
7.0	0.8068	0.1319	0.0360	0.0128	0.0055	0.0027
6.0	0.8281	0.1215	0.0305	0.0104	0.0044	0.0021
5.0	0.8532	0.1075	0.0245	0.0079	0.0032	0.0015
4.0	0.8830	0.0890	0.0180	0.0055	0.0021	0.0010
3.0	0.9170	0.0655	0.0115	0.0033	0.0013	0.0006
2.5	0.9352	0.0520	0.0085	0.0024	0.0009	0.0004
2.0	0.9534	0.0380	0.0057	0.0016	0.0006	0.0003
1.9	0.9570	0.0352	0.0052	0.0014	0.0005	0.0002
1.8	0.9605	0.0324	0.0047	0.0013	0.0005	0.0002
1.7	0.9640	0.0297	0.0043	0.0011	0.0004	0.0002
1.6	0.9674	0.0270	0.0038	0.0010	0.0004	0.0002
1.5	0.9707	0.0243	0.0034	0.0009	0.0003	0.0002
1.4	0.9739	0.0217	0.0030	0.0008	0.0003	0.0001
1.3	0.9770	0.0192	0.0026	0.0007	0.0002	0.0001
1.2	0.9800	0.0167	0.0022	0.0006	0.0002	0.0001
1.1	0.9828	0.0144	0.0019	0.0005	0.0002	0.0001
1.0	0.9855	0.0122	0.0016	0.0004	0.0002	0.0001
0.9	0.9880	0.0101	0.0013	0.0003	0.0001	
0.80	0.9904	0.0081	0.0010	0.0003	0.0001	
0.70	0.9925	0.0064	0.0008	0.0002	0.0001	
0.60	0.9944	0.0048	0.0006	0.0002		
0.50	0.9960	0.0034	0.0004	0.0001		
0.40	0.9974	0.0022	0.0003	0.0001		
0.30	0.9985	0.0013	0.0001			
0.20	0.9993	0.0006	0.0001			
0.15	0.9996	0.0003				
0.10	0.9998	0.0001				
0.09	1.0000	0.0001				
0.08	1.0000	0.0001				

The coefficient  $K_\psi$  is, therefore, referred to as the heat utilization coefficient, and equals

$$K_\psi = \Delta Q_v / (\Delta Q_v)_{k \rightarrow 0} \quad (2-8-31)$$

From formula (2-8-9) the amount of heat accumulated by a half-space (an infinitely thick wall  $l = 2R \rightarrow \infty$ )

$$(\Delta Q_v)_{l \rightarrow \infty} = 2T_{mn}c\rho \sqrt{\frac{a}{\omega}} \frac{1}{l} = 2T_{mn}c\rho \frac{1}{2\sqrt{\text{Pd}}} \quad (2-8-32)$$

Table 2-8

Values of  $B_n$  as a Function of Bi for a Cylinder

$$B_n = \frac{4\text{Bi}^2}{\mu_n^2(\mu_n^2 + \text{Bi}^2)}, \quad n = 1, 2, 3, \dots, 6$$

Bi	$B_1$	$B_2$	$B_3$	$B_4$	$B_5$	$B_6$
$\infty$	0.6917	0.1313	0.0534	0.0288	0.0179	0.0122
50.0	0.7183	0.1350	0.0540	0.0248	0.0172	0.0113
30.0	0.7348	0.1359	0.0529	0.0268	0.0155	0.0097
10.0	0.8039	0.1260	0.0387	0.0152	0.0070	0.0036
9.0	0.8133	0.1229	0.0361	0.0137	0.0061	0.0030
8.0	0.8244	0.1188	0.0331	0.0120	0.0052	0.0006
7.0	0.8375	0.1132	0.0296	0.0102	0.0043	0.0021
6.0	0.8532	0.1056	0.0254	0.0084	0.0034	0.0016
5.0	0.8721	0.0953	0.0207	0.0064	0.0025	0.0012
4.0	0.8950	0.0813	0.0156	0.0045	0.0017	0.0008
3.0	0.9224	0.0625	0.0102	0.0028	0.0010	0.0004
2.0	0.9536	0.0388	0.0053	0.0013	0.0005	0.0002
1.5	0.9696	0.0259	0.0032	0.0008	0.0003	0.0001
1.0	0.9843	0.0136	0.0015	0.0004	0.0001	
0.90	0.9869	0.0114	0.0012	0.0003	0.0001	
0.80	0.9893	0.0093	0.0010	0.0002	0.0001	
0.70	0.9916	0.0074	0.0008	0.0002	0.0001	
0.60	0.9936	0.0056	0.0006	0.0001		
0.50	0.9954	0.0040	0.0004	0.0001		
0.40	0.9970	0.0026	0.0002			
0.30	0.9983	0.0015	0.0001			
0.20	0.9992	0.0007	0.0001			
0.15	0.9995	0.0004				
0.10	0.9998	0.0002				
0.08	0.9999	0.0001				

In this case the heat accumulation coefficient  $K_\psi = \frac{1}{2\sqrt{\text{Pd}}}$ .

For a half-space, as a characteristic of heat accumulation the amount of heat absorbed by a unit area of the wall in a half-period is calculated, i.e

$$\Delta Q_s = \Delta Q_v l = 2c\rho T_{mn} \sqrt{\frac{a}{\omega}} \quad (2-8-33)$$

The physical significance of relation (2-8-33) may be explained as follows.  $\Delta Q_s$  equals the amount of heat received by a unit area of the wall layer of thickness  $\sqrt{\frac{a}{\omega}}$  (the quantity  $\sqrt{\frac{a}{\omega}}$  is measured in the units of length) uni-

formly heated from  $-T_{mn}$  to  $T_{mn}$  across the whole thickness. Thus,  $\sqrt{\frac{a}{\omega}}$  is the conventional thickness of a uniformly heated layer of a semi-infinite body in the periodically steady state. The quantity  $\sqrt{\frac{a}{\omega}}$ , which will further be denoted by  $\varepsilon$ , is in its physical significance the coefficient of heat utilization of a homogeneous wall. The coefficient  $\varepsilon$  is about 9 times less than the temperature wave length  $\Lambda$  ( $\varepsilon = 0.11 \Lambda$ ), as

$$\Lambda = \sqrt{8\pi} \sqrt{\frac{a}{\omega}} = \sqrt{8\pi} \varepsilon \quad (2-8-34)$$

The heat-utilization coefficient is inversely proportional to  $\sqrt{\omega}$  and directly proportional to  $\sqrt{a}$ . The heat-utilization coefficient, therefore, decreases with an increase in temperature oscillation frequency. At high frequencies the heat-utilization coefficient is small. At a constant frequency ( $\omega = \text{const}$ ) the heat-utilization coefficient depends only on the thermal-diffusivity coefficient. For example, for an oscillation period of 24 Hz ( $\omega = \pi/12$ ), the heat-utilization coefficient of cork slabs ( $\varepsilon = 0.039$  m) is about 3.5 times less than that of marble slabs ( $\varepsilon = 0.137$  m).

If heat transfer through an air layer takes place by conduction, then the heat-utilization coefficient is very large ( $\varepsilon = 0.543$ ,  $\omega = \pi/12$ ,  $T = 293^\circ\text{K}$ ).

The heat-utilization coefficient characterizes the temperature oscillation damping intensity in the bulk of the wall.

The depth  $X_n$  at which temperature oscillations decrease  $n$  times in comparison with the oscillation at the surface is

$$X_n = \sqrt{2} \varepsilon \log_e n = \Lambda f(n) \quad (2-8-35)$$

The values of function  $f(n)$  are given in Table 2-9. It is seen from the table that for  $n = 2$   $f(n) = 0.110$ . Therefore, for  $n = 2$ ,  $X_n = 0.110 \Lambda = \varepsilon$ , i.e. the coefficient  $\varepsilon$  is numerically equal to the depth of the layer  $X_n$ , where temperature oscillations have decreased twofold in comparison with those at the surface.

Table 2-9

Function  $f(n)$  of Temperature Oscillation Damping

$n$	2	4	10	20	50	100	1000
$f(n)$	0.110	0.221	0.367	0.477	0.623	0.733	1.100



The amount of heat accumulated by a unit area of a semi-infinite wall in a half-period at  $\text{Bi}^* \neq 0$  will be equal to

$$(Q_s)_{t \rightarrow \infty} = 2c\rho T_{mc} \theta_A \sqrt{\frac{a}{\omega}} = 2c\rho T_{mn} \sqrt{\frac{a}{\omega}} \quad (2-8-36)$$

i.e. we obtain an expression similar to (2-8-33) as the temperature oscillation at the wall surface equals  $T_{mc} \theta_A$ .

Now we shall consider temperature waves in materials with memory. Linear theory for such materials provides a differential heat-conduction equation in the integro-differential form [2-25]. Let us consider plane travelling waves

$$C(0)\dot{T}(\vec{r}, \tau) + \int_0^\infty \alpha'(\theta)\dot{T}(\vec{r}, \tau - \theta) d\theta = \lambda(0)\nabla^2 T(\vec{r}, \tau) + \int_0^\infty \lambda'(\theta)\nabla^2 T(\vec{r}, \tau - \theta) d\theta \quad (2-8-37)$$

Let a temperature field be expressed by the relation

$$T(\vec{r}, \tau) - T_0 = A \operatorname{Re} \left\{ \exp(-\xi \vec{r}, \vec{n}^1) \exp \left[ i\omega \left( \tau - \frac{1}{w} \vec{r} \cdot \vec{n}^1 \right) \right] \right\} \quad (2-8-38)$$

where  $\vec{r} = \vec{x} - \vec{x}_0$ ,  $\xi$  is the attenuation coefficient,  $\vec{n}^1$  is a unit normal to the surface, and  $A$  is the amplitude.

Solution (2-8-38) satisfies equation (2-8-37) if the external heat source equals zero and the following relation holds [2-25]

$$i\omega\{C(0) + C'_F/\omega\} = (\xi + i\omega/w^2)\{\lambda(0) + \lambda'_F(\omega)\} \quad (2-8-39)$$

where  $C'_F(\omega)$  and  $\lambda'_F(\omega)$  are the Fourier transforms for  $C'(\omega)$  and  $\lambda'(\omega)$ . All the values of  $w = w(\omega) \geq 0$  and  $\xi = \xi(\omega) \geq 0$  satisfy relation (2-8-39) if they are respectively equal to

$$w^2 = \frac{2\omega[\lambda(0) + \lambda'_F(\omega)]}{[C(0) + C'_F(\omega)]} \{\sec[v(\omega) - \varphi(\omega)]\} \{\tan[v(\omega) - \varphi(\omega)] + \sec[v(\omega) + \varphi(\omega)]\} \quad (2-8-40)$$

$$\xi = \frac{\omega}{w} \{\tan[v(\omega) - \varphi(\omega)] + \sec[v(\omega) + \varphi(\omega)]\} \quad (2-8-41)$$

where

$$\tan v(\omega) = \frac{I_m[\lambda(0) + \lambda'_F(\omega)]}{\operatorname{Re}[\lambda(0) + \lambda'_F(\omega)]}; \quad \tan \varphi(\omega) = \frac{I_m[C(0) + C'_F(\omega)]}{\operatorname{Re}[C(0) + C'_F(\omega)]}$$

For the classical theory  $C(\theta) = C(\infty) = \text{const}$ ,  $\lambda(\theta) = \lambda(\infty) = \text{const}$ ; then from (2-8-40) we get

$$w = \left[ \frac{2\omega\lambda(\infty)}{C(\infty)} \right]^{\frac{1}{2}} = \sqrt{2\omega a}; \quad \xi(\omega) = \left[ \frac{\omega C(\infty)}{2\lambda(\infty)} \right]^{\frac{1}{2}} = \sqrt{\frac{\omega}{2a}} \quad (2-8-42)$$

From the Fourier transforms it follows that

$$\lim_{\omega \rightarrow 0} C'_F(\omega) = C(\infty) - C(0); \quad \lim_{\omega \rightarrow 0} \omega C'_F(\omega) = 0 \quad (2-8-43)$$

and

$$\lim_{\omega \rightarrow \infty} C'_F(\omega) = 0; \quad \lim_{\omega \rightarrow \infty} C'_F(\omega) = -iC'(0) \quad (2-8-44)$$

Similar formulas also hold for  $\lambda'_F(\omega)$ . Using formulas (2-8-43) and (2-8-44), from (2-8-39)-(2-8-40) we get

$$\text{when } \omega \rightarrow 0 \quad w(\omega) = w_{cl}; \quad \xi(\omega) = \xi_{cl} \quad (2-8-45)$$

$$\text{when } \omega \rightarrow \infty \quad w(\omega) = \left[ \frac{2\omega\lambda(0)}{C(0)} \right]^{\frac{1}{2}}; \quad \xi(\omega) = \left[ \frac{\omega C(0)}{2\lambda(0)} \right]^{\frac{1}{2}} \quad (2-8-46)$$

where the subscript  $cl$  refers to classical values of these quantities. Hence, for low frequencies ( $\omega \rightarrow 0$ ) the velocity  $w(\omega)$  and the attenuation coefficient  $\xi(\omega)$  tend to their classical values. For high frequencies ( $\omega \rightarrow \infty$ ) the temperature wave propagation velocity and the attenuation coefficient are positive and differ from their classical values.

The quantity  $a(0)$  [ $a(0) = \lambda(0)/C(0)$ ] may be greater or smaller than the classical value of thermal conductivity [ $a(0) \gtrless a$ ], and hence, the wave propagation velocity  $w \gtrless w_{cl}$ .

In [2-36] the differential equation for a half-space has been solved ( $0 < x < \infty, \tau > 0$ )

$$\begin{aligned} C \frac{\partial^2 T(x, \tau)}{\partial \tau^2} + f(0) \frac{\partial T(x, \tau)}{\partial \tau} + \int_0^\infty \beta'(\theta) \frac{\partial T(x, \tau - \theta)}{\partial \tau} d\theta &= k(0) \nabla^2 T(x, \tau) \\ + \int_0^\infty k'(\theta) \nabla^2 T(x, \tau - \theta) d\theta \end{aligned}$$

where  $k'(\tau)$  and  $\beta'(\tau)$  are the relaxation functions of heat flux and internal energy, respectively.

The boundary conditions are

$$T(0, \tau) = f(\tau)H(\tau); \quad \dot{T}(x, 0) = T(x, 0) = 0 \quad (2-8-47)$$

where  $H(\tau)$  is the Heaviside unit vector.

The general solution takes the form

$$T = \frac{2H(\tau)}{\pi} \int_0^\infty \left\{ \int_0^\tau f(\theta) I(\tau - \theta, \xi) d\theta \right\} \xi \sin x\xi d\xi \quad (2-8-48)$$

Let us assume that

$$k(\tau) = k + k_1\tau^\mu; \quad \beta(\tau) = B + B_1\tau^\nu; \quad \mu \geq 1; \quad \nu \geq 1 \quad (2-8-49)$$

Then for sufficiently small values of time we shall have [2-36]

$$T(x, \tau) \approx H(\tau - x/w)f(\tau - x/w) \frac{\lambda}{2\pi^{1/2}(\tau - x/w)^{1/2}} \exp \left[ -\frac{\lambda^2}{4(\tau - x/w)} \right] \quad (2-8-50)$$

where

$$\lambda = x\omega\pi^{1/2}/4w$$

Here,  $w$  is the heat propagation velocity

$$w = \sqrt{K(0)/C} \quad (2-8-51)$$

For large values of time

$$k(\tau) \approx k\tau^\nu; \quad \beta(\tau) \approx B\tau^\nu \quad (0 > \nu > -1)$$

we shall have

$$T \approx f(\tau) \oplus \frac{x}{2} \sqrt{\frac{B}{K\pi\tau^3}} \exp \left( -\frac{Bx^2}{4K\tau} \right) \quad (2-8-52)$$

where the symbol  $\oplus$  indicates convolution of the function.

Let us assume that

$$K(\tau) = Ke^{-\omega\tau} \text{ and } \beta(\tau) = Be^{-\sigma\tau} \quad (2-8-53)$$

Then for large  $\tau$ , we obtain

$$T(x, \tau) \approx \frac{x\lambda}{2(\pi\tau^3)^{1/2}} \exp \left( -\frac{x^2\lambda^2}{4\tau} \right) \oplus f(\tau) \quad (2-8-54)$$

If  $f(\tau) = T_0 = \text{const}$ , then from (2-8-54) we shall have

$$T(x, \tau) \approx T_0 \operatorname{erfc} \frac{x\lambda}{2\tau^{1/2}} \quad (2-8-55)$$

For the case  $f(\tau) = T_0 = \text{const}$  and  $\omega = \sigma$  an exact solution is obtained [2-36]

$$\frac{T(x, \tau)}{T_0} = \left\{ e^{-\delta m} + \delta m \int_0^\tau e^{-\delta \theta} \frac{I_1[\delta(\theta^2 - m^2)^{1/2}]}{(\theta^2 - m^2)^{1/2}} d\theta \right\} H(\tau - m) \quad (2-8-56)$$

where

$$m = x/w; \quad 2C\delta = B + C\sigma$$

The Veron number is the criterion of validity of the differential heat-conduction equation with due regard for the finite heat-propagation velocity, and equals

$$\text{Ve} = \frac{a}{wl} \quad (2-8-57)$$

where  $l$  is the characteristic size of the body.

For metal shells ( $l \approx 1 \text{ mm}$ ) the Veron number varies between  $10^{-4}$  and  $10^{-5}$  ( $10^{-4} < \text{Ve} < 10^{-5}$ ).

## 2-9. BOUNDARY CONDITIONS OF THE FOURTH KIND

Boundary conditions of the fourth kind correspond to heating or cooling of a set of bodies in contact (ideal thermal contact). Besides, strict formulation of convective heat-transfer problems amounts to solving a conjugate problem with boundary conditions of the fourth kind.

Let us now consider some typical problems.

(a) A system of two semi-infinite cylindrical bodies in contact.

We have

$$\frac{\partial T_i(x, \tau)}{\partial \tau} = a_i \frac{\partial^2 T_i(x, \tau)}{\partial x^2} \quad (2-9-1)$$

$\tau > 0$ ,  $i = 1, 2$ ; when  $i = 1$   $x > 0$ , when  $i = 2$   $x < 0$ .

The boundary conditions are as follows:

$$T_1(x, 0) = T_{01}; \quad T_2(x, 0) = T_{02} \quad (2-9-2)$$

$$T_1(+0, \tau) = T_2(-0, \tau) \quad (2-9-3)$$

$$\frac{\partial T_1(0, \tau)}{\partial x} = \frac{\lambda_2}{\lambda_1} \frac{\partial T_2(0, \tau)}{\partial x} \quad (2-9-4)$$

$$\frac{\partial T_1(+\infty, \tau)}{\partial x} = \frac{\partial T_2(-\infty, \tau)}{\partial x} = 0 \quad (2-9-5)$$

The solution takes the form

$$Q_1 = \frac{T_1(x, \tau) - T_{01}}{T_{01} - T_{02}} = \frac{K_\varepsilon}{1 + K_\varepsilon} \left( 1 + \frac{1}{K_\varepsilon} \operatorname{erfc} \frac{x}{2\sqrt{a_1\tau}} \right) \quad (2-9-6)$$

$$Q_2 = \frac{T_2(x, \tau) - T_{02}}{T_{01} - T_{02}} = \frac{K_\varepsilon}{1 + K_\varepsilon} \operatorname{erfc} \frac{|x|}{2\sqrt{a_2\tau}} \quad (2-9-7)$$

where  $K_\varepsilon$  is the criterion which characterizes thermal activity  $\varepsilon_1$  of the first body with respect to the second one  $\varepsilon_2$

$$K_\varepsilon = \frac{\varepsilon_1}{\varepsilon_2} = \sqrt{\frac{\lambda_1 c_1 \rho_1}{\lambda_2 c_2 \rho_2}} = \frac{\lambda_1}{\lambda_2} \sqrt{\frac{a_2}{a_1}} \quad (2-9-8)$$

An analysis of the solutions reveals that when  $\tau \rightarrow \infty$  (in the steady state) the relative temperature of both cores will be similar and equal to

$$\theta(x, \infty) = \frac{K_\varepsilon}{1 + K_\varepsilon} \quad (2-9-9)$$

If the thermal activities of the cores are similar ( $K_\varepsilon = 1$ ), then the relative temperature in the steady state will be equal to  $\theta = \frac{1}{2}$ .

At the contact boundary this temperature is attained immediately after the bodies come in contact with each other and remains constant throughout the heat-transfer process, because

$$\theta(0, \tau) = \theta(x, \infty) = \frac{K_\varepsilon}{1 + K_\varepsilon} = \text{const} \quad (2-9-10)$$

If the thermal activity of one body is much greater than that of the other ( $\varepsilon_1 \gg \varepsilon_2$ ), then  $K_\varepsilon \gg 1$ . In this case,  $\theta(0, \tau)$  will be maximum and equal to  $\theta(0, \tau) = 1$ . If the relative thermal activity of the body is small ( $K_\varepsilon \rightarrow 0$ ), then  $\theta(0, \tau) = 0$ . Hence,  $\theta(0, \tau)$  ranges from zero (minimum thermal activity) to unity (maximum thermal activity).

Thus,  $\theta(0, \tau)$  describes the decrease in the relative temperature of a semi-infinite body when it comes into contact with another semi-infinite body. Therefore

$$1 - \theta(0, \tau) = \frac{1}{1 + K_\varepsilon} = X \quad (2-9-11)$$

may be called the cooling effect. If  $\theta(0, \tau) = 1$  ( $X = 0$ ), the cooling effect is zero; on the contrary, if  $\theta(0, \tau) = 0$  ( $X = 1$ ), the cooling effect is maximum.

If at the contact boundary a heat source of power  $q_0$  is operating, the solutions will take the form

$$\theta_1 = \frac{2Ki_1K_e}{1+K_e} \sqrt{Fo_1} \operatorname{ierfc} \frac{1}{2\sqrt{Fo_1}} \quad (2-9-12)$$

$$\theta_2 = \frac{2Ki_2}{1+K_e} \sqrt{Fo_2} \operatorname{ierfc} \frac{1}{2\sqrt{Fo_2}} \quad (2-9-13)$$

The following notations have been used

$$Fo_1 = a_1\tau/x^2, \quad Fo_2 = a_2\tau/x^2; \quad Ki_1 = q_0x/\lambda_0T_0; \quad Ki_2 = q_0x/\lambda_2T_0;$$

$$\operatorname{ierfc} z = \frac{1}{\sqrt{\pi}} \exp(-z^2) - z \operatorname{erfc} z$$

(b) An infinite solid cylinder of length  $R$  is brought into contact with a semi-infinite solid cylinder having different thermophysical properties. The lateral surfaces of solid cylinders are thermally insulated.

The boundary conditions are as follows

$$\begin{aligned} T_1(x, 0) &= T_0; \quad T_2(x, 0) = 0; \quad T_1(R, \tau) = T_2(R, \tau); \\ \frac{\lambda_1}{\lambda_2} \frac{\partial T_1(R, \tau)}{\partial x} &= \frac{\partial T_2(R, \tau)}{\partial x}; \quad T_1(0, \tau) = T_c = \text{const}; \quad T_2(\infty, \tau) = 0 \end{aligned} \quad (2-9-14)$$

The solutions take the form

$$\begin{aligned} \theta_1 &= \frac{T_1(x, \tau) - T_0}{T_c - T_0} = \operatorname{erfc} \frac{x}{2\sqrt{a_1\tau}} \mp h \sum_{n=1}^{\infty} h^{n+1} \operatorname{erfc} \frac{2nR \mp x}{2\sqrt{a_1\tau}} \\ &\mp \frac{T_0}{(1+K_e)(T_c - T_0)} \sum_{n=1}^{\infty} h^{n-1} \operatorname{erfc} \frac{(2n-1)R \mp x}{2\sqrt{a_1\tau}} \end{aligned} \quad (2-9-15)$$

$$\begin{aligned} \theta_2 &= \frac{T_2(x, \tau)}{T_c - T_0} = \frac{2K_e}{1+K_e} \sum_{n=1}^{\infty} h^{n-1} \operatorname{erfc} \left[ \frac{x - R + (2n-1)K_a^{-1/2}R}{2\sqrt{a_2\tau}} \right] \\ &+ \frac{T_0K_e}{(T_c - T_0)(1+K_e)} \operatorname{erfc} \frac{x - R}{2\sqrt{a_2\tau}} \\ &+ \frac{2K_eT_0}{(1+K_e)^2(T_c - T_0)} \sum_{n=1}^{\infty} h^{n-1} \operatorname{erfc} \left[ \frac{x - R + 2nK_a^{-1/2}R}{2\sqrt{a_2\tau}} \right] \end{aligned} \quad (2-9-16)$$

where

$$h = (1 - K_e)/(1 + K_e); \quad K_a = a_1/a_2; \quad K_a^{-1/2} = \sqrt{a_2/a_1}$$

Here the following notation is used

$$\pm \Phi(\mp z) = -\Phi(-z) + \Phi(+z)$$

(c) There are two infinite plates  $R_1$  and  $R_2$  which touch each other and are at the same initial temperature. One of the free surfaces is kept at temperature  $T_c$ , and the other at  $T_0$ .

The solution of this problem is available in [2-18]. If heat transfer between the plate surfaces ( $x = -R_1$ ) follows Newton's law  $q_s = \alpha[T_c - T(-R, \tau)]$ , then the temperature at the contact boundary of the plates ( $x = 0$ ) and at the surface ( $x_1 = -R_1$ ) will be determined from the graphs of Figs. 2-9 and 2-10. The graphs have been plotted for the case when  $\lambda_2 \rightarrow \infty$  and for different values of  $\mu$

$$\mu = [n + (n + 1)]/\text{Bi}_1 \quad (2-9-17)$$

where

$$n = c_2 \rho_2 R_2 / c_1 \rho_1 R_1, \quad \text{Bi}_1 = \alpha R_1 / \lambda_1, \quad \text{Fo}_1 = a_1 \tau / R_1^2$$

In Fig. 2-9 the value

$$\theta^* = 1 - \frac{\mu \text{Bi}}{(n + 1)} \left[ 1 - \frac{T(0, \tau) - T_0}{T_c - T_0} \right] \quad (2-9-18)$$

is laid off on the ordinate.

In Fig. 2-10 the value

$$\theta_s = \frac{[T(-R, \tau) - T_0]}{(T_c - T_0)} \quad (2-9-19)$$

is plotted on the ordinate.

(d) Given an infinite plate of thickness  $2R$  at temperature  $T_0$ . At the initial moment it is placed in a medium at temperature  $T_c < T_0$ . The plate is cooled by conduction. The initial conditions are

$$T_1(x, 0) = T_0; \quad T_2(x, 0) = T_0 \quad (2-9-20)$$

If the origin of coordinates lies in the middle of the plate, then the boundary conditions may be written as

$$\pm K_\lambda \frac{\partial T_1(\pm R, \tau)}{\partial x} = \pm \frac{\partial T_2(\pm R, \tau)}{\partial x} \quad (2-9-21)$$

$$T_1(\pm R, \tau) = T_2(\pm R, \tau); \quad \frac{\partial T_1(0, \tau)}{\partial x} = 0; \quad \frac{\partial T_2(\pm \infty, \tau)}{\partial x} = 0 \quad (2-9-22)$$

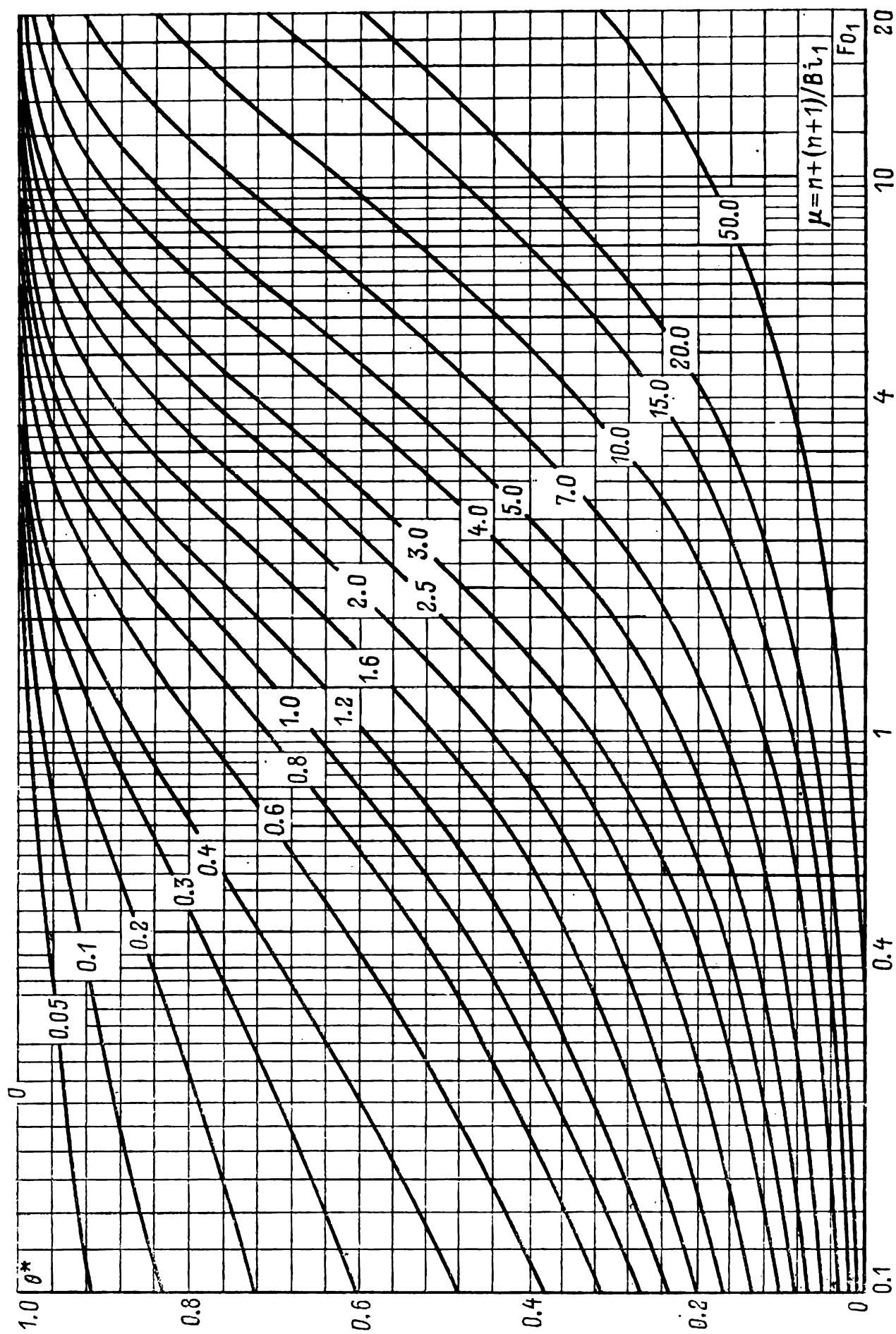


Fig. 2-9. Plot of generalized variable  $\theta^*$  versus  $Fo_1$  at the contact boundary of plates ( $x = 0$ ) for different values of  $\mu$  ( $n = c_2 \gamma_2 R_2 / c_1 \gamma_1 R_1$ ;  $Bi_1 = \alpha R_1 / \lambda_1$ ;  $\lambda_2 \rightarrow \infty$ )



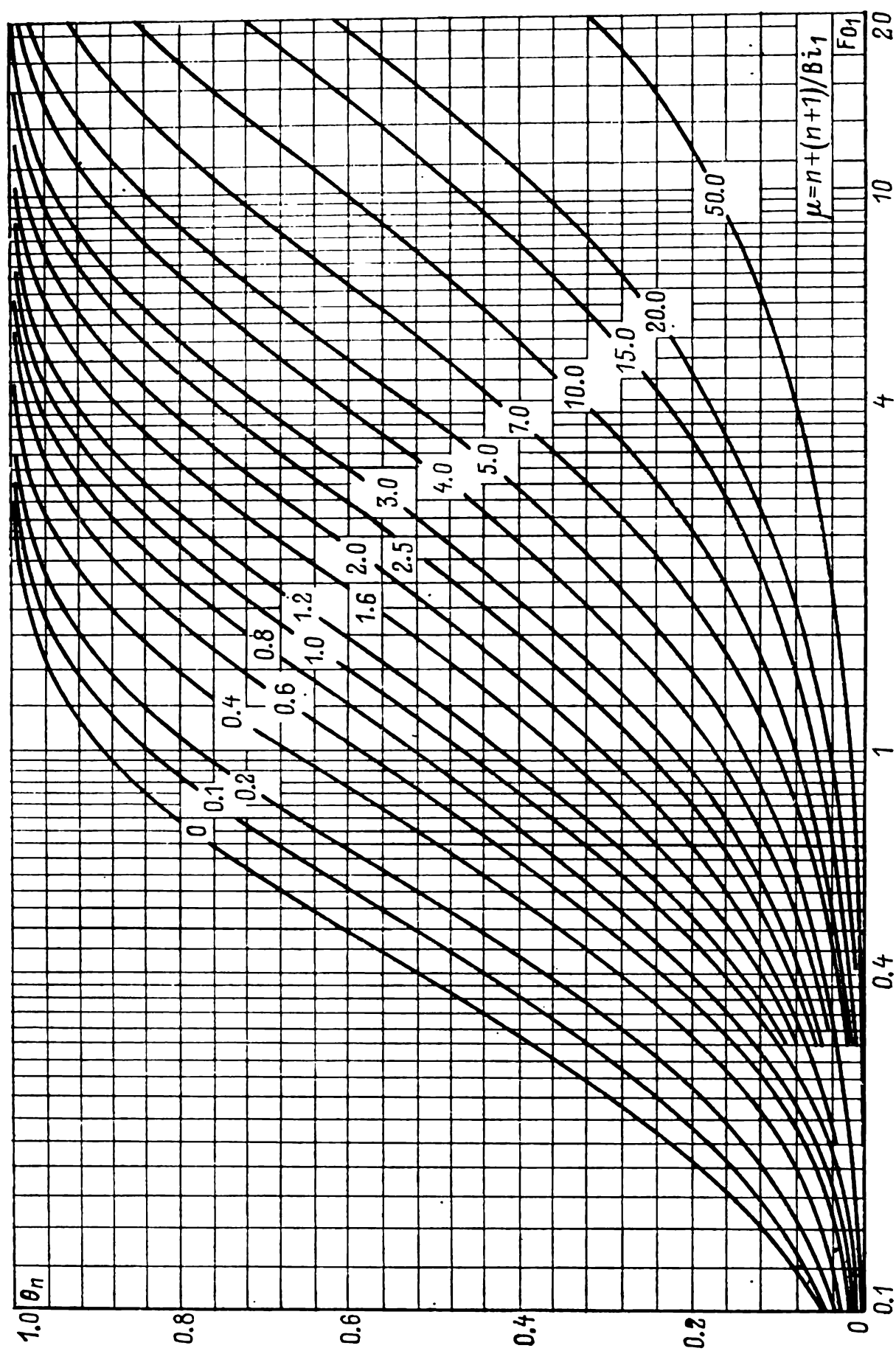


Fig. 2-10. Plot of generalized variable  $\theta_n$  versus  $Fo_1$  number for the plate surface ( $x = -R_1$ ) at different values of  $\mu$  in a two-layer plate ( $n = c_2\gamma_2R_2/c_1\gamma_1R_1$ ;  $Bi_1 = \alpha R_1/\lambda_1$ ;  $\lambda_2 \rightarrow \infty$ )

The solutions take the form

$$\theta_1 = \frac{T_1(x, \tau) - T_c}{T_0 - T_c} = 1 - \frac{1}{1 + K_\varepsilon} \sum_{n=1}^{\infty} (-h)^{n-1} \left\{ \operatorname{erfc} \frac{(2n-1)R - x}{2\sqrt{a_1\tau}} + \operatorname{erfc} \frac{(2n+1)R + x}{2\sqrt{a_1\tau}} \right\} \quad (2-9-23)$$

$$\theta_2 = \frac{T_2(x, \tau) - T_c}{T_0 - T_c} = \frac{K_\varepsilon}{1 + K_\varepsilon} \operatorname{erfc} \frac{x - R}{2\sqrt{a_2\tau}} - \frac{K_\varepsilon(1+h)}{1 + K_\varepsilon} \sum_{n=1}^{\infty} (-h)^{n-1} \operatorname{erfc} \frac{x - R + 2nR\sqrt{a_2/a_1}}{2\sqrt{a_2\tau}} \quad (2-9-24)$$

The assumption  $K_\varepsilon \rightarrow 0$  (the thermal activity of the plate is infinitesimal compared to that of the surroundings) infers instantaneous heat transfer from the heated plate accompanied by abrupt decrease of the surface temperature down to the ambient temperature. If we assume that  $K_\varepsilon = 0$  ( $h = 1$ ), then  $\theta_2(x, \tau) = 0$ , and the first solution will take the following form

$$\theta_1 = 1 - \sum_{n=1}^{\infty} (-1)^{n-1} \left[ \operatorname{erfc} \frac{(2n-1)R - x}{2\sqrt{a_1\tau}} + \operatorname{erfc} \frac{(2n+1)R + x}{2\sqrt{a_1\tau}} \right] \quad (2-9-25)$$

Equation (2-9-25) is the solution of the problem on cooling an infinite plate when its surface temperature instantaneously drops down to the ambient temperature, and thereafter remains constant throughout the cooling process (the first-kind boundary condition).

The temperature at the contact boundary is

$$\theta_s = \frac{K_\varepsilon}{1 + K_\varepsilon} - \frac{2K_\varepsilon}{(1 + K_\varepsilon)^2} \sum_{n=1}^{\infty} (-h)^{n-1} \operatorname{erfc} \frac{nR}{\sqrt{a_1\tau}} \quad (2-9-26)$$

Thus,  $\theta(R, \tau)$  continuously decreases; the decrease is maximum at the initial moment (when  $\tau \rightarrow 0$  the function  $\operatorname{erfc} \frac{nR}{\sqrt{a_1\tau}}$  tends to zero), then

$$\theta_{s, 0} = \frac{K_\varepsilon}{1 + K_\varepsilon} \quad (2-9-27)$$

is the relative temperature which is attained at the contact boundary of two semi-infinite bodies. The quantity

$$1 - \frac{K_\varepsilon}{1 + K_\varepsilon} = 1 - \theta(R, 0) = X \quad (2-9-28)$$

is called the cooling effect. The rate of change of  $\theta_{s,0}$  depends on the thermal diffusivity and the plate thickness. When  $Fo_1 = \frac{a_1 \tau}{R^2}$  is small, the change of  $\theta_{s,0}$  is insignificant.

An analysis of (2-9-23) and (2-9-24) reveals that temperature at any point of the medium first increases, reaches its maximum and then decreases. The temperature at the contact boundary ( $x = R$ ) attains its maximum instantaneously, and as  $x$  ( $x > R$ ) increases the time necessary to reach the maximum increases.

For small values of  $Fo_2 = \frac{a_2 \tau}{R^2}$ , only the first term of (2-9-24) may be considered, the approximate equality may then be written as

$$\theta_2 \approx \frac{K_e}{1 + K_e} \operatorname{erfc} \frac{\frac{x}{R} - 1}{1 \sqrt{Fo_2}} - \frac{2K_e}{(1 + K_e)^2} \operatorname{erfc} \frac{\frac{x}{R} - 1 + 2K_a^{-1/2}}{2 \sqrt{Fo_2}} \quad (2-9-29)$$

If the derivative  $\frac{\partial \theta_2}{\partial Fo_2}$  is equated to zero, then for the determination of  $(Fo_2)_m$  — the Fourier number corresponding to the maximum temperature, we get the equation

$$(Fo_2)_m = \frac{1 + \left( \frac{x}{R} - 1 \right) K_a^{1/2}}{K_a \log_e \left\{ \frac{2}{1 + K_e} \left[ 1 + \frac{2}{K_a^{1/2} \left( \frac{x}{R} - 1 \right)} \right] \right\}} \quad (2-9-30)$$

It is evident from (2-9-30) that the time required for the temperature maximum to be reached increases with the relative coordinate;  $(Fo_2)_m = 0$  for  $x = R$ .

(e) Problems relating to cooling of a sphere and infinite cylinder in an infinite medium are presented in [2-18].

## 2-10 TWO- AND THREE-DIMENSIONAL PROBLEMS

### (a) Solution of Problems in Cartesian Coordinates

Most of unsteady heat-conduction problems may be solved by finite integral transformations. This may be illustrated by solving a two-dimensional problem.

We have

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{Q(x, y, \tau)}{\lambda} = \frac{1}{a} \frac{\partial T}{\partial \tau} \quad (2-10-1)$$

$$\left. \begin{aligned} -\lambda_1 \frac{\partial T}{\partial x} + \alpha_1 T &= f_1(y, \tau) \text{ when } x = 0 \\ \lambda_2 \frac{\partial T}{\partial x} + \alpha_2 T &= f_2(y, \tau) \text{ when } x = l_1 \end{aligned} \right\} \quad (2-10-2)$$

$$\left. \begin{aligned} -\lambda_3 \frac{\partial T}{\partial y} + \alpha_3 T &= f_3(x, \tau) \text{ when } y = 0 \\ \lambda_4 \frac{\partial T}{\partial y} + \alpha_4 T &= f_4(x, \tau) \text{ when } y = l_2 \end{aligned} \right\} \quad (2-10-3)$$

$$T = \varphi(x, y) \text{ when } \tau = 0; \quad T \equiv T(x, y, \tau) \quad (2-10-4)$$

Let us make use of the integral transform

$$T_I(\beta_n, \gamma_m, \tau) = \int_0^{l_1} \int_0^{l_2} K(\beta_n, x) K(\gamma_m, y) T(x, y, \tau) dx dy \quad (2-10-5)$$

The differential equation then takes the form

$$\frac{dT_I(\beta_n, \gamma_m, \tau)}{d\tau} + a(\beta_n^2 + \gamma_m^2) T_I(\beta_n, \gamma_m, \tau) = A(\beta_n, \gamma_m, \tau) \quad (2-10-6)$$

where

$$\begin{aligned} A(\beta_n, \gamma_m, \tau) &= \frac{a}{\lambda} \int_0^{l_1} \int_0^{l_2} K(\beta_n, x) K(\gamma_m, y) Q(x, y, \tau) dx dy \\ &+ \frac{a}{\lambda_1} [K(\beta_n, x)]_{x=0} \int_0^{l_2} K(\gamma_m, y) f_1(y, \tau) dy \\ &+ \frac{a}{\lambda_2} [K(\beta_n, x)]_{x=l_1} \int_0^{l_2} K(\gamma_m, y) f_2(y, \tau) dy \end{aligned}$$

$$\begin{aligned}
& + \frac{a}{\lambda_3} [K(\gamma_m, y)]_{\gamma=0} \int_0^{l_1} K(\beta_n, x) f_3(x, \tau) dx \\
& + \frac{a}{\lambda_4} [K(\gamma_m, y)]_{\gamma=l_2} \int_0^{l_1} K(\beta_n, x) f_4(x, \tau) dx
\end{aligned} \tag{2-10-7}$$

The solution of equation (2-10-6) may be inverted using the inversion formula

$$T(x, y, \tau) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} K(\beta_n, x) K(\gamma_m, y) T_I(\beta_n, \gamma_m, \tau) \tag{2-10-8}$$

$$\begin{aligned}
T(x, y, \tau) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \exp[-a(\beta_n^2 + \gamma_m^2) \tau] K(\beta_n, x) K(\gamma_m, y) \\
&\times \left\{ \varphi_I(\beta_n, \gamma_m) + \int_0^{\tau} \exp[a(\beta_n^2 + \gamma_m^2) \tau] A(\beta_n, \gamma_m, \tau) d\tau \right\}
\end{aligned} \tag{2-10-9}$$

where

$$\begin{aligned}
\varphi_I(\beta_n, \gamma_m) &= [T_I(\beta_n, \gamma_m, \tau)]_{\tau=0} \\
&\quad l_1 \quad l_2 \\
&= \int_0^{l_1} \int_0^{l_2} K(\beta_n, x) K(\gamma_m, y) \varphi(x, y) dx dy
\end{aligned} \tag{2-10-10}$$

Solutions for boundary conditions of the first kind may be obtained from the solutions for boundary conditions of the third kind. To accomplish this,  $\frac{1}{\alpha_k} \frac{dK}{dx_i}$  should be substituted for  $\frac{K}{\lambda_k}$  at  $\lambda_k = 0$  ( $k = 1, 2, 3, 4$ ;  $i = 1, 2$ ;  $x_1 = x$ ;  $x_2 = y$ ).

Integral transformation (2-10-5) with the inversion formula may be used for semi-infinite and infinite bodies [2-21]. The transformation and inversion formulas are presented in Tables 2-10 and 2-11.

When solving steady-state problems, the principle of superposition of temperature fields may be used, for example, in the solution of the Laplace equation for boundary conditions of the third kind

$$\nabla^2 T = 0; \quad \lambda_j \frac{\partial T}{\partial n_j} + \alpha_j T = f_j \text{ at the boundary } S_j. \tag{2-10-11}$$

The solution may be represented as the sum of the solutions of auxiliary problems

$$T = \sum_{k=1}^k T_k \tag{2-10-12}$$

Fourier Integral Transforms for Two- and Three-Dimensional Heat-Conduction Problems in Cartesian Coordinates [2-21]

Range	Transformation formula	Inversion formula	Transformation $\nabla^2 T$
$0 \leq x \leq l_1$ $0 \leq y \leq l_2$	$T_I = \int_0^{l_2} \int_0^{l_1} TK(\beta_n, x) \times K(\gamma_m, y) dx dy$	$T = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} K(\beta_n, x) \times K(\gamma_m, y) T_I$	$-(\beta_n^2 + \gamma_m^2) T_I + \sum_{j=1}^4 \int_{(l_j)} \frac{1}{\lambda_j} L_j f_j dl_j; \\ L = K(\beta_n, x) K(\gamma_m, y)$
$0 \leq x \leq l_1$ $0 \leq y \leq l_2$ $0 \leq z \leq l_3$	$T_I = \int_0^{l_3} \int_0^{l_2} \int_0^{l_1} TK(\beta_n, x) \times K(\gamma_m, y) K(\varepsilon_k, z) dx dy dz$	$T = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} K(\beta_n, x) \times K(\gamma_m, y) K(\varepsilon_k, z) T_I$	$-(\beta_n^2 + \gamma_m^2 + \varepsilon_k^2) T_I + \sum_{j=1}^6 \iint_{(S_j)} \frac{1}{\lambda_j} L_j f_j dS_j; \\ L = K(\beta_n, x) K(\gamma_m, y) K(\varepsilon_k, z)$
$0 \leq x \leq \infty$ $0 \leq y \leq \infty$	$T_I = \int_0^{\infty} \int_0^{\infty} TK(\beta, x) \times K(\gamma, y) dx dy$	$T = \int_0^{\infty} \int_0^{\infty} K(\beta, x) K(\gamma, y) \times T_I d\beta d\gamma$	$-(\beta^2 + \gamma^2) T_I + \sum_{j=1}^2 \int_{(l_j)} \frac{1}{\lambda_j} L_j f_j dl_j; \\ L = K(\beta, x) K(\gamma, y)$

$\begin{array}{l} 0 \leq x \leq \infty \\ 0 \leq y \leq \infty \\ 0 \leq z \leq \infty \end{array}$	$T_I = \int_0^\infty \int_0^\infty \int_0^\infty TK(\beta, x) \times K(\gamma, y) K(\epsilon, z) dx dy dz$	$T = \int_0^\infty \int_0^\infty \int_0^\infty K(\beta, x) K(\gamma, y) \times K(\epsilon, z) T_I d\beta d\gamma d\epsilon$	$\begin{array}{l} -(\beta^2 + \gamma^2 + \epsilon^2) T_I \\ + \sum_{j=1}^3 \iint_{(S_j)} \frac{1}{\lambda_j} L_j f_j dS_j; \\ L = K(\beta, x) K(\gamma, y) K(\epsilon, z) \end{array}$
$\begin{array}{l} -\infty < x < \infty \\ -\infty < y < \infty \\ -\infty < z < \infty \end{array}$	$T_I = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp[i(\beta x + \gamma y + \epsilon z)] T dx dy dz$	$T = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp[-i(\beta x + \gamma y + \epsilon z)] T_I d\beta d\gamma d\epsilon$	$\begin{array}{l} -(\beta^2 + \gamma^2 + \epsilon^2) T_I \\ \\ \\ \end{array}$
$\begin{array}{l} 0 < y < l_2 \\ 0 < x < \infty \end{array}$	$T_I = \int_0^{l_2} \int_0^\infty TK(\beta, x) \times K(\gamma_m, y) dx dy$	$T = \sum_{m=1}^\infty \int_0^\infty K(\beta, x) \times K(\gamma_m, y) T_I d\beta$	$\begin{array}{l} -(\beta^2 + \gamma_m^2) T_I \\ + \sum_{j=1}^2 \int_{(l_j)} \frac{1}{\lambda_j} L_j f_j dl_j; \\ L = K(\beta, x) K(\gamma_m, y) \end{array}$
$\begin{array}{l} -\infty \leq x \leq \infty \\ 0 \leq y \leq \infty \end{array}$	$T_I = \int_{-\infty}^{+\infty} \int_0^\infty e^{i\beta x} K(\gamma, y) T dx dy$	$T = \frac{1}{2\pi} \int_0^{+\infty} \int_{-\infty}^{+\infty} e^{-i\beta x} K(\gamma, y) \times T_I d\beta d\gamma$	$\begin{array}{l} -(\beta^2 + \gamma^2) T_I \\ + \int_{(l_j)} \frac{1}{\lambda_j} L_j f_j dl_j; \\ L = e^{i\beta x} K(\gamma, y) \end{array}$

Table 2-10 (continued)

Range	Transformation formula	Inversion formula	Transformation $\nabla^2 T$
$0 \leq x \leq \infty$ $0 \leq y \leq l_2$ $0 \leq z \leq l_3$	$T_I = \int_0^{\infty} \int_0^{l_2} \int_0^{l_3} TK(\beta, x) \times K(\gamma_m, y) \times K(\epsilon_k, z) dx dy dz$	$T = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \int_0^{\infty} K(\beta, x) \times K(\gamma_m, y) K(\epsilon_k, z) T_I d\beta$	$-(\beta^2 + \gamma_m^2 + \epsilon_k^2) T_I + \sum_{j=1}^5 \iint_{(S_j)} \frac{1}{\lambda_j} L_j f_j dS_j;$ $L = K(\beta, x) K(\gamma_m, y) K(\epsilon_k, z)$
$-\infty \leq x \leq \infty$ $-\infty \leq y \leq \infty$ $0 \leq z \leq l_3$	$T_I = \int_0^{\infty} \int_0^{\infty} \int_0^{l_3} K(\epsilon_k, z) \times \exp(i\beta x + i\gamma y) T dx dy dz$	$T = \frac{1}{(2\pi)^2} \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(\epsilon_k, z) \times \exp[-i(\beta x + \gamma y)] T_I d\beta d\gamma$	$-(\beta^2 + \gamma^2 + \epsilon_k^2) T_I + \sum_{j=1}^2 \iint_{(S_j)} \frac{1}{\lambda_j} L_j f_j dS_j;$ $L = K(\epsilon_k, z) \exp[i(\beta x + \gamma y)]$
$0 \leq x \leq \infty$ $0 \leq y \leq \infty$ $0 \leq z \leq l_3$	$T_I = \int_0^{\infty} \int_0^{\infty} \int_0^{l_3} K(\beta, x) K(\gamma, y) \times K(\epsilon_k, z) T dx dy dz$	$T = \sum_{k=1}^{\infty} \int_0^{\infty} \int_0^{\infty} K(\beta, x) K(\gamma, y) \times K(\epsilon_k, z) T_I d\beta d\gamma$	$-(\beta^2 + \gamma^2 + \epsilon_k^2) T_I + \sum_{j=1}^4 \iint_{(S_j)} \frac{1}{\lambda_j} L_j f_j dS_j;$ $L = K(\beta, x) K(\gamma, y) K(\epsilon_k, z)$

Note. Kernel  $K$  is taken from Table 2-5; for boundary conditions of the first kind  $\frac{K_j}{\lambda_j}$  should be substituted by  $\frac{1}{\alpha_j} \frac{dK_j}{dn_j}$ , where  $n_j$  is the direction of the normal to surface  $S_j$ .



Table 2-11

Kernels of an Integral Transform for a Half-Space Problem ( $0 \leq x < \infty$ )

Boundary conditions at $x = 0$	Kernel $K(\beta x)$
of the third kind $H_1 = \frac{\alpha_1}{\lambda_1} = \text{const}$	$\left(\frac{2}{\pi}\right)^{1/2} \frac{\beta \cos \beta x + H_1 \sin \beta x}{\sqrt{\beta^2 + H_1^2}}$
of the second kind $H_1 = 0 \quad (\alpha_1 = 0)$	$\left(\frac{2}{\pi}\right)^{1/2} \cos \beta x$
of the first kind $H_1 = \infty \quad (\lambda_1 = 0)$	$\left(\frac{2}{\sqrt{\pi}}\right)^{1/2} \sin \beta x$

Note. Transformation formula:  $T_I(\beta) = \int_0^\infty K(\beta, x) T(x) dx$

Inversion formula:  $T(x) = \int_0^\infty K(\beta, x) T_I(\beta) d\beta$

Boundary conditions:  $-\lambda_1 \frac{dT}{dx} + \alpha_1 T = 0$

where  $T_k$  are the solutions of auxiliary problems

$$\nabla^2 T_k = 0, \quad \lambda_j \frac{\partial T_k}{\partial n_j} + \alpha_j T_k = \delta_{jk} f_j \quad (2-10-13)$$

in which  $j = 1, 2, \dots, k$ ;  $k = 1, 2, \dots, k$ ;  $\delta_{jk}$  is the delta-function.

Figure 2-11 is an illustration of the superposition principle of a two-dimensional temperature field for boundary conditions of the first kind.

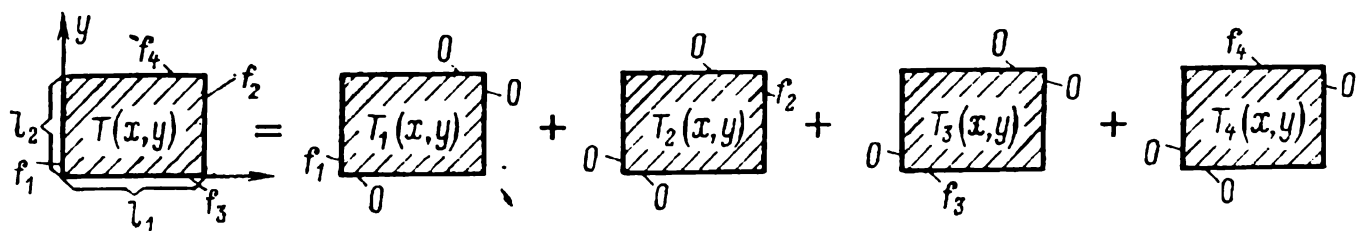


Fig. 2-11. Superposition of two-dimensional temperature fields

### (b) Solution of Problems in Cylindrical Coordinates

Finite integral transformations for cylindrical coordinates are generally known as Hankel transformations.

Inversion formulas take the form

$$F(r) = \sum_{n=1}^{\infty} K_v(\beta_n, r) F_I(\beta_n) \quad (2-10-14)$$

$$F_I(\beta_n) = \int_{R_1}^{R_2} r K_v(\beta_n, r) F(r) dr \quad (2-10-15)$$

where  $K_v(\beta_n, r)$  is a normalized eigenfunction

$$K_v(\beta_n, r) = \frac{U_v(\beta_n, r)}{\sqrt{N}} \quad (2-10-16)$$

$U_v(\beta_n, r)$  is the eigenfunction of the equation

$$\frac{d}{dr} \left( r \frac{dU}{dr} \right) + \left( \beta^2 r - \frac{v^2}{r} \right) U = 0, \quad R_1 < r < R_2 \quad (2-10-17)$$

with homogeneous boundary conditions

$$-\lambda_1 \frac{\partial U}{\partial r} + \alpha_1 U = 0 \text{ when } r = R_1 \quad (2-10-18)$$

$$\lambda_2 \frac{\partial U}{\partial r} + \alpha_2 U = 0 \text{ when } r = R_2 \quad (2-10-19)$$

Equation (2-10-17) is the Bessel equation of the  $v$ -th order which is a particular case of the Sturm-Liouville system of equations; eigenfunctions  $U_v(\beta_n, r)$  are orthogonal in the interval  $R_1 \leq r \leq R_2$  with the weight function  $r(x) = r$ , i.e.

$$\int_{R_1}^{R_2} r U_v(\beta_n, r) U_v(\beta_m, r) dr = 0 \text{ when } m \neq n \quad (2-10-20)$$

For a solid cylinder ( $0 \leq r \leq R$ ), it must be assumed that  $\lambda_1 = \alpha_1 = 0$ ,  $R_2 = R$ ,  $R_1 = 0$ . In this case

$$U_v(\beta_n, r) = J_v(\beta_n r) \quad (2-10-21)$$

From the boundary condition (2-10-19) the characteristic equation

$$\lambda_2 \beta J'_v(\beta R) + \alpha_2 J_v(\beta R) = 0 \quad (2-10-22)$$

is obtained.

The norm  $N$  is

$$N = \int_0^R r J_v^2(\beta_n r) dr = \frac{R^2}{2} \left[ J_v'^2(\beta_n R) + \left( 1 - \frac{v^2}{\beta_n^2 R^2} \right) J_v^2(\beta_n R) \right] \\ = \frac{R^2}{2} \left[ \frac{H^2}{\beta_n^2} + \left( 1 - \frac{v^2}{\beta_n^2 R^2} \right) \right] J_v^2(\beta_n R) \quad (2-10-23)$$

where  $H = \frac{\alpha_2}{\lambda_2}$  [in equality (2-10-23) the characteristic equation (2-10-22) was used].

Then the kernel of the integral transform equals

$$K_v(\beta_n, r) = \frac{U_v(\beta_n, r)}{\sqrt{N}} = \frac{\sqrt{2}}{R} \left[ \frac{H^2}{\beta_n^2} + \left( 1 - \frac{v^2}{\beta_n^2 R^2} \right) \right]^{-1/2} \frac{J_v(\beta_n r)}{J_v(\beta_n R)} \quad (2-10-24)$$

Kernels of the integral transform for a solid cylinder are given in Table 2-12.

Table 2-12

**Kernels  $K_v(\mu_n, r/R)$  of the Hankel Integral Transformation for a Finite Range ( $0 \leq r \leq R$ ) in Cylindrical Coordinates**

Boundary conditions at $r = R$	Kernel $K_v(\mu_n, r/R)$	Characteristic equations
of the first kind $H = \frac{\alpha}{\lambda} = \infty$ ( $\lambda = 0$ )	$\frac{\sqrt{2}}{R} \frac{J_v(\mu_n r/R)}{J_v'(\mu_n)}$	$J_v(\mu) = 0$
of the second kind $H = 0$ ( $\alpha = 0$ )	$\frac{\sqrt{2}}{R} \left[ 1 - \frac{v^2}{\mu_n^2} \right]^{-1/2} \frac{J_v(\mu_n r/R)}{J_v(\mu_n)}$	$J_v'(\mu) = 0$
of the third kind $H = \text{const}$	$\frac{\sqrt{2}}{R} \left[ \left( \frac{HR}{\mu_n} \right)^2 + \left( 1 - \frac{v^2}{\mu_n^2} \right) \right]^{-1/2} \times \frac{J_v(\mu_n r/R)}{J_v(\mu_n)}$	$\mu J_v'(\mu) + HR J_v(\mu) = 0$

Note.

$$\mu_n = \beta_n R; \quad \mu = \beta R; \quad H = \frac{\alpha}{\lambda} \equiv \frac{\alpha_2}{\lambda_2}$$

In a similar way the solution may be found for a hollow cylinder.

In Table 2-13 eigenfunctions  $U_v(\beta_n, r)$ , norms  $N$  and characteristic equations for a hollow cylinder are presented. If the cylinder temperature depends on two coordinates  $T(r, z)$  or  $T(r, \theta)$ , then for such two-dimensional problems the inversion formula contains functions of two variables (Table 2-14). The solution procedure remains the same. However, if the

## Kernels of the Hankel Integral Transformation for a Hollow Cylinder [2-21]

$$(R_1 \leq r \leq R_2) K_v(\mu_n, r/R_1) = \frac{1}{\sqrt{N}} U_v(\mu_n, r/R_1), \quad k = R_2/R_1, \quad \mu_n = \beta_n R_1$$

Boundary conditions		$U_v(\mu_n, r/R_1)$ and $N$	$\mu_n$ (roots of the characteristic equation)
at $r = R_1$	at $r = R_2$		
of the third kind $\lambda_1 = \text{const}$ $\alpha_1 = \text{const}$	of the third kind $\lambda_2 = \text{const}$ $\alpha_2 = \text{const}$	$U_v(\mu_n, r/R_1) = \frac{J_v(\mu_n r/R_1) R_1}{\lambda_2 \mu_n J'_v(\mu_n k) + \alpha_2 R_1 J_v(\mu_n k)} - \frac{Y_v(\mu_n r/R_1) R_1}{\lambda_2 \mu_n Y'_v(\mu_n k) + \alpha_2 R_1 Y_v(\mu_n k)};$ $N = \frac{R_2^2}{2} \left[ \frac{\alpha_2^2 R_1^2}{\lambda_2^2 \mu_n^2} + \left( 1 - \frac{v^2 R_1^2}{\mu_n^2 R_2^2} \right) \right] U_v^2(\mu_n, k) - \frac{R_1^2}{2} \left[ \frac{\alpha_1^2 R_1^2}{\lambda_1^2 \mu_n^2} + \left( 1 - \frac{v^2}{\mu_n^2} \right) \right] U_v^2(\mu_n, 1)$	$-\frac{\mu \lambda_1 J'_v(\mu) + \alpha_1 R_1 J_v(\mu)}{\mu \lambda_2 J'_v(\mu k) + \alpha_2 R_1 J_v(\mu k)} - \frac{\mu \lambda_1 Y'_v(\mu) + \alpha_1 R_1 Y_v(\mu)}{\mu \lambda_2 Y'_v(\mu k) + \alpha_2 R_1 Y_v(\mu k)} = 0$
of the third kind $\lambda_1 = \text{const}$ $\alpha_1 = \text{const}$	of the second kind $\alpha_2 = 0$ $\lambda_2 = 1$ $U'_v(\mu_n, k) = 0$	$U_v(\mu_n, r/R_1) = \frac{J_v(\mu_n r/R_1) R_1}{Y_v(\mu_n r/R_1) R_1} - \frac{\mu_n Y'_v(\mu_n k)}{R_2^2 \left( 1 - \frac{v^2}{\mu_n^2 k^2} \right) U_v^2(\mu_n, k)};$ $N = \frac{R_2^2}{2} \left( 1 - \frac{v^2}{\mu_n^2 k^2} \right) U_v^2(\mu_n, k) - \frac{R_1^2}{2} \left[ \frac{\alpha_1^2 R_1^2}{\lambda_1^2 \mu_n^2} + \left( 1 - \frac{v^2}{\mu_n^2} \right) \right] U_v^2(\mu_n, 1)$	$-\frac{\mu \lambda_1 J'_v(\mu) + \alpha_1 R_1 J_v(\mu)}{J'_v(\mu k)} - \frac{-\mu \lambda_1 Y'_v(\mu) + \alpha_1 R_1 Y_v(\mu)}{Y'_v(\mu k)} = 0$

<p>of the third kind</p> <p><math>\lambda_1 = \text{const}</math></p> <p><math>\alpha_1 = \text{const}</math></p>	<p>of the first kind</p> <p><math>\alpha_2 = 1</math></p> <p><math>\lambda_2 = 0</math></p> <p><math>U_\nu(\mu_n, k) = 0</math></p>	$U_\nu(\mu_n, r/R_1)$ $= \frac{J_\nu(\mu_n r/R_1)}{J_\nu(\mu_n k)} - \frac{Y_\nu(\mu_n r/R_1)}{Y_\nu(\mu_n k)};$ $N = \frac{R_2^2}{2} U_\nu^{1/2}(\mu_n, k)$ $- \frac{R_1^2}{2} \left[ \frac{\alpha_1^2 R_1^2}{\lambda_1^2 \mu_n^2} + \left( 1 - \frac{\nu^2}{\mu_n^2} \right) \right] U_\nu^2(\mu_n, 1)$	$\frac{-\mu\lambda_1 J'_\nu(\mu) + \alpha_1 R_1 J_\nu(\mu)}{J_\nu(\mu k)}$ $\frac{-\mu\lambda_1 Y'_\nu(\mu) + \alpha_1 R_1 Y_\nu(\mu)}{Y'_\nu(\mu k)} = 0$
<p>of the second kind</p> <p><math>\alpha_1 = 0</math></p> <p><math>\lambda_1 = 1</math></p> <p><math>U'_\nu(\mu, 1) = 0</math></p>	<p>of the third kind</p> <p><math>\alpha_2 = \text{const}</math></p> <p><math>\alpha_2 = \text{const}</math></p>	$U_\nu(\mu_n, r/R_1)$ $= \frac{J_\nu(\mu_n r/R_1) R_1}{\lambda_2 \mu_n J'_\nu(\mu_n k) + \alpha_2 R_1 J_\nu(\mu_n k)}$ $- \frac{Y_\nu(\mu_n r/R_1) R_1}{\lambda_2 \mu_n Y'_\nu(\mu_n k) + \alpha_2 R_1 Y_\nu(\mu_n k)};$ $N = \frac{R_2^2}{2} \left[ \frac{\alpha_2^2 R_1^2}{\lambda_2^2 \mu_n^2} + \left( 1 - \frac{\nu^2}{\mu_n^2 k^2} \right) \right] U_\nu^2(\mu_n, k)$ $- \frac{R_1^2}{2} \left( 1 - \frac{\nu^2}{\mu_n^2} \right) U_\nu^2(\mu_n, 1)$	$\frac{J'_\nu(\mu)}{\mu\lambda_2 J'_\nu(\mu k) + \alpha_2 R_1 J_\nu(\mu k)}$ $\frac{Y'_\nu(\mu)}{\mu\lambda_2 Y'_\nu(\mu k) + \alpha_2 R_1 Y_\nu(\mu k)} = 0$

Table 2-13 (continued)

Boundary conditions		$U_v(\mu_n, r/R_1)$ and $N$	$\mu_n$ (roots of the characteristic equation)
at $r = R_1$	at $r = R_2$		
of the second kind $\alpha_1 = 0$ $\lambda_1 = 1$ $U'_v(\mu, 1) = 0$	of the second kind $\alpha_2 = 0$ $\lambda_2 = 1$ $U'_v(\mu, k) = 0$	$U_v(\mu_n, r/R_1) = \frac{J_v(\mu_n r/R_1) R_1}{\mu_n J'_v(\mu_n k)}$ $N = -\frac{Y_v(\mu_n r/R_1) R_1}{\mu_n Y'_v(\mu_n k)};$ $N = -\frac{R_2^2}{2} \left( 1 - \frac{v^2}{\mu_n^2 k^2} \right) U_v^2(\mu_n, k)$ $- \frac{R_1^2}{2} \left( 1 - \frac{v^2}{\mu_n^2} \right) U_v^2(\mu_n, 1)$	$\frac{J'_v(\mu)}{J'_v(\mu k)} - \frac{Y'_v(\mu)}{Y'_v(\mu k)} = 0$
of the second kind $\alpha_1 = 0$ $\lambda_1 = 1$ $U'_v(\mu, 1) = 0$	of the first kind $\alpha_2 = 1$ $\lambda_2 = 0$ $U_v(\mu, k) = 0$	$U_v(\mu_n, r/R_1) = -\frac{J_v(\mu_n r/R_1)}{J_v(\mu_n k)}$ $N = -\frac{Y_v(\mu_n r/R_1)}{Y_v(\mu_n k)};$ $N = \frac{R_2^2}{2} U_v'^2(\mu_n, k)$ $- \frac{R_1^2}{2} \left( 1 - \frac{v^2}{\mu_n^2} \right) U_v^2(\mu_n, 1)$	$\frac{J'_v(\mu)}{J'_v(\mu k)} - \frac{Y'_v(\mu)}{Y'_v(\mu k)} = 0$

of the first kind $\alpha_1 = 1$ $\lambda_1 = 0$ $U_v(\mu, 1) = 0$	of the third kind $\alpha_2 = \text{const}$ $\lambda_2 = \text{const}$	$U_v(\mu_n, r/R_1) = \frac{J_v(\mu_n r/R_1) R_1}{\lambda_2 \mu_n J'_v(\mu_n k) + \alpha_2 R_1 J_v(\mu_n k)}$ $- \frac{Y_v(\mu_n r/R_1) R_1}{\lambda_2 \mu_n Y'_v(\mu_n k) + \alpha_2 R_1 Y_v(\mu_n k)};$ $N = \frac{R_2^2}{2} \left[ \frac{\alpha_2^2 R_1^2}{\lambda_2^2 \mu_n^2} + \left( 1 - \frac{v^2}{\mu_n^2 k^2} \right) \right] U_v^2(\mu_n, k)$ $- \frac{R_1^2}{2} U_v'^2(\mu_n, 1)$	$\frac{J_v(\mu)}{\mu \lambda_2 J'_v(\mu k) + \alpha_2 R_1 J_v(\mu k)}$ $- \frac{Y_v(\mu)}{\mu \lambda_2 Y'_v(\mu k) + \alpha_2 R_1 Y_v(\mu k)} = 0$
of the first kind $\alpha_1 = 1$ $\lambda_1 = 0$ $U_v(\mu, 1) = 0$	of the second kind $\alpha_2 = 0$ $\lambda_2 = 1$ $U'_v(\mu, k) = 0$	$U_v(\mu_n, r/R_1) = \frac{J_v(\mu_n r/R_1) R_1}{\mu_n J_v(\mu_n k)}$ $- \frac{Y_v(\mu_n r/R_1) R_1}{\mu_n Y_v(\mu_n k)};$ $N = \frac{R_2^2}{2} \left( 1 - \frac{v^2}{\mu_n^2 k^2} \right) U_v^2(\mu_n, k)$ $- \frac{R_1^2}{2} U_v'^2(\mu_n, 1)$	$\frac{J_v(\mu)}{J'_v(\mu k)} - \frac{Y_v(\mu)}{Y'_v(\mu k)} = 0$
of the first kind $\alpha_1 = 1$ $\lambda_1 = 0$ $U_v(\mu, 1) = 0$	of the first kind $\alpha_2 = 1$ $\lambda_2 = 0$ $U_v(\mu, k) = 0$	$U_v(\mu_n, r/R_1)$ $= \frac{J_v(\mu_n r/R_1)}{J_v(\mu_n k)} - \frac{Y_v(\mu_n r/R_1)}{Y_v(\mu_n k)};$ $N = \frac{R_2^2}{2} U_v'^2(\mu_n, k) - \frac{R_1^2}{2} U_v'^2(\mu_n, 1)$	$\frac{J_v(\mu)}{J_v(\mu k)} - \frac{Y_v(\mu)}{Y_v(\mu k)} = 0$

Hankel Integral Transformations for Two-Dimensional Heat-Conduction Problems in Cylindrical Coordinates [2-21]

Range	Transformation formula	Inversion formula	Transform $\nabla^2 T$
Solid cylinder $0 \leq r \leq R_3$	$T_I = \int_0^{R_3} r K_0(\beta_n, r) T dr$	$T = \sum_{n=1}^{\infty} K_0(\beta_n, r) T_I$	$-\beta_n^2 T_I + \left[ \frac{K_0(\beta_n, r)}{\lambda_3} \right]_{r=R_3} f_{3I}$
$0 \leq r < \infty$	$T_I = \int_0^{\infty} r J_0(\beta r) T dr$	$T = \int_0^{\infty} \beta J_0(\beta r) T_I d\beta$	$-\beta^2 T_I$
$0 \leq r \leq R_3$ $0 \leq \theta \leq 2\pi$	$T_I = \int_0^{R_3} \int_0^{2\pi} r K_v(\beta_n, r) \times \cos v(\theta - \theta') T d\theta' dr$	$T = \frac{1}{\pi} \sum_{v=0}^{\infty} \sum_{n=1}^{\infty} K_v(\beta_n, r) T_I$	$-\lambda_n^2 T_I + R_3 \left[ \frac{K_v(\beta_n, r)}{\lambda_3} \right]_{r=R_3} f_{3I}(v)$
$0 \leq r \leq R_3$ $0 \leq z \leq l$	$T_I = \int_0^{R_3} \int_0^l r K_0(\beta_n, r) \times K(\gamma_m, z) T dz dr$	$T = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} K_0(\beta_n, r) \times K(\gamma_m, z) T_I$	$-\beta_n^2 T_I + R_3 \left[ \frac{K_0(\beta_n, r)}{\lambda_3} \right]_{r=R_3} f_{3I}(\gamma_m) + \left[ \frac{K(\gamma_m, z)}{\lambda_3} \right]_{z=0} f_{3I}(\beta_n) + \left[ \frac{K(\gamma_m, z)}{\lambda_3} \right]_{z=l} f_{4I}(\beta_n)$
$0 \leq r \leq R_3$ $0 \leq z \leq \infty$	$T_I = \int_0^{R_3} \int_0^{\infty} r K_0(\beta_n, r) \times K(\gamma, z) T dz dr$	$T = \int_0^{\infty} \sum_{n=1}^{\infty} K_0(\beta_n, r) \times K(\gamma, z) T_I d\gamma$	$-\lambda_n^2 T_I + R_3 \left[ \frac{K_0(\beta_n, r)}{\lambda_3} \right]_{r=R_3} f_{3I}(\gamma) + \left[ \frac{K(\gamma, z)}{\lambda_3} \right]_{z=0} f_{3I}(\beta_n)$



$0 \leq r \leq R_3$ $-\infty < z < \infty$	$T_I = \int_0^{R_3} \int_0^\infty r K_0(\beta_n, r) \times e^{i\gamma z} T dz dr$	$T = \frac{1}{2\pi} \int_{-\infty}^\infty \sum_{n=1}^\infty K_0(\beta_n, r) \times e^{-i\gamma z} T_I d\gamma$	$-\lambda_n^2 T_I + R_3 \times \left[ \frac{K_0(\beta_n, r)}{\lambda_3} \right]_{r=R_3} f_{3I}(\gamma)$
$0 \leq r < \infty$ $0 \leq \theta \leq 2\pi$	$T_I = \int_0^{2\pi} \int_0^\infty r J_\nu(\beta r) \times T \cos \nu(\theta - \theta') d\theta' dr$	$T = \frac{1}{\pi} \sum_{\nu=0}^\infty \int_0^\infty \beta J_\nu(\beta r) T_I d\beta$	$-\beta^2 T_I$
$0 \leq r < \infty$ $0 \leq z \leq l$	$T_I = \int_0^l \int_0^\infty r J_0(\beta r) \times K(\gamma_m, z) T dz dr$	$T = \sum_{m=1}^\infty \int_0^\infty \beta J_0(\beta r) \times K(\gamma_m, z) T_I d\beta$	$-\lambda_m^2 T_I + \left[ \frac{K(\gamma_m, z)}{\lambda_3} \right]_{z=0} f_{3I}(\beta) + \left[ \frac{K(\gamma_m, z)}{\lambda_4} \right]_{z=l} f_{4I}(\beta)$
$0 \leq r < \infty$ $0 \leq z < \infty$	$T_I = \int_0^\infty \int_0^\infty r J_0(\beta r) \times K(\gamma, z) T dz dr$	$T = \int_0^\infty \int_0^\infty \beta J_0(\beta r) \times K(\gamma, z) T_I d\gamma d\beta$	$-\lambda^2 T_I + \left[ \frac{K(\gamma, z)}{\lambda_3} \right]_{z=0} f_{3I}(\beta)$
$0 \leq r < \infty$ $-\infty < z < \infty$	$T_I = \int_0^\infty \int_0^\infty r J_0(\beta r) \times e^{i\gamma z} T dz dr$	$T = \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty \beta J_0(\beta r) \times e^{-i\gamma z} T_I d\gamma d\beta$	$-\lambda^2 T_I$

Table 2-14 (continued)

Range	Transformation formula	Inversion formula	Transform $\nabla^2 T$
Hollow cylinder $R_1 \leq r \leq R_2$	$T_I = \int_{R_1}^{R_2} r K_0(\beta_n, r) T dr$	$T = \sum_{n=1}^{\infty} K_0(\beta_n, r) T_I$	$\begin{aligned} & -\beta_n^2 T_I + R_1 \\ & \times \left[ \frac{K_0(\beta_n, r)}{\lambda_1} \right]_{r=R_1} f_{1I} + R_2 \\ & \times \left[ \frac{K_0(\beta_n, r)}{\lambda_2} \right]_{r=R_2} f_{2I} \end{aligned}$
$R_1 \leq r \leq R_2$ $0 \leq \theta \leq 2\pi$	$\begin{aligned} T_I &= \int_{R_1}^{R_2} \int_0^{2\pi} r K_\nu(\beta_n, r) \\ & \times \cos \nu(\theta - \theta') T d\theta' dr \end{aligned}$	$T = \frac{1}{\pi} \sum_{\nu=0}^{\infty} \sum_{n=1}^{\infty} K_\nu(\beta_n, r) T_I$	$\begin{aligned} & -\beta_n^2 T_I + R_1 \\ & \times \left[ \frac{K_0(\beta_n, r)}{\lambda_1} \right]_{r=R_1} f_{1I}(\nu) \\ & + R_2 \left[ \frac{K_0(\beta_n, r)}{\lambda_2} \right]_{r=R_2} f_{2I}(\nu) \end{aligned}$
$R_1 \leq r \leq R_2$ $0 \leq z \leq l$	$\begin{aligned} T_I &= \int_{R_1}^{R_2} \int_0^l r K_0(\beta_n, r) \\ & \times K(\gamma_m, z) T dz dr \end{aligned}$	$\begin{aligned} T &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} K_0(\beta_n, r) \\ & \times K(\gamma_m, z) T_I \end{aligned}$	$\begin{aligned} & -\lambda_{nm}^2 T_I + R_1 \\ & \times \left[ \frac{K_0(\beta_n, r)}{\lambda_1} \right]_{r=R_1} f_{1I}(\gamma_m) \\ & + R_2 \left[ \frac{K_0(\beta_n, r)}{\lambda_2} \right]_{r=R_2} f_{2I}(\gamma_m) \\ & + \left[ \frac{K_0(\gamma_m, z)}{\lambda_3} \right]_{z=0} f_{3I}(\beta_n) \\ & + \left[ \frac{K(\gamma_m, z)}{\lambda_4} \right]_{z=l} f_{4I}(\beta_n) \end{aligned}$

$R_1 \leq r \leq R_2$ $0 \leq z < \infty$	$T_I = \int_{R_1}^{R_2} \int_0^\infty r K_0(\beta_n, r) \times K(\gamma, z) T dz dr$	$T = \int_0^\infty \sum_{n=1}^\infty K_0(\beta_n, r) \times K(\gamma, z) T_I d\gamma$	$ \begin{aligned} & -\lambda_n^2 T_I + R_1 \\ & \times \left[ \frac{K_0(\beta_n, r)}{\lambda_1} \right]_{r=R_2} f_{1I}(\gamma) \\ & + R_2 \left[ \frac{K_0(\beta_n, r)}{\lambda_3} \right]_{r=R_2} f_{2I}(\gamma) \\ & + \left[ \frac{K(\gamma, z)}{\lambda_3} \right]_{z=0} f_{3I}(\beta_n) \end{aligned} $
$R_1 \leq r \leq R_2$ $-\infty < z < \infty$	$T_I = \int_{R_1}^{R_2} \int_{-\infty}^\infty r K_0(\beta_n, r) \times e^{i\gamma z} T dz dr$	$T = \frac{1}{2\pi} \int_{-\infty}^\infty \sum_{n=1}^\infty K_0(\beta_n, r) \times e^{-i\gamma z} T_I d\gamma$	$ \begin{aligned} & -\lambda_n^2 T_I + R_1 \\ & \times \left[ \frac{K_0(\beta_n, r)}{\lambda_1} \right]_{r=R_1} f_{1I}(\gamma) \\ & + R_2 \left[ \frac{K_0(\beta_n, r)}{\lambda_3} \right]_{r=R_2} f_{2I}(\gamma) \end{aligned} $

Notes: 1. Roots for variable  $r$  are given in Tables 2-12 and 2-13.

2. For boundary conditions of the first kind  $K/\lambda_j$  should be substituted by  $\frac{1}{\alpha_j} \frac{dK}{dn_j}$ ;  $f_I(\gamma)$ ,  $f_I(\beta)$ ,  $f_{1I}(\gamma)$  are the transforms of function  $f$  with respect

to variables  $\varphi$ ,  $r$  and  $z$ .

3. When  $\nu = 0$   $\pi$  is substituted by  $2\pi$ .

4.  $\lambda_{nm}^2 \equiv \beta_n^2 + \gamma_m^2$ ;  $\lambda_n^2 \equiv \beta_n^2 + \gamma^2$  and so on.

heat-transfer coefficient is a function of time, the problem cannot be solved by the finite integral transformation; it is solved by the heat-potential method [2-35].

### (c) Solution of Problems in Spherical Coordinates

The differential heat-conduction equation in spherical coordinates may be written as follows

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \mu} \left[ (1 - \mu^2) \frac{\partial T}{\partial \mu} \right] + \frac{1}{r^2(1 - \mu^2)} \frac{\partial^2 T}{\partial \theta^2} = \frac{1}{a} \frac{\partial T}{\partial \tau} \quad (2-10-25)$$

where  $\mu = \cos \Phi$ .

Equation (2-10-25) may be solved by the method of separation of variables

$$T(r, \mu, \theta, \tau) = \mathfrak{I}(\tau) U(r) M(\mu) \Theta(\theta)$$

Equation (2-10-25) is then split into the following four equations

$$\frac{d\mathfrak{I}}{d\tau} + a\beta^2 \mathfrak{I} = 0, \quad \frac{d^2 \Theta}{d\theta^2} + m^2 \Theta = 0 \quad (2-10-26)$$

$$\frac{d^2 U}{dr^2} + \frac{2}{r} \frac{dU}{dr} + \left[ \beta^2 - \frac{n(n+1)}{r^2} \right] U = 0 \quad (2-10-27)$$

$$\frac{d}{d\mu} \left[ (1 - \mu^2) \frac{dM}{d\mu} \right] + \left[ n(n+1) - \frac{m^2}{1 - \mu^2} \right] M = 0 \quad (2-10-28)$$

The solution of equations (2-10-26) through (2-10-28) will be

$$\mathfrak{I} \sim e^{-a\beta^2 \tau}; \quad \Theta \sim \sin m\theta; \quad \cos m\theta \quad (2-10-29)$$

$$U \sim (\beta r)^{-\frac{1}{2}} J_{n+1/2}(\beta r); \quad (\beta r)^{-1/2} Y_{n+1/2}(\beta r) \quad (2-10-30)$$

$$M \sim P_n^m(\mu); \quad Q_n^m(\mu) \quad (2-10-31)$$

where  $P_n^m(\mu)$  and  $Q_n^m(\mu)$  are the  $n$ -th power Legendre functions of the  $m$ -th order of the first and second kinds, respectively.

Introduction of the new variable  $W = (\beta r)^{-1/2}U$  reduces equation (2-10-27) to the form

$$\frac{d^2 W}{dr^2} + \frac{1}{r} \frac{dW}{dr} + \left[ \beta^2 - \frac{1}{r^2} \left( n + \frac{1}{2} \right)^2 \right] W = 0 \quad (2-10-32)$$

Equation (2-10-32) is the Bessel equation of the  $\left( n + \frac{1}{2} \right)$ -th order.

The solution of this equation are the Bessel functions.

Solution of equation (2-10-32) will certainly involve Bessel spherical functions of the first and second kinds ( $n = 0, 1, 2, 3$ )

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x); \quad y_n(x) = \sqrt{\frac{\pi}{2x}} Y_{n+1/2}(x) \quad (2-10-33)$$

In particular, it should be noted that at  $n = 0.1$

$$\left. \begin{aligned} j_0(x) &= \frac{\sin x}{x}; & y_0(x) &= -\frac{\cos x}{x}; & j_1(x) &= \frac{\sin x - x \cos x}{x^2} \\ y_1(x) &= \frac{\cos x - x \sin x}{x^2} \end{aligned} \right\} \quad (2-10-34)$$

$\mu$  varies from  $-1$  to  $+1$ , as the angle  $\Phi$  ranges from  $0$  to  $\pi$ .

Expansion of function  $F(\mu)$  with respect to functions  $P_n^m(\mu)$  in the range  $(-1 < \mu < 1)$  gives transformation formulas of the form

$$F(\mu) = \sum_{n=0}^{\infty} K_n^m(\mu) F_I(n, m) \quad (2-10-35)$$

$$F_I(n, m) = \int_{-1}^{+1} K_n^m(\mu) F(\mu) d\mu \quad (2-10-36)$$

where

$$K_n^m(\mu) = \sqrt{\frac{2n+1}{2} \frac{(n-m)!}{(n+m)!}} P_n^m(\mu) \quad (2-10-37)$$

As the Legendre functions  $P_n^m(\mu)$  are orthogonal in the interval  $(-1 < \mu < 1)$  with the weight function equal to unity, then

$$\int_{-1}^{+1} P_n^m(\mu) P_k^m(\mu) d\mu = \begin{cases} 0 & \text{at } n \neq k \\ \frac{2}{(2n+1)} \frac{(n+m)!}{(n-m)!} & \text{at } n = k \end{cases} \quad (2-10-38)$$

In Table 2-15 the transformation and inversion formulas are given for  $\mu$  in different intervals.

The differential operator of the variable  $\mu$

$$D \equiv \frac{\partial}{\partial \mu} \left[ (1 - \mu^2) \frac{\partial T}{\partial \mu} \right]$$

entering into equation (2-10-28) changes into the Legendre differential equation

$$\frac{d}{d\mu} \left[ (1 - \mu^2) \frac{dK_n}{d\mu} \right] + n(n+1) K_n = 0 \quad (2-10-39)$$

where  $K_n(\mu)$  is the normalized Legendre function.

This equation is solved by the Legendre integral transformation

$$T_I = \int K_n(\mu) T d\mu; \quad T = \sum_{n=0}^{\infty} K_n(\mu) T_I \quad (2-10-40)$$

The form of the transformation kernel  $K_n(\mu)$  depends on the variation range of  $\mu$  (see Table 2-15).

If the operator  $D$  has the form

$$D \equiv \frac{\partial}{\partial \mu} (1 - \mu^2) \frac{\partial T}{\partial \mu} + \frac{1}{1 - \mu^2} \frac{\partial^2 T}{\partial \theta^2}; \quad -1 < \mu < 1, \quad 0 < \theta < 2\pi \quad (2-10-41)$$

then the following formulas

$$T_I(\mu, m, \theta) = \int_0^{2\pi} \cos m(\theta' - \theta) T(\mu, \theta') d\theta' \quad (2-10-42)$$

$$T(\mu, \theta) = \frac{1}{\pi} \sum_{m=0}^{\infty} T_I(\mu, m, \theta) \quad (2-10-43)$$

are used to solve the problem. In formulas (2-10-42) and (2-10-43)  $m = 0, 1, 2, 3, \dots$

The transformation and inversion formulas and also the integral transformation  $\nabla^2 T$  are tabulated in Table 2-16. Tables 2-15 and 2-16 may be used for solving particular problems.

In this chapter, problems with the boundary conditions of the fourth kind have not been considered. The problems on a system of solids, which are in thermal interaction, may be solved by the finite integral transformation method [2-21] or by simultaneous application of the Laplace and Fourier-Hankel transformations [2-18]. The convective heat-transfer problems with the boundary conditions of the fourth kind will be dealt with in Chapter 4.

Table 2-15

Legendre Integral Transformation and Corresponding Inversion Formulas for Different Ranges of  $\mu$

Range of $\mu$	Integral transformation	Inversion formula	Kernel of the transform
$-1 \leq \mu \leq 1$	$T_I(n) = \int_{-1}^1 K_n(\mu) T d\mu$	$T = \sum_{n=0}^{\infty} K_n(\mu) T_I(n)$	$K_n(\mu) = \sqrt{\frac{2n+1}{2}} P_n(\mu)$
$0 \leq \mu \leq 1$	$T_I(2n+1) = \int_0^1 K_{2n+1}(\mu) T d\mu$	$T = \sum_{n=0}^{\infty} K_{2n+1}(\mu) T_I(2n+1)$	$K_{2n+1}(\mu) = \sqrt{4n+3} P_{2n+1}(\mu)$
$0 \leq \mu \leq 1$	$T_I(2n) = \int_0^1 K_{2n}(\mu) T d\mu$	$T = \sum_{n=0}^{\infty} K_{2n}(\mu) T_I(2n)$	$K_{2n}(\mu) = \sqrt{4n+1} P_{2n}(\mu)$
$-1 \leq \mu \leq 1$	$T_I(n, m) = \int_{-1}^1 K_n^m(\mu) T d\mu$	$T = \sum_{n=0}^{\infty} K_n^m(\mu) T_I(n, m)$	$K_n^m(\mu) = \sqrt{\frac{2n+1}{2} \frac{(n-m)!}{(n+m)!}} P_n^m(\mu)$

Integral Transformation (Legendre transformation) for Laplace Operator in Spherical Coordinates

Range of variation	Boundary condition	Integral transformation	Inversion formula	$\nabla^2 T$	Transform $\nabla^2 T$
$-1 \leq \mu \leq 1$	Periodical temperature	$T_I = \int_{-1}^1 K_n(\mu) T d\mu$	$T = \sum_{n=0}^{\infty} K_n(\mu) T_I$	$\frac{\partial}{\partial \mu} \left[ (1 - \mu^2) \frac{\partial T}{\partial \mu} \right]$	$-n(n+1) T_I$
$-1 \leq \mu \leq 1$ ; $0 \leq \theta \leq 2\pi$	Periodical temperature with period $2\pi$	$T_I = \int_0^{2\pi} \int_{-1}^1 K_n^m(\mu) T \times$ $\times \cos m(\theta - \theta') d\mu d\theta'$	$T = \frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} K_n^m(\mu) \times$ $\times T_I$ (at $m=0$ $\pi$ is substituted by $2\pi$ )	$\frac{\partial}{\partial \mu} \left[ (1 - \mu^2) \frac{\partial T}{\partial \mu} \right] + \frac{m^2}{1 - \mu^2} \frac{\partial^2 T}{\partial \theta^2}$	$-n(n+1) T_I$
$0 \leq \mu \leq 1$	$\frac{\partial T}{\partial \mu} \Big _{\mu=0} = f$	$T_I = \int_0^1 K_{2n}(\mu) T d\mu$	$T = \sum_{n=0}^{\infty} K_{2n}(\mu) T_I$	$\frac{\partial}{\partial \mu} \left[ (1 - \mu^2) \frac{\partial T}{\partial \mu} \right]$	$-2n(2n+1) T_I$ $- [K_{2n}(\mu)]_{\mu=0} f$
$0 \leq \mu \leq 1$	$T _{\mu=0} = f$	$T_I = \int_0^1 K_{2n+1}(\mu) T d\mu$	$T = \sum_{n=0}^{\infty} K_{2n+1}(\mu) T_I$	$\frac{\partial}{\partial \mu} \left[ (1 - \mu^2) \frac{\partial T}{\partial \mu} \right]$	$-(2n+1)(2n+2) T_I$ $+ \left[ \frac{dK_{2n+1}(\mu)}{d\mu} \right]_{\mu=0} f$
$-1 \leq \mu \leq 1$ $0 \leq r \leq R_I$	$T _{r=R_I} = f(\mu)$	$T_I = \int_0^{R_I} \int_{-1}^1 r^2 K_{np}(r, \mu) \times$ $\times T dr d\mu$	$T = \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} K_{np}(r, \mu) \times$ $\times T_I$	$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \mu} \left[ (1 - \mu^2) \frac{\partial T}{\partial \mu} \right]$	$-v_{np}^2 T_I - \int_{-1}^1 R_I^2 f(\mu)$ $\times \left[ \frac{dK_{np}(r, \mu)}{dr} \right]_{r=R_I} d\mu$

Note. Kernels in the present table are as follows:

$$K_n(\mu) = \sqrt{\frac{2n+1}{2}} P_n(\mu); K_{2n}(\mu) = \sqrt{4n+1} P_{2n}(\mu); K_{np}(\mu, r) = \frac{r^{-1/2} J_n + 1/2 (v_{np}') P_n(\mu)}{\sqrt{N_r N_\mu}},$$

$$K_n^m(\mu) = \sqrt{\frac{2n+1}{2}} \frac{(n-m)!}{(n+m)!} P_n^m(\mu) \quad (m \leq n); K_{2n+1}(\mu) = \sqrt{4n+3} P_{2n+1}(\mu), \text{ where } N_r N_\mu = -\frac{R_I^2}{2n+1} J_n -$$

$$-1/2 (v_{nn} R_I) J_n + 3/2 (v_{nn} R_I); v_{nn} - \text{positive kernels of equation } J_n + \frac{1}{2} \times (v_n R_I) = 0.$$



# CONVECTIVE HEAT TRANSFER

## 3

### 3-1. HEAT AND MASS TRANSFER IN A FLOW PAST A FLAT PLATE

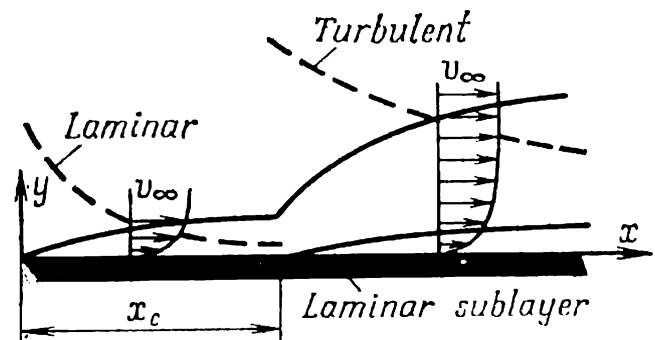
Heat and mass transfer between a solid surface and the surrounding fluid \* can take place under the effect of external forces and also due to the difference in densities caused by local heating in the gravity field. In the former case, heat and mass transfer takes place under conditions of forced convection; in the latter case it is caused by free convection.

Let us consider a very simple case of a steady-state flow past a plate of sufficient length  $l$ . If the plate is oriented along the flow so that the  $x$ -axis coincides with the flow direction, the  $y$ -axis is normal to the plate, and the  $z$ -axis lies along the plate width, then  $v_z = 0$ ;  $\partial/\partial z = 0$  and  $\partial/\partial \tau = 0$ , i.e. a parallel plane flow is considered.

#### (a) *Hydrodynamics of a Flow past a Flat Plate*

If in a flow past a flat plate the longitudinal fluid velocity  $v_x$  is measured along the  $y$ -axis, then we obtain the longitudinal velocity distribution curve  $v_x(y)$  or the velocity profile  $v_x(y)$  presented in Fig. 3-1.

Fig. 3-1. Simplified model showing transition in a boundary layer from laminar to turbulent flow over a flat plate



The fluid velocity increases along the  $y$ -axis, and then starting from a certain distance  $\delta$  from the surface it remains almost constant. Therefore, the fluid velocity drop from a value equal to zero to a value  $v_\infty \equiv v_c$ , corresponding to a velocity far from the wall (the main fluid flow), occurs

\* Generally, in hydrodynamics the word "fluid" denotes a gas or a mixture of gases; in case it is necessary to emphasize the liquid state of the substance, we shall simply write liquid.

in a layer of thickness  $\delta$ . This layer is known as the boundary layer. The reason for the decrease in velocity from  $v_c$  to 0 is the viscosity, i.e. the ability of fluid to resist relative motion of fluid particles due to random molecular motion.

Hence, the whole fluid flux may be divided into two regions, namely, the boundary layer which exhibits the viscous force effect, and the main flow where viscosity may be neglected, and the fluid may be considered ideal.

A fluid flow, by its nature, is divided into laminar and turbulent flows. Heat transfer in a turbulent flow takes place more intensively than in a laminar flow due to the random motion of fluid particles (microvolumes). The flow in a boundary layer may also be turbulent.

Development of a boundary layer along the surface of the body in the flow direction is shown in Fig. 3-1. The boundary layer thickness increases continuously along the plate surface starting from zero at the leading edge. At a distance  $x_k$  from the leading edge the laminar flow changes into a turbulent one. As the velocity  $v_c$  increases, the critical distance  $x_k$  decreases, but the product  $v_c x_k$  remains constant.

Experiments with fluids of different viscosities may reveal that transition from a laminar flow to a turbulent one occurs at a definite value of  $\frac{v_c x_k}{\nu}$ , i.e. transition is characterized by the critical Reynolds number  $\left( Re = \frac{v_c x_k}{\nu} \right)$ .

$Re = 3 \cdot 10^6$  is the upper limit for a laminar boundary layer of a flat plate, and  $Re \approx 8 \cdot 10^4$  is the lower limit. Upon creating stronger disturbances in a flow the lower limit for the transition from one state into another may be somewhat lowered. It may be assumed, in practice, that for  $Re > 5 \cdot 10^5$  the flow in the boundary layer is turbulent. When a fluid flows through a pipe the critical Reynolds number is practically constant and equals 2300 [3-1].

For an incompressible fluid ( $\rho = \text{const}$ ;  $d\rho/d\tau = 0$ ;  $\text{div } \vec{v} = 0$ ) without an external force field ( $F_k = 0$ ) and for a steady-state parallel plane flow ( $\partial/\partial\tau = 0$ ) past a plate, the Navier-Stokes equation may be written as follows:

in the  $x$ -direction

$$\left( \rho v_x \frac{\partial v_x}{\partial x} + \rho v_y \frac{\partial v_x}{\partial y} \right) = - \frac{\partial p}{\partial x} + \eta \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right) \quad (3-1-1)$$

in the  $y$ -direction

$$\left( \rho v_x \frac{\partial v_y}{\partial x} + \rho v_y \frac{\partial v_y}{\partial y} \right) = - \frac{\partial p}{\partial y} + \eta \left( \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right) \quad (3-1-2)$$

The continuity equation

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \quad (3-1-3)$$

The boundary conditions:

fluid adhesion to the wall

$$v_x = v_y = 0 \text{ when } y = 0 \quad (3-1-4)$$

coincidence of velocity  $v_x$  far from the boundary layer with the main flow velocity

$$v_x = v_\infty \text{ when } y \rightarrow \infty \quad (3-1-5)$$

It may be demonstrated that in a thin boundary layer the variation of  $v_x$  in the  $y$ -direction is maximum, and insignificant in the  $x$ -direction. In the direction of flow,  $x$  varies between 0 and  $l$ , where  $l$  is the characteristic dimension of the body (in our case  $l$  represents the plate length). In the  $y$ -direction the flow parameters vary within the limits of the boundary layer. The boundary layer thickness  $\delta$  is considerably less than the plate length ( $\delta \ll l$ ).

In accordance with the well-known estimates of boundary-layer theory, the terms of the order  $\delta^* = \delta/l$  ( $\delta^* \ll 1$ ) may be neglected. Then out of the two Navier-Stokes equations only one remains and equations (3-1-1) to (3-1-3) may be written as:

$$\frac{\partial v_x}{\partial \tau} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 v_x}{\partial y^2} \quad (3-1-6)$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \quad (3-1-7)$$

The system of equations (3-1-6) and (3-1-7) is called the Prandtl equations for a boundary layer. The boundary conditions will be

$$\text{at } y = 0 \quad v_x = v_y = 0 \quad (3-1-8)$$

$$\text{at } y = \infty \quad v_x = v_\infty(x, \tau) \quad (3-1-9)$$

The solution of the problem (3-1-6) to (3-1-9) is very difficult even for a steady-state flow. In the particular case of a very thin plate of infinite length ( $l \rightarrow \infty$ ) equation (3-1-9) was solved by Blasius [3-2].

In the considered case the velocity of a potential flow is constant ( $v_c = \text{const}$ ), therefore from the Bernoulli equation it is inferred that  $\partial p / \partial x = 0$ .

Let us now present the solution procedure. We shall introduce the quantity  $\psi$ , called the stream function of the fluid, defined by relations

$$v_x = \frac{\partial \psi}{\partial y} \text{ and } v_y = - \frac{\partial \psi}{\partial x} \quad (3-1-10)$$

Then  $\psi$  will be the function of dimensionless quantity  $\xi$  defined by

$$\xi = \frac{y}{(xv/v_\infty)^{1/2}} = y \left( \frac{v_\infty}{xv} \right)^{1/2} = \frac{y}{x} \sqrt{\text{Re}_x} \quad (3-1-11)$$

where  $\text{Re}_x = v_\infty x / v$

$$\psi = \int_0^y v_x dy = \int_0^\xi v_\infty \left( \frac{v_x}{v_\infty} \right) \left( \frac{xv}{v_\infty} \right)^{1/2} d\xi = (vxv_\infty)^{1/2} \int_0^\xi \left( \frac{v_x}{v_\infty} \right) d\xi \quad (3-1-12)$$

In the solution of this equation the following relations were used

$$\left( \frac{\partial \xi}{\partial y} \right)_x = \left( \frac{xv}{v_\infty} \right)^{-\frac{1}{2}}; \quad \left( \frac{\partial \xi}{\partial x} \right)_y = -\frac{y}{2x} \left( \frac{xv}{v_\infty} \right)^{-\frac{1}{2}} \quad (3-1-13)$$

Then

$$f(\xi) = \int_0^\xi \left( \frac{v_x}{v_\infty} \right) d\xi = \frac{\psi}{(vxv_\infty)^{1/2}} \quad (3-1-14)$$

$$v_x = v_\infty \frac{\partial f}{\partial \xi} = v_\infty f' \quad (3-1-15)$$

Besides, we have

$$\begin{aligned} v_y &= -\frac{\partial \psi}{\partial x} = -\frac{1}{2} \left( \frac{vv_\infty}{x} \right)^{1/2} f - (vxv_\infty)^{1/2} \frac{\partial \xi}{\partial x} \frac{df}{d\xi} \\ &= \frac{1}{2} \left( \frac{vv_\infty}{x} \right)^{1/2} (\xi f' - f) = \frac{v_\infty}{2\sqrt{\text{Re}_x}} (\xi f' - f) \end{aligned} \quad (3-1-16)$$

Substitution of relations (3-1-15) and (3-1-16) into

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = v \frac{\partial^2 v_x}{\partial y^2}$$

yields

$$-\frac{v_c^2}{2x} \xi f' f'' + \frac{v_c^2}{2x} f'' (\xi f' - f) = v \frac{v_c^2}{xv} f'''$$

hence

$$f' f'' + 2f''' = 0 \quad (3-1-17)$$

Boundary conditions (3-1-8) and (3-1-9) are written as:

$$\left. \begin{array}{l} \text{at } \xi = 0 \quad f = f' = 0 \\ \text{at } \xi = \infty \quad f' = 1 \end{array} \right\} \quad (3-1-18)$$

Blasius obtained the solution of equation (3-1-17) in the form of convergent series for small values of  $\xi$  and the asymptotic approximation for large values of  $\xi$ . Howarth [3-3] presented a numerical solution of equation (3-1-17). The results are summarized in Table 3-1. In Fig. 3-2 the plot

Table 3-1

Blasius Function  $f(\xi)$  for a Flat Plate

$\xi = y \left( \frac{v_\infty}{\nu x} \right)^{1/2}$	$f$	$f' = \frac{v_x}{v_\infty}$	$f''$	$\xi = y \left( \frac{v_\infty}{\nu x} \right)^{1/2}$	$f$	$f' = \frac{v_x}{v_\infty}$	$f''$
0	0	0	0.33206	4.6	2.88826	0.98269	0.02948
0.2	0.00664	0.06641	0.33199	4.8	3.08534	0.98779	0.02187
0.4	0.02656	0.13277	0.33147	5.0	3.28329	0.99155	0.01591
0.6	0.05974	0.19894	0.33008	5.2	3.48189	0.99425	0.01134
0.8	0.10611	0.26471	0.32739	5.4	3.68094	0.99616	0.00793
1.0	0.16557	0.32979	0.32301	5.6	3.88031	0.99748	0.00543
1.2	0.23795	0.39378	0.31659	5.8	4.07990	0.99838	0.00365
1.4	0.32298	0.45627	0.30787	6.0	4.27964	0.99898	0.00240
1.6	0.42032	0.51676	0.29667	6.2	4.47948	0.99937	0.00155
1.8	0.52952	0.57477	0.28293	6.4	4.67938	0.99961	0.00098
2.0	0.65003	0.62977	0.26675	6.6	4.87931	0.99977	0.00061
2.2	0.78120	0.68132	0.24835	6.8	5.07928	0.99987	0.00037
2.4	0.92230	0.72899	0.22800	7.0	5.27926	0.99992	0.00022
2.6	1.07252	0.77246	0.20646	7.2	5.47025	0.99996	0.00013
2.8	1.23099	0.81152	0.18401	7.4	5.67924	0.99998	0.00007
3.0	1.39682	0.84605	0.16136	7.6	5.87924	0.99999	0.00004
3.2	1.56911	0.87609	0.13913	7.8	6.07923	1.00000	0.00002
3.4	1.74696	0.90177	0.11788	8.0	6.27923	1.00000	0.00001
3.6	1.92954	0.92333	0.09809	8.2	6.47923	1.00000	0.00001
3.8	2.11605	0.94112	0.08013	8.4	6.67923	1.00000	0.00000
4.0	2.30576	0.95552	0.06424	8.6	6.87923	1.00000	0.00000
4.2	2.49806	0.96696	0.05052	8.8	7.07923	1.00000	0.00000
4.4	2.69238	0.97587	0.03897				

$v_x/v_\infty = f'(\xi)$  is presented. From Table 3-1 and Fig. 3-2 it is evident that at  $\xi \geq 5$ ,  $v_x$  is practically equal to the flow velocity  $v_\infty$  (at  $\xi \geq 5$   $f' = 0.99$ ).

Figure 3-3 is a plot of the variation of transverse velocity  $v_y$  along the  $y$ -axis which shows that when  $\xi \rightarrow \infty$   $\frac{v_y}{v_\infty} \sqrt{\text{Re}_x}$  tends to the specific limit 0.8604. Therefore, at the external boundary layer the transverse velocity equals

$$(v_y)_{y \rightarrow \infty} = \frac{0.8604}{\sqrt{\text{Re}_x}} \quad (3-1-19)$$

If the complex profile  $v_y(y)$  shown in Fig. 3-2 is replaced by a straight line, i.e. by the straight line which joins the origin of coordinates with the point  $\xi = 5.2$ , then the approximate solution will be written as:

$$\frac{v_y}{v_\infty} = 0.165 \frac{y}{x}; \quad 0 < \xi < 5.2 \quad (3-1-20)$$

Hence, in the external boundary layer there is a velocity component normal to the plate. This may be attributed to the fact that the fluid flowing along

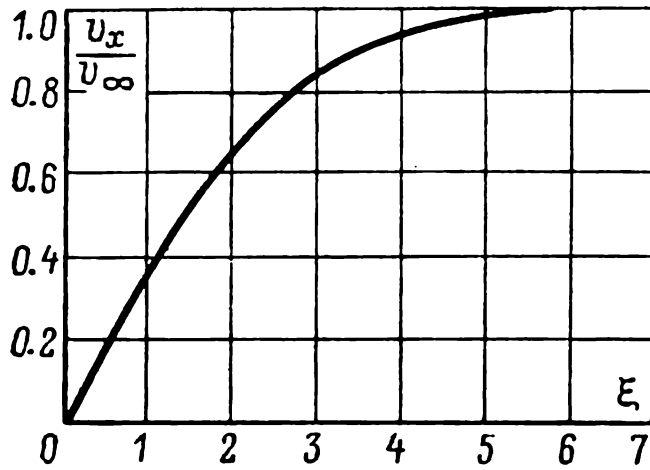


Fig. 3-2. Distribution of longitudinal velocity  $v_x$  in the  $y$ -direction in the boundary layer in the case of flow over a flat plate

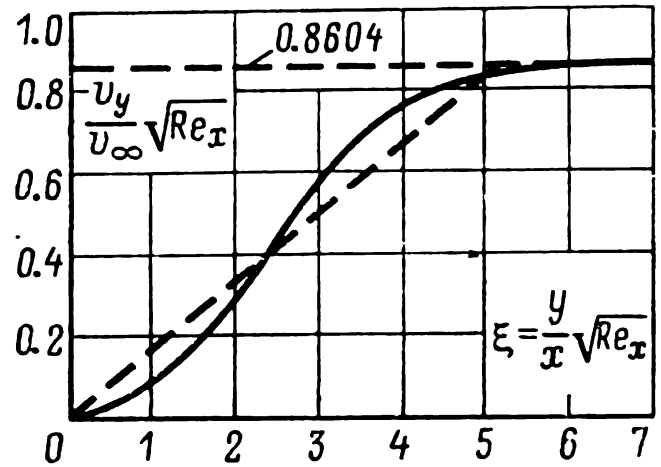


Fig. 3-3. Distribution of transverse velocity  $v_y$  along  $y$ -axis in the boundary layer in the case of flow over a flat plate

the plate gets slightly pushed aside from the surface due to the increase in the boundary layer thickness  $\delta(x)$  in the direction of flow.

The shear stress at the surface will be equal to

$$\begin{aligned} \sigma_0 &= -\eta \left( \frac{\partial v_x}{\partial y} \right)_{y=0} = -\eta v_\infty \left( \frac{v_\infty}{x\nu} \right)^{1/2} f''(0) = -0.33206 \eta v_\infty \left( \frac{v_\infty}{x\nu} \right)^{1/2} \\ &= -0.33206 \rho v_\infty^2 \text{Re}_x^{-1/2} \end{aligned} \quad (3-1-21)$$

Generally, a dimensionless quantity, called the local friction factor, is introduced

$$C_x = \frac{-\sigma_0}{(\rho v_\infty^2/2)} = \frac{0.6641}{\sqrt{\text{Re}_x}} \quad (3-1-22)$$

The mean friction factor  $C$  is

$$C = \bar{C}_x = \frac{1}{l} \int_0^l C_x dx = \frac{0.6641}{l \sqrt{\nu_c/\nu}} \int_0^l x^{-1/2} dx = \frac{1.328}{\sqrt{\text{Re}}} \quad (3-1-23)$$

An integral boundary-layer equation may be obtained from differential equations (3-1-6) and (3-1-7).

For a steady-state flow ( $\partial/\partial\tau = 0$ ), integration of equation (3-1-6) with respect to  $y$  over the interval 0 to  $\delta$  gives

$$\int_0^\delta v_x \frac{\partial v_x}{\partial x} dy + \int_0^\delta v_y \frac{\partial v_x}{\partial y} dy = \nu \int_0^\delta \frac{\partial^2 v_x}{\partial y^2} dy - \frac{1}{\rho} \int_0^\delta \frac{\partial p}{\partial x} dy \quad (3-1-24)$$

Using the relations

$$\frac{\partial}{\partial y} \int_0^\delta v_x v_y dy = \int_0^\delta v_x \frac{\partial v_y}{\partial y} dy + \int_0^\delta \frac{\partial v_x}{\partial y} v_y dy$$

$$\frac{\partial}{\partial y} \int_0^\delta v_x v_y dy = v_y v_x \Big|_0^\delta = - \int_0^\delta v_c \frac{\partial v_x}{\partial x} dy$$

$$\frac{\partial v_x}{\partial x} = - \frac{\partial v_y}{\partial y}; \quad v_y = - \int_0^\delta \frac{\partial v_x}{\partial x} dy$$

we obtain

$$\frac{\partial}{\partial x} \int_0^\delta v_x v_x dy - \frac{\partial}{\partial x} \int_0^\delta v_c v_x dy + \frac{\partial v_c}{\partial x} \int_0^\delta v_x dy = - \nu \left( \frac{\partial v_x}{\partial y} \right)_{y=0} - \frac{\partial p}{\partial x} \delta \quad (3-1-25)$$

for the boundary conditions

$$v_x = v_y = 0 \quad \text{at } y = 0$$

$$v_x = v_\infty \quad \text{at } y = \delta$$

Moreover, it was assumed that pressure along the  $y$ -axis remained unchanged.

Equation (3-1-24) may now be rewritten as:

$$\frac{\partial}{\partial x} \int_0^\delta (v_\infty - v_x) v_x dy - \frac{\partial v_\infty}{\partial x} \int_0^\delta v_x dy = \frac{1}{\rho} \sigma_0 + \frac{\partial p}{\partial x} \delta \quad (3-1-26)$$

Equation (3-1-26) is known as an integral boundary-layer equation for a laminar flow past a flat plate. It is sometimes written in a different form involving two linear parameters, namely, velocity displacement thickness  $\delta_{v,d}$  and momentum displacement thickness  $\delta_{m,d}$ , defined by

the relations

$$\delta_{v,d} = \frac{1}{v_\infty} \int_0^\infty (v_\infty - v_x) dy = \int_0^\infty \left(1 - \frac{v_x}{v_\infty}\right) dy \quad (3-1-27)$$

$$\delta_{m,d} = \frac{1}{v_\infty^2} \int_0^\infty (v_\infty - v_x)v_x dy = \int_0^\infty \left(1 - \frac{v_x}{v_\infty}\right) \frac{v_x}{v_\infty} dy \quad (3-1-28)$$

A geometrical interpretation of the velocity displacement thickness  $\delta_{v,d}$  is shown in Fig. 3-4. An infinite boundary layer ( $\delta \rightarrow \infty$ ) is assumed

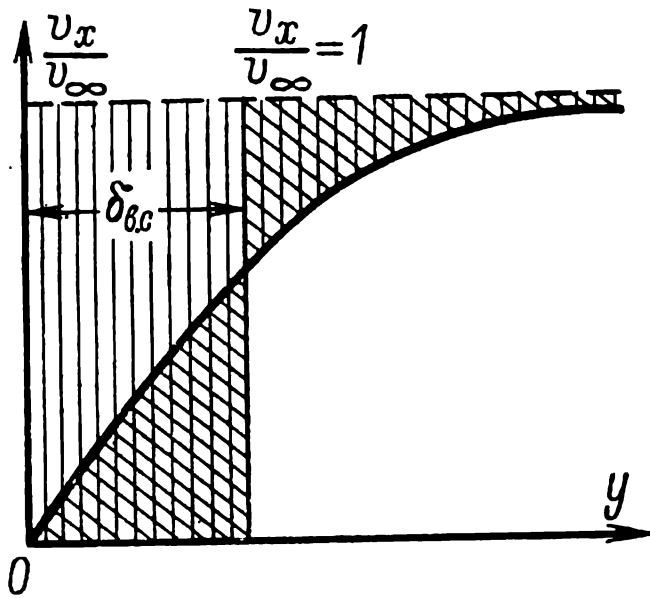


Fig. 3-4. Calculation of velocity displacement thickness  $\delta_{v,d}$ .

and it is substituted by a conventional finite layer of thickness  $\delta_{v,d}$  so that the area over the curve  $v_x/v_\infty$  would equal the area of rectangle  $\delta_{v,d} \cdot 1$ .

Physically this implies that in a frictionless fluid flow without a boundary layer, the velocity field remains the same as in a real fluid flow with a boundary layer only when the wall is displaced by  $\delta_{v,d}$ .

If the velocity distribution  $v_x(y)$  were linear, then  $\delta_{v,d} = \frac{1}{2} \delta_x$

$\delta_{m,d} = \frac{1}{6} \delta$  provided that at  $y = \delta$   $v_x = v_c$ , i.e. the limit  $\infty$  is replaced by  $\delta$ .

From the Bernoulli equation we find:

$$\delta \frac{\partial p}{\partial x} = -\rho \frac{\partial v_c}{\partial x} \int_0^\delta v_c dy \quad (3-1-29)$$

The integral boundary-layer equation will then take the form:

$$\frac{\partial}{\partial x} (\delta_{m,d} v_\infty^2) + v_\infty \delta_{v,d} \frac{\partial v_\infty}{\partial x} = \frac{\sigma_0}{\rho} \quad (3-1-30)$$



From the equation of motion (3-1-6) the energy equation may be obtained by multiplying the former by  $v_x$ . In a similar way the integral energy equation is obtained:

$$\frac{\partial}{\partial x} \int_0^\delta v_x (v_\infty^2 - v_x^2) dy = 2\nu \int_0^\delta \left( \frac{\partial v_x}{\partial y} \right)^2 dy \quad (3-1-31)$$

By introducing the dimensionless energy displacement thickness  $\delta_{e,d}$  and the dissipative function  $\Phi^*$  defined by the relations

$$\delta_{e,d} = \frac{1}{v_\infty^3} \int_0^\infty v_x (v_\infty^2 - v_x^2) dy \quad (3-1-32)$$

$$\Phi^* = \int_0^\infty \left[ \frac{\partial(v_x/v_\infty)}{\partial(y/\delta)} \right]^2 d\left(\frac{y}{\delta}\right) \quad (3-1-33)$$

We get the integral energy equation for a boundary layer in the form

$$\frac{dv_\infty^3 \delta_{e,d}}{dx} = \frac{2\nu v_\infty^2}{\delta} \Phi^* \quad (3-1-34)$$

The integral boundary layer equation is used for approximate solutions based on prescribed velocity profiles. For example, let us assume that the velocity profile  $v_x(y)$  is described by the polynomial

$$v_x = A + By + Cy^2 + Dy^3 \quad (3-1-35)$$

where  $A, B, C, D$  are constants determined from the following boundary conditions:

$$\text{at } y = 0 \quad v_x = 0 \quad \frac{\partial^2 v_x}{\partial y^2} = 0 \quad (3-1-36)$$

$$\text{at } y = \delta \quad v_x = v_c \quad \frac{\partial v_x}{\partial y} = 0 \quad (3-1-37)$$

These boundary conditions imply liquid adhesion at the wall ( $v_x = 0$ ), constant friction at the wall ( $\partial v_x / \partial y = \text{const}$ ), and constant longitudinal velocity  $v_x$  along  $y$  at distances greater than  $\delta$  ( $v_x = \text{const}$  when  $y \geq \delta$ ).

By satisfying these conditions, we obtain the values of constants

$$A = 0; \quad B = \frac{3}{2} \frac{v_c}{\delta}; \quad C = 0; \quad D = -\frac{1}{2} \frac{v_c}{\delta^3} \quad (3-1-38)$$

The velocity distribution curve will then take the form

$$\frac{v_x}{v_c} = \frac{3}{2} \frac{y}{\delta} - \frac{1}{2} \left( \frac{y}{\delta} \right)^3 \quad (3-1-39)$$

i.e. the transverse velocity distribution follows a semi-cubic parabola.

Let us now find the momentum displacement thickness

$$\delta_{m.d} = \frac{1}{v_\infty^2} \int_0^\delta (v_\infty - v_x) v_x dy = \frac{39}{280} \delta \quad (3-1-40)$$

The shear stress at the wall

$$\sigma_0 = -\eta \left( \frac{\partial v_x}{\partial y} \right)_{y=0} = -\frac{3}{2} \eta \frac{v_\infty}{\delta} \quad (3-1-41)$$

Then, assuming in the integral equation (3-1-29) that  $\partial v_\infty / \partial x = 0$ , we get

$$\frac{39}{280} v_\infty^2 \frac{d\delta}{dx} = -\frac{3}{2} v \frac{v_\infty}{\delta} \quad (3-1-42)$$

Integration of (3-1-42) gives

$$\delta = 4.64 \sqrt{\frac{v x}{v_\infty}} \quad (3-1-43)$$

since at  $x = 0$ ,  $\delta = 0$ .

The boundary layer thickness is thus proportional to  $\sqrt{x}$ . Relation (3-1-43) may be written as:

$$\frac{\delta}{x} = \frac{4.64}{\sqrt{\text{Re}_x}} \quad (3-1-44)$$

Substitution of (3-1-44) into formula (3-1-41) gives

$$\sigma_0 = -\frac{0.323 \rho v_\infty^2}{\sqrt{\text{Re}_x}} \quad (3-1-45)$$

Formula (3-1-45) differs from (3-1-21), obtained from the Blasius solution, by a numerical factor only. Here the numerical factor equals 0.323 instead of 0.33206 in (3-1-21), i.e. it is less by about 3%. It may be mentioned that if the velocity profile  $v_x(y)$  is governed by formula (3-1-39), then the velocity displacement thickness  $\delta_{v,d} = 0.375 \delta$  and

$$\delta_{m.d} = \frac{39}{280} \delta \approx \frac{1}{7} \delta [3-4] \quad (3-1-46)$$

The numerical coefficients appearing in formulas for boundary layer thickness and friction depend on velocity distribution in the boundary layer (see Table 3-2).

Table 3-2

**Effect of Velocity Profile in the Boundary Layer on the Boundary Layer Thickness and Friction Factor**

Velocity profile	Boundary conditions		$\frac{\delta}{x} \sqrt{\text{Re}_x}$	$C \sqrt{\text{Re}}$
	$y = 0$	$y = \delta$		
$\frac{v_x}{v_\infty} = \frac{y}{\delta}$	$v_x = 0$	$v_x = v_\infty$	3.46	1.156
$\frac{v_x}{v_\infty} = 2 \frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$	$v_x = 0$	$v_x = \infty$ $dv_x/dy = 0$	5.47	1.462
$\frac{v_x}{v_\infty} = \frac{3}{2} \frac{y}{\delta} - \frac{1}{2} \left(\frac{y}{\delta}\right)^3$	$v_x = 0$ $\frac{d^2v}{dy^2} = 0$	$v_x = \infty$ $\frac{dv_x}{dy} = 0$	4.64	1.292
Blasius solution	—	—	5.0	1.328

**(b) Heat Transfer in a Flow past a Flat Plate**

When fluid flows past a flat plate, the plate temperature varies between  $T_w$  (temperature at the plate surface) and  $T_\infty$  (fluid temperature at a distance from the surface). This variation in temperature  $T_\infty - T_w$  takes place in a small layer of thickness  $\delta_t$ , which characterizes the thermal boundary layer thickness different from the hydrodynamic boundary layer thickness  $\delta$ . It may be larger or smaller than  $\delta$  ( $\delta_t \gtrless \delta$ ) depending on the Prandtl number

$$\frac{\delta_t}{\delta} \sim \frac{1}{\sqrt{\text{Pr}}} \quad (3-1-47)$$

Therefore if  $\text{Pr} \geq 1$ , then  $\delta_t \leq \delta$ , if  $\text{Pr} \leq 1$ , then  $\delta_t \geq \delta$ .

The distribution curves of longitudinal velocity  $v_x$  and temperature  $T$  in a boundary layer for limiting Prandtl numbers are presented in Fig. 3-5.

Having estimated function  $\Phi^*$ , it may be demonstrated that all the terms of this function are approximately equal to  $\delta^{*2}$ , except the first term which equals 1. If all the terms appearing in the Fourier-Kirchhoff

equation for a boundary layer are estimated, then in the steady-state conditions ( $\partial T/\partial \tau = 0$ ) when fluid flows past the flat plate we shall get

$$c_p \rho \left( v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} \right) = \lambda \frac{\partial^2 T}{\partial y^2} + \eta \left( \frac{\partial v_x}{\partial y} \right)^2 \quad (3-1-48)$$

For a low velocity flow, the momentum equation (3-1-11) at constant pressure  $\partial p/\partial x = 0$  and the heat transfer equation (3-1-48) without pure dissipative function are similar. If the kinematic viscosity coefficient is

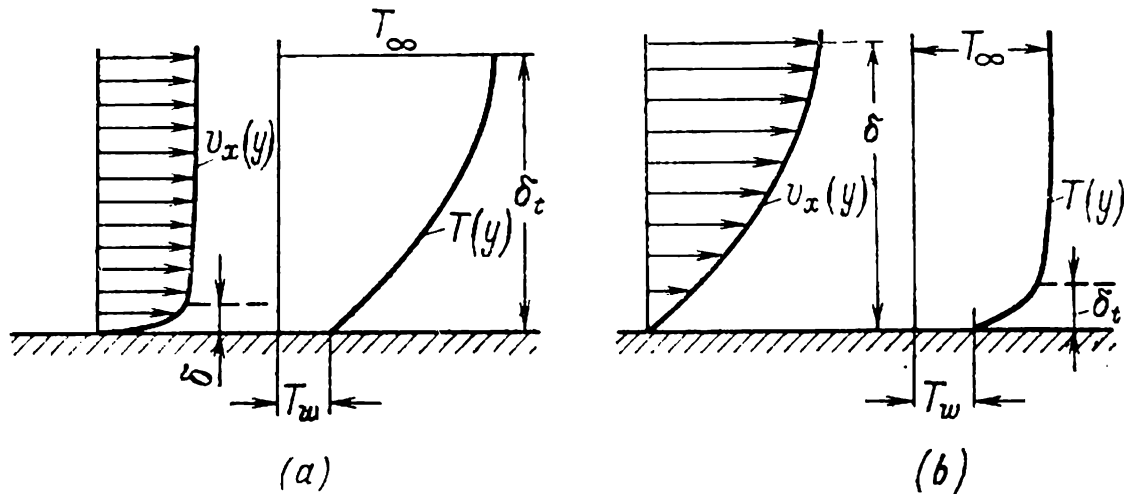


Fig. 3-5. Velocity  $v_x(y)$  and temperature  $T(y)$  profiles in a laminar boundary layer for limiting Pr

equal to the thermal diffusivity ( $\alpha = \nu$ ,  $Pr = 1$ ), then the solution of equations will be similar (the profiles of the velocity field and the temperature field in a boundary layer coincide), and the thicknesses of velocity and thermal boundary layers will be equal ( $\delta = \delta_t$ ).

It is of interest to estimate the value of the dissipative term  $\eta(\partial v_x/\partial y)^2$ . Calculations show that if  $v_c^2/c_p \Delta T < 1$ , then this term may be neglected. For air, for example, when the temperature difference  $\Delta T = 5.6^\circ\text{C}$  the viscous dissipation may be neglected provided  $v_c < 152 \text{ m/s}$ . In practice the fluid velocities are considerably less, therefore  $\eta(\partial v_x/\partial y)^2$  may be neglected.

Longitudinal velocity  $v_x$  and transverse velocity  $v_y$  depend on  $y$ . The integral equation of a thermal boundary layer is derived from equation (3-1-48) by integration with respect to  $y$  over the interval 0 to  $\delta_t$ . For this the following relations

$$\frac{\partial}{\partial x} \int_0^{\delta_t} v_x T dy = \int_0^{\delta_t} T \frac{\partial v_x}{\partial x} dy + \int_0^{\delta_t} v_x \frac{\partial T}{\partial x} dy \quad (3-1-49)$$

$$\frac{\partial}{\partial y} \int_0^{\delta_t} v_y T dy = \int_0^{\delta_t} v_y \frac{\partial T}{\partial y} dy - \int_0^{\delta_t} T \frac{\partial v_x}{\partial x} dy \quad (3-1-50)$$

and the boundary conditions

at  $y = 0$   $v_x = v_y = 0$ ; at  $y \rightarrow \infty$   $T = T_\infty$

$$v_x = v_\infty; \quad \left( \frac{\partial T}{\partial y} \right)_{y \rightarrow \infty} = 0 \quad (3-1-51)$$

are used.

As a result, we get

$$-\frac{\partial}{\partial x} \int_0^{\delta_t} (T_\infty - T) v_x dy = a \left( \frac{\partial T}{\partial y} \right)_{y=0} \quad (3-1-52)$$

The local heat transfer coefficient defined by the relation

$$\alpha_x = \frac{q_w}{(T_\infty - T_w)} \quad (3-1-53)$$

will be equal to

$$\alpha_x = - \frac{\lambda(\partial T/\partial y)_{y=0}}{T_\infty - T_w} = \frac{1}{c_p \rho (T_\infty - T_w)} \int_0^{\delta_t} (T_\infty - T) v_x dy \quad (3-1-54)$$

Equation (3-1-48) may be solved analytically or by numerical methods. If the functions  $\xi$  and  $f$  are introduced in accordance with (3-1-11) and (3-1-14), then equation (3-1-48) may be rewritten as

$$\frac{\partial^2 T}{\partial \xi^2} + \frac{1}{2} \text{Pr} f \frac{dT}{d\xi} = - \text{PrEc} T_\infty (f'')^2 \quad (3-1-55)$$

where Ec is the Eckert number ( $\text{Ec} = v_c^2/c_p T_\infty$ ).

The boundary conditions are as follows:

$$\text{at } \xi = 0 \quad T = T_w; \quad \text{at } \xi = \infty \quad T = T_\infty \quad (3-1-56)$$

Equation (3-1-55) is an ordinary differential equation and is solved in a conventional manner.

If the dissipative function is neglected ( $\Phi = 0$ ), then the equation takes the simplified form

$$\theta'' + \frac{1}{2} \text{Pr} f \theta' = 0 \quad (3-1-57)$$

where  $\theta$  is a dimensionless temperature

$$\theta = (T - T_\infty)/(T_w - T_\infty)$$

and the boundary conditions become

$$\text{at } \xi = 0 \quad \theta = 1; \quad \text{at } \xi = \infty \quad \theta = 0 \quad (3-1-58)$$

The solution of equation (3-1-57) for boundary conditions (3-1-58) was originally obtained by Schlichting [3-1] and is expressed as:

$$\theta = \frac{\int_{\xi}^{\infty} (f'')^{\text{Pr}} d\xi}{\int_0^{\infty} (f'')^{\text{Pr}} d\xi} \quad (3-1-59)$$

If  $\text{Pr} = 1$ , the temperature curve  $\theta(\xi)$  and the velocity curve  $v_x^*(y)$  will coincide, because  $\theta = 1 - f' = 1 - v_x/v_{\infty}$ .

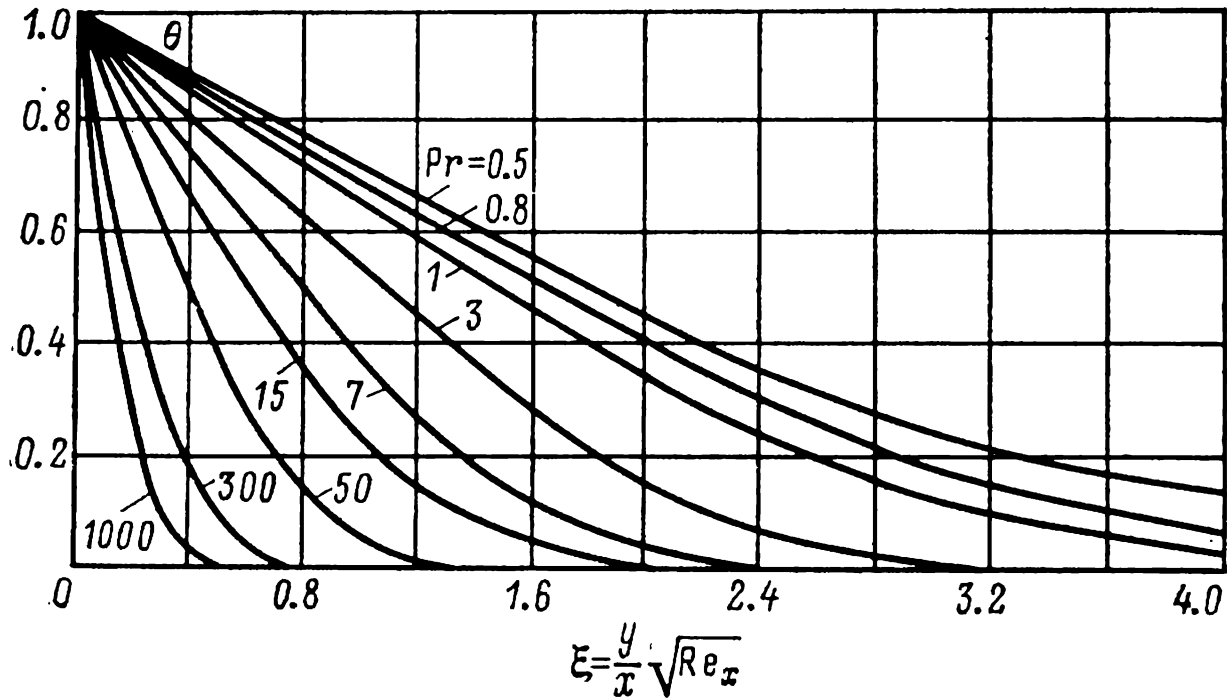


Fig. 3-6. Temperature  $\theta$  distribution in a laminar boundary layer at different  $\text{Pr}$  numbers

In Fig. 3-6 the dimensionless temperature distribution over generalized coordinate  $\xi$  is plotted for different Prandtl numbers varying between 0.5 and 1000.

Let us now introduce a variable  $\vartheta$  defined by the relation

$$\vartheta = \frac{T - T_{\infty}}{v_{\infty}^2 / 2c_p}$$

then equation (3-1-55) will take the form

$$\vartheta'' + \frac{1}{2} \text{Pr} f \vartheta' = -2\text{Pr} (f'')^2 \quad (3-1-60)$$

The solution of equation (3-1-60) may be obtained by the quadrature method.

If

$$\text{at } \xi = 0 \quad \vartheta' = 0; \quad \text{at } \xi = \infty \quad \vartheta = 0 \quad (3-1-61)$$

then the solution takes the form:

$$\vartheta = 2\text{Pr} \int_{\xi}^{\infty} (f'')^{\text{Pr}} \left[ \int_0^{\xi} (f'')^{2-\text{Pr}} d\xi \right] d\xi \quad (3-1-62)$$

For adiabatic conditions when  $T_w = T_a$

$$\vartheta(0) = \frac{T_a - T_{\infty}}{v_{\infty}^2 / 2c_p} \quad (3-1-63)$$

The values of  $\vartheta(0)$  for various Pr numbers are presented in Table 3-3 and Fig. 3-7.

Table 3-3

Effect of Prandtl Number on Temperature Distribution During Adiabatic Flow past a Flat Plate

Pr	0.6	0.7	0.8	0.9	1.0	1.1	7.0	10.0	15.0
$\vartheta(0)$	0.770	0.835	0.895	0.950	1.000	1.050	2.515	2.965	3.535

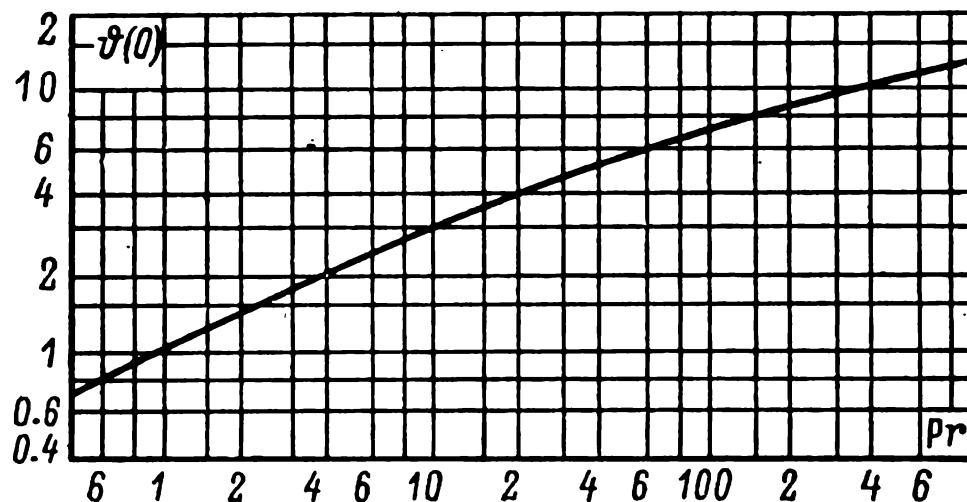


Fig. 3-7. Plot of  $\vartheta(0)$  versus Pr

The curves representing temperature distribution in a boundary layer for an adiabatically heated plate are shown in Fig. 3-8 for different Pr numbers.

The solution of equation (3-1-55) takes the form

$$T_\infty - T = (T_a - T_w)\theta + \frac{v_\infty^2}{2c_p} \vartheta \quad (3-1-64)$$

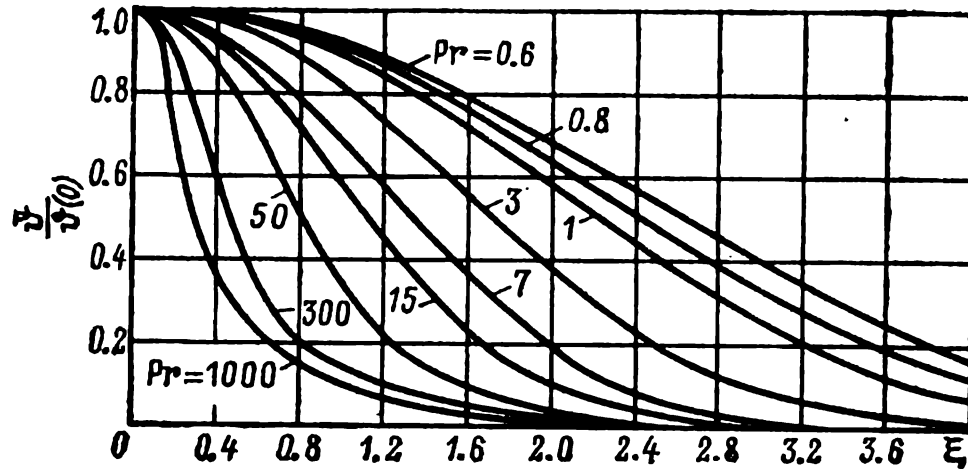


Fig. 3-8. Plot of  $\vartheta/\vartheta(0) [(T_\infty - T)/(T_\infty - T_a)]$  versus generalized variable  $\xi \left( \xi = \frac{y}{x} \sqrt{\text{Re}_x} \right)$  for  $\text{Pr} = 0.6$  to 1000

The heat flux is

$$q = -\frac{\lambda}{x} (T_a - T_w) \sqrt{\text{Re}_x} \theta'(0) \quad (3-1-65)$$

The heat transfer equation (3-1-57) was solved numerically by Evans [3-5]. Merk [3-6] presented the solution in the following form

$$\frac{\text{Nu}_x}{\sqrt{\text{Re}_x}} = \sqrt{\frac{\text{Pr}}{\pi}} \left[ 1 - 1.7207 \left( \frac{\text{Pr}}{\pi} \right)^{1/2} + \dots \right], \quad \text{Pr} \leq 0.001 \quad (3-1-66)$$

$$\frac{\text{Nu}_x}{\sqrt{\text{Re}_x}} = 0.33875 \text{Pr}^{1/3} \left[ 1 - \frac{1}{45\text{Pr}} + \frac{2}{675 \text{Pr}^2} - \dots \right], \quad \text{Pr} \geq 0.5 \quad (3-1-67)$$

where  $\text{Nu}_x = \alpha_x x / \lambda$  is the local Nusselt number.

For the Prandtl number varying between 0.5 and 50 ( $0.5 \leq \text{Pr} \leq 50$ ) the formula for the Nusselt number takes the following form:

$$\frac{\text{Nu}_x}{\sqrt{\text{Re}_x}} = 0.332 \text{Pr}^{1/3} \quad (3-1-68)$$

The solution of the equation of heat transfer in a laminar boundary layer may be obtained as follows. For small  $\text{Pr}$  numbers the velocity boundary layer thickness is small (see Fig. 3-5). By virtue of this the transverse velocity  $v_y$  is much less than the longitudinal one, as from the approximate relation

$$\frac{v_y}{v_\infty} \approx 0.165 \frac{y}{x} \quad (3-1-69)$$

it follows that at  $y = \delta \rightarrow 0$   $v_y \rightarrow 0$ .



Therefore, the heat transfer equations may be written as:

$$v_x \frac{\partial T}{\partial x} = a \frac{\partial^2 T}{\partial y^2} \quad \text{at } \text{Pr} \rightarrow 0 \quad (3-1-70)$$

The boundary conditions are:

$$\left. \begin{array}{l} \text{at } y = 0 \quad v_x = 0, \quad T = T_w \\ \text{at } x = 0 \quad v_x = v_\infty, \quad T = T_\infty \\ \text{at } y \rightarrow \infty \quad v_x = v_\infty, \quad T = T_\infty \end{array} \right\} \quad (3-1-71)$$

As an approximation it may also be assumed that  $v_x = \bar{v}_x = \text{const}$ . The solution of equation (3-1-70) will then take the form

$$\frac{T - T_w}{T_\infty - T_w} = \text{erf} \frac{y \sqrt{\bar{v}x}}{2 \sqrt{ax}} \quad (3-1-72)$$

where  $\text{erf } z$  is the Gaussian error integral function.

The local Nusselt number is

$$\text{Nu}_x = \frac{\alpha_x x}{\lambda} = \frac{x}{(T_\infty - T_w)} \left( \frac{\partial T}{\partial y} \right)_{y=0} = \frac{1}{\sqrt{\pi}} \text{Pr}^{1/2} \sqrt{\text{Re}_x} \quad (3-1-73)$$

where  $\text{Re}_x = \frac{\bar{v}_x x}{\nu}$  is the local Reynolds number.

Solution of (3-1-73) will coincide with (3-1-66) if in the latter we restrict ourselves to the first term of the expansion series.

A relationship between  $\text{Nu}_x$  and  $\text{Re}_x$  and  $\text{Pr}$  numbers may be obtained from the integral equation (3-1-52) of heat transfer in a boundary layer.

Let us now consider a general case when the distribution curves  $v_x(y)$  and  $T(y)$  do not coincide. In this case the initial length  $x_0$  of the plate is

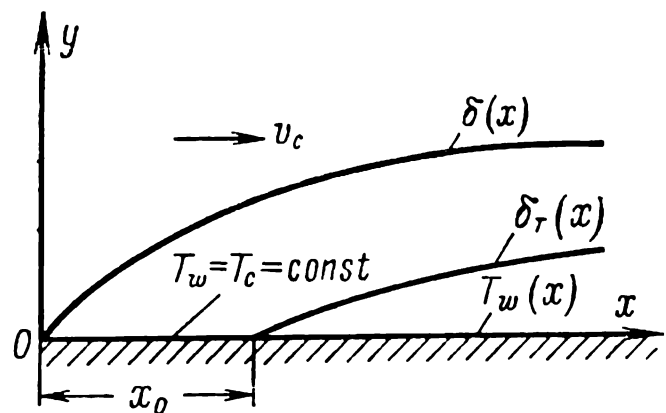


Fig. 3-9. Velocity and thermal laminar boundary layers past a flat plate

not heated, but is maintained at a constant temperature  $T_c = T_\infty = \text{const}$ . The velocity layer begins at the leading edge of the plate, and the thermal one at the boundary of the heated section of the plate (Fig. 3-9).

Under steady-state conditions  $q_w = \text{const}$ , therefore  $\frac{\partial^2 T}{\partial y^2} = 0$ , since  $\frac{\partial T}{\partial y} = \text{const}$ . In this case, the temperature distribution curve will also follow a cubic parabola [3-4]

$$\frac{\vartheta}{\vartheta_\infty} = \frac{3}{2} \frac{y}{\delta_T} - \frac{1}{2} \left( \frac{y}{\delta_T} \right)^3 \quad (3-1-74)$$

where

$$\vartheta = T - T_w, \quad \vartheta_\infty = T_\infty - T_w$$

The integral heat transfer equation may be written as

$$\frac{\partial}{\partial x} \int_0^{\delta_T} (\vartheta_\infty - \vartheta) v_x dy = a \left( \frac{\partial \vartheta}{\partial y} \right)_0 \quad (3-1-75)$$

The integral on the left-hand side of equation (3-1-52) is

$$\int_0^{\delta_T} (\vartheta_\infty - \vartheta) v_x dy = \vartheta_\infty v_\infty \delta \left( \frac{3}{20} \Delta^2 - \frac{3}{280} \Delta^4 \right) \quad (3-1-76)$$

where  $\Delta = \delta_T/\delta$ . Here, relation (3-1-39) has been used. If  $\Delta < 1$ ,  $\Delta^4$  may be neglected. Then

$$\frac{3}{20} \vartheta_\infty v_\infty \frac{d}{dx} (\Delta^2 \delta) = \frac{3}{2} a \frac{\vartheta_\infty}{\delta \Delta} \quad (3-1-77)$$

since

$$(\partial \vartheta / \partial y)_0 = \frac{3}{2} a \vartheta_\infty / \delta_T$$

Relation (3-1-44) is used to obtain  $\delta$

$$\Delta = \frac{1}{1.026 \text{Pr}^{1/3}} \left[ 1 - \left( \frac{x_0}{x} \right)^{3/4} \right]^{1/3} \quad (3-1-78)$$

For viscous oils  $\text{Pr} \approx 1000$ , then at  $x_0 = 0$  (the plate is heated along the length)  $\Delta \approx 0.1$ . For gases ( $\text{Pr} < 1$ ) the formula is not valid in a strict sense. The local Nusselt number

$$\text{Nu}_x = 0.331 \text{Pr}^{1/3} \text{Re}_x^{1/2} \left[ 1 - \left( \frac{x_0}{x} \right)^{3/4} \right]^{-1/3} \quad (3-1-79)$$

When the plate is heated along its full length ( $x_0 = 0$ ), formula (3-1-79) becomes almost the same as (3-1-68).

The mean Nusselt number is

$$\text{Nu} = \frac{\alpha l}{\lambda} = 2 \text{Nu}_x \quad (3-1-80)$$

and for  $x_0 = 0$ , we get

$$\text{Nu} = 0.662 \text{Pr}^{1/3} \sqrt{\text{Re}} \quad (3-1-81)$$

For liquids the relation between  $\text{Nu}$ ,  $\text{Re}$  and  $\text{Pr}$ , obtained by an analysis of a great body of experimental data, takes the form

$$\text{Nu} = 0.76 \sqrt{\text{Re}} \text{Pr}_w^{0.43} \left( \frac{\text{Pr}_\infty}{\text{Pr}_w} \right)^{0.25} \quad (3-1-82)$$

When analyzing experimental data on temperature distributions in boundary layers, the local Nusselt number may be easily determined from the conventional thermal boundary layer thickness  $\delta'_T$ , which is equal to the straight-line  $\vartheta_\infty = \text{const}$  cut by the tangent to the temperature distribution curve  $\vartheta(y)$ , at  $y = 0$  (Fig. 3-10).

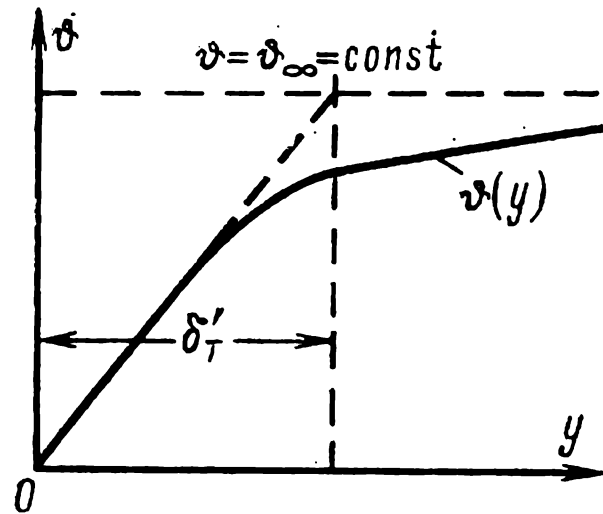


Fig. 3-10. Determination of Nusselt number in terms of conventional thickness of thermal boundary layer

Then

$$\text{Nu}_x = \frac{\alpha_x x}{\lambda} = \frac{x}{\vartheta_\infty} \left( \frac{\partial \vartheta}{\partial y} \right)_{y=0} = \frac{x}{\delta'_T} \quad (3-1-83)$$

This graphical method of determining  $\text{Nu}_x$  is applicable only for small flow velocities when  $\delta_T$  is commensurable with  $x$ .

### (c) Wedge Flows

We have considered heat transfer in flows past a flat plate, or the so-called gradientless flows ( $\partial p / \partial x = 0$ ).

For flows past wedges having expansion angle  $\pi\beta$  (Fig. 3-11), the pressure gradient along the  $x$ -axis (the flow direction) is not zero. In this case, the relation between  $v_c$  and coordinate  $x$  is of the form

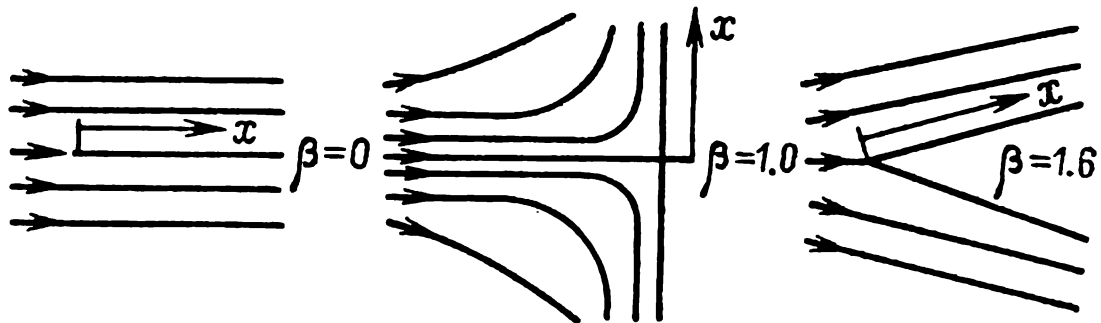
$$v_c = Ax^m \quad (3-1-84)$$

where  $m$  is the exponent

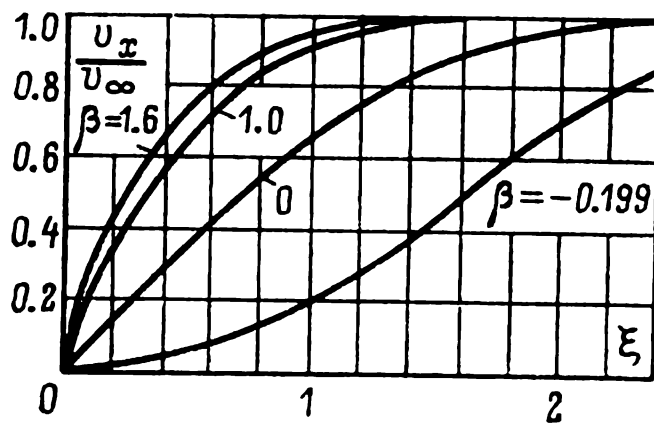
$$m = \frac{\beta}{2 - \beta} \quad (3-1-85)$$

For a flat plate,  $\beta = 0$ , thus  $m = 0$  and  $v_c = v_\infty = \text{const}$ ; for a plate normal to the flow,  $\beta = 1.0$ ,  $m = 1$ . For wedges,  $0 < \beta < 1$  and  $0 < m < 1$ . In this case, the differential boundary layer equation

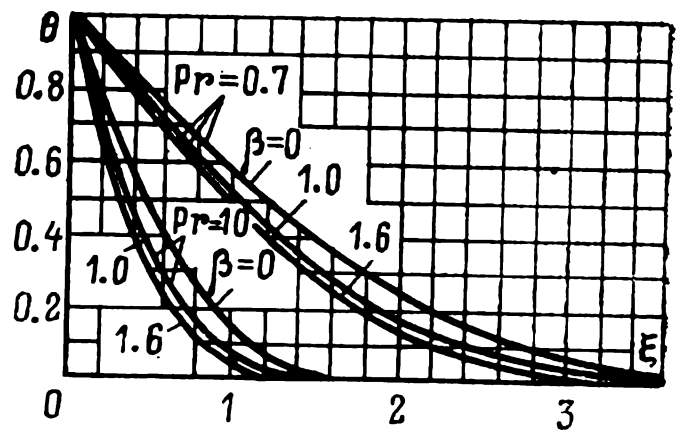
$$\left. \begin{aligned} v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} &= - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 v_x}{\partial y^2} \\ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} &= 0 \end{aligned} \right\} \quad (3-1-86)$$



(a)



(b)



(c)

Fig. 3-11. Wedge-type flow

(a) schematic of the flow; (b) velocity profiles (plot of  $u_x/v_\infty$  versus  $\xi_1$ ); (c) temperature profiles (plot of  $\theta$  versus  $\xi_1$ ) at various  $Pr$  and  $\beta$

upon substitution

$$\xi_1 = \frac{2}{\sqrt{2-\beta}} \xi; \quad \psi_1 = f(\xi_1) \sqrt{\frac{2}{m+1} \nu x v_\infty} \quad (3-1-87)$$

takes the form

$$2f''' + f''f + \beta(1 - f'^2) = 0 \quad (3-1-88)$$

Equation (3-1-88) differs from equation (3-1-17) by the presence of the third term. D. Hartree solved this equation by numerical methods [3-7]. The solutions for some wedges are presented in Fig. 3-11.

The local mass-transfer number in this case is

$$\text{Nu}_x = 0.56 \frac{(\beta + 0.2)^{0.1}}{(2 - \beta)^{0.5}} \text{Re}_x^{0.5} \text{Pr}^{0.333 + 0.067\beta - 0.026\beta^2} \quad (3-1-89)$$

The dimensionless temperature distribution  $\theta \left( \theta = \frac{T_w - T}{T_w - T_\infty} \right)$  over the coordinate  $\xi_1 \left( \xi_1 = \frac{2}{\sqrt{2 - \beta}} \xi \right)$  is shown in Fig. 3-11c for different expansion angles  $\pi\beta$ .

#### (d) Mass Transfer in a Flow past a Flat Plate

The differential equation of mass transfer in a boundary layer of a flow past a flat plate is similar to the heat transfer equation. For a two-dimensional steady-state flow ( $\partial/\partial\tau = 0$ ,  $\partial/\partial z = 0$ ,  $v_z = 0$ ), the differential mass transfer equation for the  $k$ -th species takes the form

$$\rho v_x \frac{\partial \rho_k}{\partial x} + \rho v_y \frac{\partial \rho_k}{\partial y} = \rho D \frac{\partial^2 \rho_{k0}}{\partial y^2} + \dot{I}_v \quad (3-1-90)$$

where  $\dot{I}_v$  is a bulk mass source caused by phase or chemical conversions. Here, it is assumed that thermodiffusion and pressure diffusion effects may be neglected. In the derivation of equation (3-1-90) the order estimation method was used for the terms appearing in the differential transfer equation.

If there is no mass source ( $\dot{I}_v = 0$ ), then from differential equation (3-1-90) an integral boundary layer equation

$$\frac{\partial}{\partial x} \int_0^{\delta_D} (\rho_{kw} - \rho_{k\infty}) v_x dy = D \rho \left( \frac{\partial \rho_{k0}}{\partial y} \right)_w \quad (3-1-91)$$

is found using a similar method.

The obtained solutions for the temperature profile may be extended to a concentration profile, since for equal Pr and Sc numbers, the profiles coincide. The Schmidt number Sc or the diffusional Prandtl number  $\text{Pr}_D$  ( $\text{Pr}_D \equiv \text{Sc}$ ) may be expressed in terms of the thermal Prandtl number with the aid of the Lewis number Le ( $\text{Le} = D/\alpha$ )

$$\text{Sc} \equiv \text{Pr}_D = \text{Pr}/\text{Le}$$

The mass transfer coefficient  $\alpha_m$  is usually determined from the relation

$$\alpha_m = \frac{j_{kw}}{(\rho_{kw} - \rho_{k\infty})} = \frac{\rho D (\partial \rho_{k0} / \partial y)_w}{(\rho_{kw} - \rho_{k\infty})} \quad (3-1-92)$$

where  $j_{kw}$  is the mass flux density of the  $k$ -th species.

The local mass-transfer Nusselt number  $(Nu_x)_D$  is then

$$(Nu_x)_D = \frac{\alpha_m x}{D} = 0.331 Sc^{1/3} \sqrt{Re_x} \quad (3-1-93)$$

If  $Pr$  and  $Sc$  are equal ( $Le = 1$ ), then  $Nu_x = (Nu_x)_D$ . Hence

$$\frac{\alpha}{\alpha_m} = \frac{a}{D} c_p \rho = c_p \rho \quad (3-1-94)$$

i.e. the ratio of the heat-transfer coefficient to the mass-transfer coefficient is equal to the volume heat capacity of the fluid at constant pressure. This relation is known as the Lewis formula and is valid for the earlier mentioned assumptions. In a number of research papers the authors have related the mass-transfer coefficient to the difference in partial pressure  $\Delta p$  ( $\Delta p = p_w - p_\infty$ )

$$j_{kw} = \alpha_p (p_w - p_\infty) \quad (3-1-95)$$

In this case

$$\alpha_m = \frac{\alpha_p M}{M_k \rho} \quad (3-1-96)$$

where  $M$  and  $M_k$  are molecular masses of the mixture and the  $k$ -th species, respectively. Strictly speaking, formulas (3-1-93) through (3-1-96) hold true for a binary mixture when the rates of diffusion of both species are equal. If the diffusion rates of species are different, then the diffusional motion is accompanied by apparent motion of the whole mixture due to the total pressure gradient.

As an example, we consider a simple case of diffusion when liquid evaporates from a vessel. As a first approximation the liquid surface temperature is assumed constant and equal to the adiabatic air saturation temperature or the wet bulb temperature  $t_b$ . The ambient (humid air) temperature is constant ( $t_c = \text{const}$ ). Humid air is assumed to be a binary mixture composed of steam ( $M_1 = 18$ ) and dry air ( $M_2 = 29$ ).

Liquid evaporation from a vessel involves not only interdiffusion of steam and dry air into humid air, but also a convective flow of humid air from the liquid surface. The point is that the liquid surface acts as an impermeable membrane for air, and as a result air diffusion tends to create excess pressure. But this excess pressure relaxes and causes motion in the mixture, directed away from the liquid surface. Let us assume that the total pressure is constant ( $p = \text{const}$ ) as the excess pressure relaxes at the speed of sound velocity

$$p = p_1 + p_2 = \text{const}$$

Steam diffuses from the liquid surface into the air by the action of the relative concentration gradient  $\partial\rho_{10}/\partial y$  (Fig. 3-12). At the same time air diffuses to the liquid surface in the reverse direction under the action of the relative concentration gradient  $\partial\rho_{20}/\partial y$  of air.

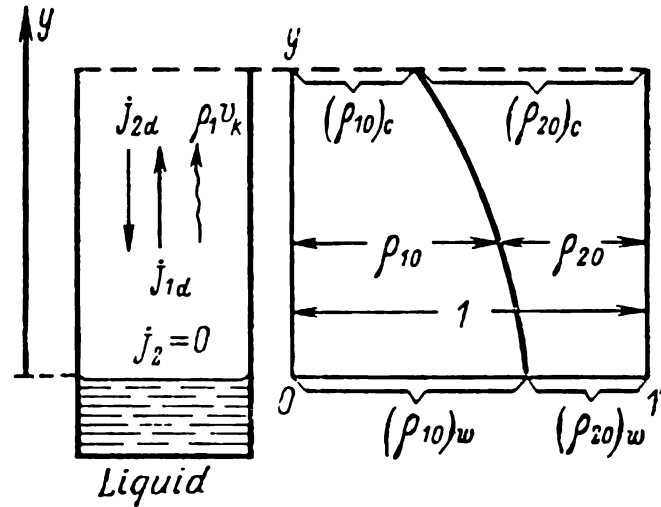


Fig. 3-12. Distribution of vapour  $\rho_{10}$  and dry air  $\rho_{20}$  concentration along vessel height  $y$  during liquid evaporation

Since

$$\rho_1 + \rho_2 = \rho; \rho_{10} + \rho_{20} = 1 \quad (3-1-97)$$

then

$$\partial\rho_{10}/\partial y = -\partial\rho_{20}/\partial y$$

If steam freely diffuses into air, then for the latter the liquid surface acts as an impermeable membrane, and as a result, the amount of air at the liquid surface would continuously increase, resulting in higher total pressure. Since pressure remains constant ( $p = \text{const}$ ), then to fulfill this condition it is necessary that the whole mixture should move as a molar (convective) flow at a certain velocity  $v_k$  ( Fig. 3-12).

The total steam flow is the sum of diffusive and convective vapour flows

$$j_1 = -D_{12}\rho \frac{\partial\rho_{10}}{\partial y} + \rho_1 v_k \quad (3-1-98)$$

The total air flow at the liquid surface is zero

$$j_2 = -D_{21}\rho \frac{\partial\rho_{20}}{\partial y} + \rho_2 v_k = 0 \quad (3-1-99)$$

From the last formula we find

$$v_k = \frac{D_{21}}{\rho_{20}} \frac{\partial\rho_{20}}{\partial y} = - \frac{D_{12}}{1 - \rho_{10}} \frac{\partial\rho_{10}}{\partial y} \quad (3-1-100)$$

since  $D_{21} = D_{12}$  and  $\partial\rho_{10}/\partial y = -\partial\rho_{20}/\partial y$ . Substituting this expression for  $v_k$  in formula (3-1-98), we obtain

$$j_1 = - \frac{1}{1 - \rho_{10}} \rho D_{12} \frac{\partial\rho_{10}}{\partial y} \quad (3-1-101)$$

Thus, the convective transfer of vapour increases by a factor of  $1/(1 - \rho_{10})$ . If the diffusive vapour transfer is assumed to be equal to one, then the fraction of convective transfer amounts to  $\rho_{10}/(1 - \rho_{10})$ .

From (3-1-100) and (3-1-101) it follows that

$$v_k = j_1/\rho \quad (3-1-102)$$

i.e. the rate of convective transfer equals the vapour flux density divided by the vapour-gas mixture density. The quantity  $v_k$  is sometimes called the Stefan flow velocity. This velocity of apparent motion should not be confused with the linear velocity of diffusive (molecular) vapour transfer ( $v_{1\text{ diff}} = j_1/\rho_1$ ). Transition from relative concentrations to partial pressures is accomplished using the formulas

$$\rho_{10} = p_{10} \frac{M_1}{M} \quad (3-1-103)$$

where  $p_{10} = p_1/p$  is the relative partial vapour pressure

$$\frac{1}{M} = \rho_{10} \left( \frac{1}{M_1} - \frac{1}{M_2} \right) + \frac{1}{M_2} \quad (3-1-104)$$

Then we obtain

$$j_1 = - \frac{D_{12}M_1}{RT} \frac{1}{(1 - p_{10})} \frac{dp_1}{dy} \quad (3-1-105)$$

where  $R$  is the universal gas constant.

If formula (3-1-105) is integrated with respect to  $y$  from 0 (the liquid surface) to  $\xi$  (the distance from the vessel edge to the liquid surface), we get the Stefan formula

$$j_{10} = - \frac{D_{12}M_1p}{\xi RT} \log_e \left[ \frac{1 - (p_{10})_c}{1 - (p_{10})_w} \right] \quad (3-1-106)$$

where subscript  $w$  refers to the surface and  $c$  denotes the surrounding medium.

In the derivation of formulas (3-1-98) and (3-1-106) thermodiffusion as well as the convective transfer of vapour in the form of thermal slip and diffusive slip were disregarded.

It is well known that a pressure gradient at the wall surface results in convective transfer at the surface itself in the direction of high temperatures (opposite to the heat flux). The linear velocity of such an apparent motion is

$$v_{mc} \approx 1.5 \frac{D_{12}}{T} \frac{\partial T}{\partial y} \quad (3-1-107)$$



A similar effect (diffusive slip) occurs at the vessel surface, if close to its walls there exists a gradient of one of the species of the binary mixture along the height. As a first approximation, this velocity  $v_{gc}$  equals

$$v_{gc} = \frac{D_{12}}{\rho_2} \left[ \left( \frac{M_2}{M_1} \right)^{1/2} - 1 \right] \frac{\partial \rho_2}{\partial y} \quad (3-1-108)$$

For a vapour-gas mixture ( $M_2 = 29$ ,  $M_1 = 18$ ) we get

$$v_{gc} = 0.270 \frac{D_{12}}{\rho_2} \frac{\partial \rho_2}{\partial y} \quad (3-1-109)$$

which corresponds to the data in [3-8].

It may be easily demonstrated that convective transfer velocities  $v_k$ ,  $v_{mc}$ ,  $v_{gc}$  are of the same order.

With reference to the linear velocity of diffusive transfer  $v_{dif}$ , they come to about 5% for ordinary conditions of fluid evaporation into the atmosphere (natural evaporation). Thus, if it is necessary to introduce a correction factor for the Stefan flow, the thermal and diffusive slip flows must be accounted for.

The Stefan formula (3-1-106) has been repeatedly verified by experiments on capillary evaporation of liquid. If heat, required for evaporation, is received from the surrounding air (glass pipes filled with liquid are in a heated air flow), then a correction factor should be introduced into the Stefan formula. The value  $(\xi - \xi_0)$  should be substituted for  $\xi$ , where  $\xi_0$  is a coefficient depending on the air velocity and temperature [3-9].

### (e) *Turbulent Transfer*

We shall discuss briefly the heat and mass transfer between a wall surface and a turbulent flow. To obtain a generalized relation for the velocity profile in a turbulent boundary layer, a dimensionless velocity  $v^*$  and a dimensionless distance from the wall  $y^*$  or from the tube surface  $r^*$  are introduced from the following formulas:

$$v^* = \frac{v}{v_0} ; \quad y^* = y \frac{v_0}{\nu} ; \quad r^* = \frac{r v_0}{\nu} \quad (3-1-110)$$

where  $v_0$  is the so-called friction velocity equal to

$$v_0 = \sqrt{\frac{\sigma_0}{\rho}} = v_\infty \sqrt{\frac{1}{2} C_f} \quad (3-1-111)$$

In the case of molecular momentum transfer, which is governed by Newton's law, the relation between  $v^*$  and  $y^*$  may be found from the solution of the differential equation

$$v_0^2 = \nu \frac{dv_x}{dy} \quad (3-1-112)$$

Hence

$$v^* = y^* \quad (3-1-113)$$

For a flow past a flat plate the equation of momentum transfer in a boundary layer is defined by the Prandtl formula

$$\frac{\sigma_0}{\rho} = (ky)^2 \left| \frac{d\bar{v}_x}{dy} \right| \frac{d\bar{v}_x}{dy} \quad (3-1-114)$$

After integrating (3-1-112) and using relations (3-1-110), we get

$$v^* = k_1 \log_e y^* + k_2 \quad (3-1-115)$$

where  $k_1$  and  $k_2$  are experimental constants.

Thus, the distribution of velocity  $v^*$  in a boundary layer in the direction normal to the surface is governed by a logarithmic law.

As Nikuradse has reported [3-10], the constants  $k_1$  and  $k_2$  are 2.5 and 5.5, respectively, i.e.

$$v^* = 2.5 \log_e y^* + 5.5 \quad (3-1-116)$$

Between the linear law of velocity variation (3-1-113) and the logarithmic law (3-1-116) there exists a transition zone for which Reichart [3-11] obtained the following formula

$$v^* = 5.0 \log_e y^* - 3.05 \quad (3-1-117)$$

by an analysis of experimental data.

The values calculated with various empirical formulas are plotted in Fig. 3-13. As shown in the figure experimental points lie close to the curves plotted using the formulas given earlier. For a flow past a flat plate and also for a flow through pipes, the velocity profile is often approximated by

$$\frac{v_x}{v_\infty} = \left( \frac{y}{\delta} \right)^{1/7} \quad (3-1-118)$$

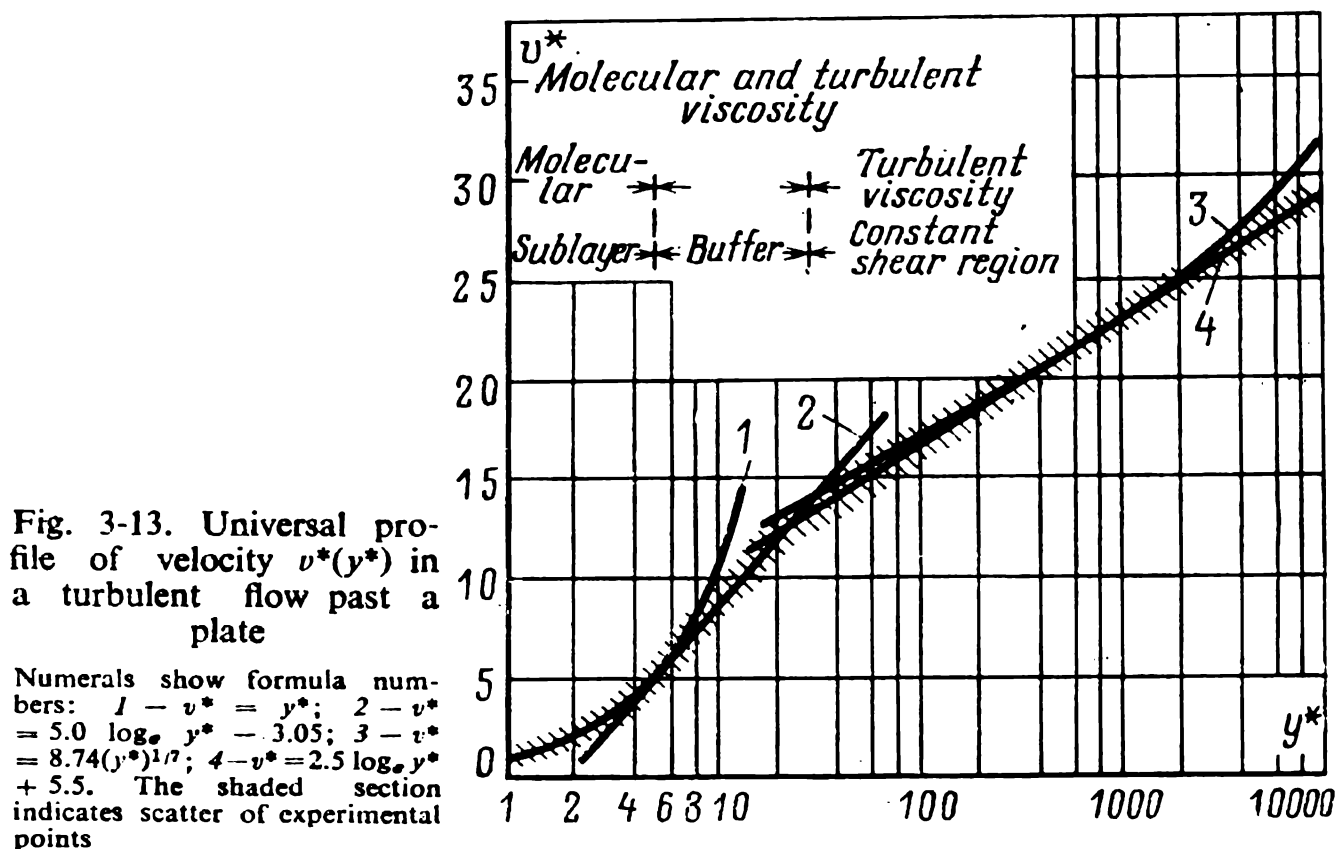
In particular, when  $C_f = 0.0791 \text{ Re}^{-0.25}$ , we get from relation (3-1-118)

$$v^* = 8.74 (y^*)^{1/7} \quad (3-1-119)$$

This relation is also plotted in Fig. 3-13. It is seen from the figure that in the range  $100 < y^* < 1200$  the curve plotted with the use of formula (3-1-119) is similar to that plotted using Nikuradse's formula (3-1-116).

The earlier given formulas for a velocity profile in a boundary layer do not hold for a wall layer, as the shear stress proportional to  $d\bar{v}_x/dy$  becomes infinite as  $y \rightarrow 0$ . This may be attributed to the fact that at the surface turbulence disappears. Therefore, for a thin wall layer the formulas given above are not valid. Prandtl suggested a hypothesis on the existence of

a laminar sublayer  $\delta_b$  with linearly distributed velocity. In the variation range of  $y$  from 0 to  $\delta_b$  the distribution  $\bar{v}_x(y)$  is linear, i.e.  $\bar{v}_x = ky$ , where  $k$  is a constant. Thereafter in the range ( $\delta_b < y < \delta$ ) the velocity profile is given by formula (3-1-118) or (3-1-115).



Using the integral boundary-layer equation with due regard for the velocity profile, defined by formula (3-1-118), we get the following formula for a turbulent boundary layer thickness:

$$\frac{\delta}{x} = 0.376 \text{Re}_x^{-0.2} \quad (3-1-120)$$

Here, it is assumed that  $x$  is read off from the very edge of the plate (a turbulent boundary layer is known to begin at a distance  $x_k$  from the plate edge). In the considered case the velocity displacement thickness  $\delta_{v,d}$  equals  $\delta/8$ . If for a critical distance  $x_k$  the thickness  $\delta$  is calculated by formulas for laminar and turbulent boundary layers, then in the latter case  $\delta$  will be greater. The boundary layer thickness cannot increase irregularly. Transition of a laminar boundary layer into the turbulent one is not an instantaneous process, but it occurs gradually and involves a transition region. In this region first fluctuations of a rather long wave appear and the motion in the boundary layer becomes unsteady. Thereafter the waves acquire irregular shapes and change into vortices, and finally disappear taking the form of a moderate turbulent motion. The transition zone does not lend itself to investigation, as irregular fluctuations take place in it.

Karman [3-12] divided the flow over a solid into three regions: (1) laminar sublayer; (2) buffer layer and (3) turbulent core. The laminar sublayer flow is fully laminar and transfer processes are molecular. In the turbulent core eddy viscosity and thermal conductivity are much greater than molecular viscosity and thermal conductivity. In the buffer sublayer, the laminar flow gradually changes into a turbulent one. Besides, Karman assumed  $Pr_\epsilon = 1$ .

In the present case the laminar sublayer thickness  $\delta_b$  is

$$\frac{\delta_b}{\delta} = 191 Re_x^{-0.7} \quad (3-1-121)$$

The shear stress at the surface is

$$\frac{\sigma}{\rho v_\infty^2} = 0.0296 Re_x^{-0.2} \quad (3-1-122)$$

For a local friction factor  $C_f$  Prandtl and Schlichting suggested the formula [3-1]

$$C_f = \frac{0.455}{(\log_{10} Re_x)^{2.58}} \quad (3-1-123)$$

For large  $Re_x$  numbers the laminar boundary layer effect may be neglected.

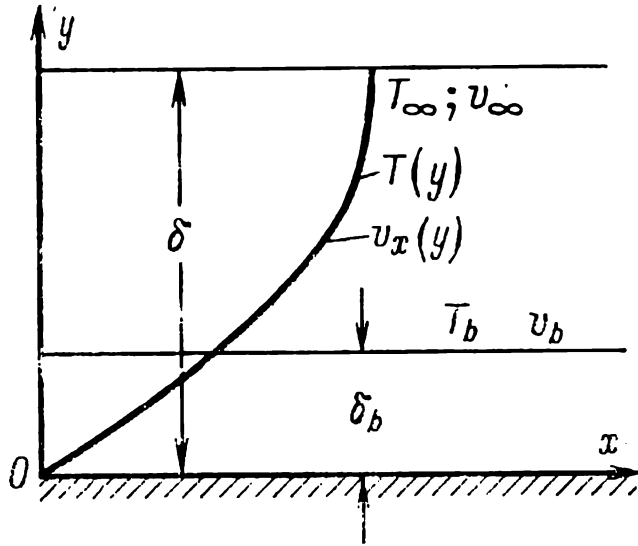


Fig. 3-14. Distribution of temperature  $T(y)$  and velocity  $v_x(y)$  in a turbulent boundary layer with a laminar sublayer ( $\delta_b$ )

Using the scheme of a laminar sublayer with a linear temperature distribution (Fig. 3-14) and the analogy between heat and momentum transfers in the form

$$q_0 = \sigma_0 c_p \frac{T_b - T_\infty}{v_\infty - v_b} \quad (3-1-124)$$

$$q_0 = \sigma_0 \frac{\lambda}{\eta v_b} (T_0 - T_b) \quad (3-1-125)$$

we may eliminate the unknown  $T_b$  from formulas (3-1-124) and (3-1-125). Then assuming  $\text{Pr}_\varepsilon = 1$ , we obtain for the heat-transfer coefficient  $\alpha$

$$\alpha = \frac{q_0}{T_\infty - T_0} = \frac{\sigma_0 c_p / v_\infty}{1 + \frac{v_b}{v_\infty} (\text{Pr} - 1)} \quad (3-1-126)$$

The shear stress at the plate may be found from the Blasius formula valid for liquid surfaces at  $\text{Re} < 10^7$

$$\sigma_0 = 0.0228 \rho v_\infty^2 \text{Re}_\delta^{-0.25} \quad (3-1-127)$$

From velocity profiles in a laminar sublayer and a turbulent boundary layer, we get

$$\frac{v_b}{v_\infty} = \frac{2.12}{\text{Re}_x^{0.1}} \quad (3-1-128)$$

As a result, we obtain

$$\text{Nu}_x = \frac{0.0296 \text{Re}_x^{0.8} \text{Pr}}{1 + 1.305 (\text{Pr} - 1) \text{Re}_x^{-0.1} \text{Pr}^{-1/6}} \quad (3-1-129)$$

For gases  $\text{Pr}$  is close to unity and formula (3-1-129) is simplified considerably. If it is assumed that the boundary layer is turbulent over the entire length, the mean Nusselt number for a plate of length  $l$  is

$$\text{Nu} = \overline{\text{Nu}}_{x=l} = 0.037 \text{Re}^{0.8} \text{Pr}^{1/3} \quad (3-1-130)$$

The validity of the above equation, obtained by numerical methods, is verified experimentally.

An analysis of experimental data on heat transfer in an air flow past a plate reported by different authors leads to the formula

$$\text{Nu} = 0.032 \text{Re}^{0.8} \quad (3-1-131)$$

For liquids the formula takes the form

$$\text{Nu} = 0.037 \text{Re}^{0.8} \text{Pr}_0^{0.43} \left( \frac{\text{Pr}_0}{\text{Pr}_\infty} \right)^{0.25} \quad (3-1-132)$$

When calculations are made according to the described procedure, a complete analogy between heat and mass transfers is ordinarily assumed. Then  $\text{Nu}_x = (\text{Nu}_x)_m$ , where the mass-transfer Nusselt number is equal to the heat-transfer Nusselt number.

The friction factor  $C$  in a flow past a plate depends on turbulence level  $\varepsilon$  or on the critical Reynolds number  $(\text{Re}_l)_{cr}$ , as they are related in a single way (see Fig. 3-15). The relation between  $C$  and  $\text{Re}_l$  may be expressed as [3-13]

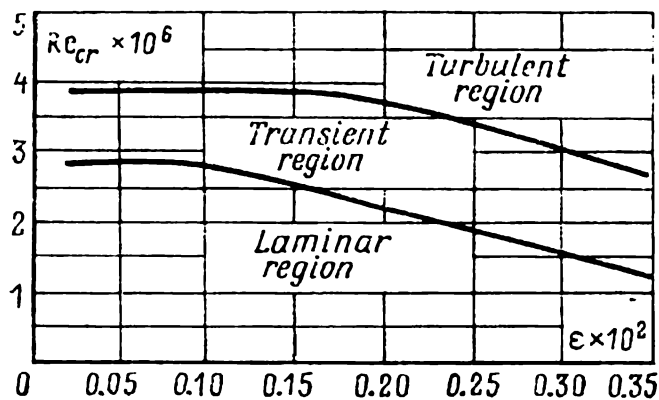
$$C = \frac{0.074}{\text{Re}_l^{0.2}} - \frac{A}{\text{Re}_l} \quad (3-1-133)$$

where  $A$  is determined from Table 3-4.

Table 3-4

Relationship Between  $A$  and  $(Re)_cr$ 

$(Re)_cr$	$1 \cdot 10^5$	$3 \cdot 10^5$	$5 \cdot 10^5$	$1 \cdot 10^6$
$A$	360	1050	1700	3300

Fig. 3-15. Plot of critical  $Re$  versus turbulence level  $\epsilon$ 

### 3.2. SIMULTANEOUS HEAT AND MASS TRANSFER IN A LAMINAR FLOW PAST A FLAT PLATE

#### (a) Injection into a Boundary Layer

Heat and mass transfer in a porous plate when coolant gas is injected into the boundary layer is one of the most important problems of modern engineering. The problem is formulated as follows: a porous plate is placed in a heated gas flow. An inert gas is injected through the pores at a constant velocity  $v_w$  to cool the plate. Calculate the velocity  $v_x(y)$ , temperature  $T(y)$  and concentration  $W(y)$  profiles in the boundary layer (Fig. 3-16). A system of differential equations of momentum, heat and mass transfer for a steady-state parallel plane flow ( $\partial/\partial\tau = 0$ ;  $v_z = 0$ ) may be written as

$$\rho v_x \frac{\partial v_x}{\partial x} + \rho v_y \frac{\partial v_x}{\partial y} = \frac{\partial}{\partial y} \left( \eta \frac{\partial v_x}{\partial y} \right) \quad (3-2-1)$$

$$\frac{\partial}{\partial x} (\rho v_x) + \frac{\partial}{\partial y} (\rho v_y) = 0 \quad (3-2-2)$$

$$\begin{aligned} c_p \rho v_x \frac{\partial T}{\partial x} + c_p \rho v_y \frac{\partial T}{\partial y} &= \frac{\partial}{\partial y} \left( \lambda \frac{\partial T}{\partial y} \right) + \eta \left( \frac{\partial v_x}{\partial y} \right)^2 \\ + \rho D_{12} (C_{p1} - C_{p2}) \frac{\partial T}{\partial y} \frac{\partial W}{\partial y} \end{aligned} \quad (3-2-3)$$

$$\rho v_x \frac{\partial W}{\partial x} + \rho v_y \frac{\partial W}{\partial y} = \frac{\partial}{\partial y} \left( \rho D_{12} \frac{\partial W}{\partial y} \right) \quad (3-2-4)$$

where  $W$  is the relative concentration ( $W \equiv \rho_{10}$ ).

The slip fluid is assumed to be a binary gaseous mixture at constant pressure  $p = \text{const}$ . Subscript 1 refers to the main flow gas, subscript 2 indicates the injected gas. The Soret and Dufour effects are neglected. The boundary conditions are as follows (Fig. 3-16):

$$\text{at } y = 0 \quad v_x = 0; \quad v_y = v_w = \text{const}; \quad T = T_w = \text{const}$$

$$\text{or } \left( \frac{\partial T}{\partial y} \right)_w = 0; \quad W = W_w = \text{const} \quad (3-2-5)$$

$$\text{at } y \rightarrow \infty \quad v_x = v_\infty; \quad T = T_\infty; \quad W = W_\infty$$

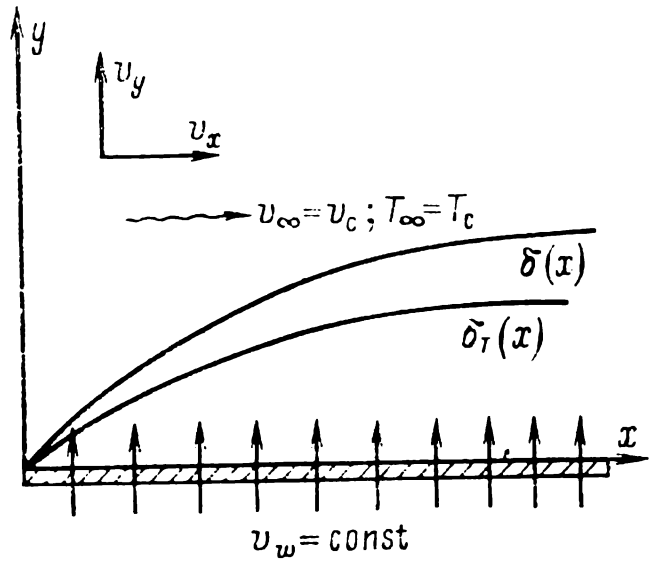


Fig. 3-16. Illustration for the calculation of heat and mass transfer during flow past a porous plate with gas injection into the boundary layer

For the solution of equations (3-2-1) through (3-2-4) the problem may be further simplified (1) by neglecting diffusion heat transfer; (2) by assuming the fluid to be incompressible; (3) by regarding transfer properties to be constant ( $\eta = \text{const}$ ,  $\lambda = \text{const}$ ,  $D_{12} = \text{const}$ ). Equations (3-2-1) through (3-2-4) then take the form

$$\rho v_x \frac{\partial v_x}{\partial x} + \rho v_y \frac{\partial v_x}{\partial y} = \eta \frac{\partial^2 v_x}{\partial y^2}; \quad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \quad (3-2-6)$$

$$v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} = a \frac{\partial^2 T}{\partial y^2} \quad (3-2-7)$$

$$v_x \frac{\partial W}{\partial x} + v_y \frac{\partial W}{\partial y} = D \frac{\partial^2 W}{\partial y^2} \quad (3-2-8)$$

From the set of differential transfer equations the integral transfer equations of a boundary layer may be obtained for boundary conditions (3-2-5).

The differential equation of momentum transfer for a gradient steady-state flow ( $\partial p / \partial x \neq 0$ ) over a flat plate may be written as

$$\rho \frac{\partial v_x^2}{\partial x} + \rho \frac{\partial (v_x v_y)}{\partial y} = \rho v_\infty \frac{dv_\infty}{dx} + \eta \frac{\partial^2 v_x}{\partial y^2} \quad (3-2-9)$$

$$\frac{\partial (\rho v_\infty v_x)}{\partial x} + \frac{\partial (\rho v_\infty v_y)}{\partial y} = v_x \frac{dv_\infty}{dx} \quad (3-2-10)$$

Here the Bernoulli equation was used

$$\rho v_{\infty} \frac{dv_{\infty}}{dx} = - \frac{dp}{dx} \quad (3-2-11)$$

Term-by-term subtraction of equation (3-2-10) from equation (3-2-9) yields

$$\frac{\partial}{\partial x} [\rho v_x (v_{\infty} - v_x)] + \frac{\partial}{\partial y} [\rho v_y (v_{\infty} - v_x)] + (v_{\infty} - v_x) \rho \frac{dv_{\infty}}{dx} = - \eta \frac{\partial^2 v_x}{\partial y^2} \quad (3-2-12)$$

Equation (3-2-12) is integrated with respect to  $y$  from 0 to  $\delta$  using equation (3-2-5)

$$\begin{aligned} & \frac{\partial}{\partial x} \int_0^{\delta} [\rho v_x (v_{\infty} - v_x)] dy + \frac{dv_x}{dx} \int_0^{\delta} (\rho_{\infty} v_{\infty} - \rho v_x) dy \\ &= \eta_w \left( \frac{dv_x}{dy} \right)_w + v_w v_{\infty} \rho_w \end{aligned} \quad (3-2-13)$$

Equation (3-2-13) differs from the equation without injection by an additional term  $\rho_w v_w v_{\infty}$ , characterizing virtual friction due to injection. The integral boundary-layer equation for heat transfer is obtained from the differential equation for the case when  $\partial p / \partial x \neq 0$ .

Integration of the heat-transfer equation with respect to  $y$  from 0 to  $\delta_T$  gives

$$\int_0^{\delta_T} c_p \rho v_x \frac{\partial T}{\partial x} dy + \int_0^{\delta_T} c_p \rho v_y \frac{\partial T}{\partial y} dy = \int_0^{\delta_T} \rho v_x v_{\infty} \frac{\partial v_{\infty}}{\partial x} dy + \int_0^{\delta_T} \frac{\partial}{\partial y} \left( \lambda \frac{\partial T}{\partial y} \right) dy \quad (3-2-14)$$

The second terms on both sides of equation (3-2-13) may be integrated using boundary conditions (3-2-5)

$$\begin{aligned} & \int_0^{\delta_T} c_p \rho v_x \frac{\partial T}{\partial x} dy + c_p \rho v_y T \Big|_0^{\delta_T} + \int_0^{\delta_T} c_p T \frac{\partial(\rho v_x)}{\partial x} dy \\ &= \int_0^{\delta_T} \rho v_x \frac{\partial}{\partial x} \left( \frac{v_{\infty}^2}{2} \right) dy - \lambda \left( \frac{\partial T}{\partial y} \right)_w \end{aligned} \quad (3-2-15)$$



Taking into consideration that

$$\int_0^{\delta_T} \rho v_x \frac{\partial}{\partial x} \left( \frac{v_\infty^2}{2} \right) dy = \int_0^{\delta_T} \rho v_x \frac{\partial}{\partial x} (c_{p\infty} T_\infty) dy$$

permutation of integration and differentiation operations results in

$$\frac{\partial}{\partial x} \int_0^{\delta_T} (c_{p\infty} T_\infty - c_p T) \rho v_x dy = \lambda \left( \frac{\partial T}{\partial y} \right)_w + \rho_w v_w (c_{p\infty} T_\infty - c_{pw} T_w) \quad (3-2-16)$$

Equation (3-2-16) differs from the integral equation of the boundary layer without injection by an additional term characterizing molar heat transfer by the injected gas. The integral mass-transfer equation is obtained in a similar way

$$\frac{\partial}{\partial x} \int_0^{\delta_w} (W_\infty - W) \rho v_x dy = \rho_w D_{12} \left( \frac{\partial W}{\partial y} \right)_w - \rho_w v_w (W_w - W_\infty) \quad (3-2-17)$$

Integral transfer equations (3-2-13)-(3-2-15) and (3-2-16) are used for approximate calculations of simultaneous heat and mass transfer in a boundary layer.

Let us now return to the system of differential equations (3-2-6)-(3-2-8). Eckert and Hartnett [3-14] solved this system by making the following assumptions: (1) fluid is incompressible; (2) properties of the species are independent of pressure and temperature, and vary little.

For boundary conditions (3-2-5) these equations may be reduced to ordinary differential equations by introducing the stream function  $\psi$  ( $v_x = \partial\psi/\partial y$ ,  $v_y = -\partial\psi/\partial x$ ) and using the following notations

$$f = \psi / \sqrt{v_\infty x}, \quad \xi = \frac{y}{x} \sqrt{v_\infty x / \nu}; \quad \theta = (T - T_w) / (T_\infty - T_w);$$

$$\varphi = \frac{W_w - W}{W_w - W_\infty} \quad (3-2-18)$$

Assuming that  $v_\infty = v_c = \text{const}$ ,  $\partial p / \partial x = 0$ ,  $f$ ,  $\theta$  and  $\varphi$  depend only on  $\xi$ , i.e. the surface temperature  $T_w$  and concentration of the injected gas  $W_w$  are uniform over the plate surface and independent of  $x$ , we get:

the momentum transfer equation

$$\frac{d^3 f}{d\xi^3} + \frac{1}{2} f \frac{d^2 f}{d\xi^2} = 0 \quad (3-2-19)$$

the mass transfer equation

$$\frac{d^2\varphi}{d\xi^2} + \frac{1}{2} \text{Sc} f \frac{d\varphi}{d\xi} = 0 \quad (3-2-20)$$

the heat transfer equation

$$\frac{d^2\theta}{d\xi^2} + \frac{1}{2} \text{Pr} f \frac{d\theta}{d\xi} = 0 \quad (3-2-21)$$

for the following boundary conditions

$$\left. \begin{array}{l} \text{at } \xi = 0 \quad \left( \frac{df}{d\xi} \right)_w = 0; \quad f_w = -2 \left( \frac{v_w}{v_\infty} \right) \sqrt{\text{Re}_x}; \quad \theta = 0; \quad \varphi = 0 \\ \text{at } \xi \rightarrow \infty \quad \frac{df}{d\xi} = 1; \quad \theta = 1; \quad \varphi = 1 \end{array} \right\} \quad (3-2-22)$$

From boundary conditions (3-2-22) it immediately follows that the condition of dependence of  $f$  on  $\xi$  alone implies that  $v_w$  is inversely proportional to  $\sqrt{x}$ .

It should be mentioned that equations (3-2-19)-(3-2-21) may be extended to a flow over an infinite wedge with an expansion angle  $\beta\pi$ , where  $\beta = 2m/(m+1)$ , if the velocity at the external boundary layer follows the relation  $v_\infty = Ax^m$ .

For a constant property flow the equation of motion is solved independently of the energy and diffusion equations over a wide range of  $f_w$  ( $f_w$  or  $1/2f_w$  is called the injection parameter), and the solutions of the dynamic problem are used to calculate energy and diffusion equations.

In the solution of the diffusion equation the following fact is very essential: if the wall is not completely permeable for both the species, then because of constant total pressure of the mixture across the boundary layer ( $\partial p / \partial y = 0$ ), molar motion appears which to some extent moves aside simultaneously both the species from the wall. For a wall impermeable to the main flow gas,  $v_w$  is

$$v_w = - \frac{D_{12}}{1 - W_w} \left( \frac{\partial W}{\partial y} \right)_w \quad (3-2-23)$$

After making simple transformations we get

$$\frac{1 - W_w}{W_w - W_\infty} = \frac{\left( \frac{d\varphi}{d\xi} \right)_w}{\text{Sc} \left( \frac{1}{2} f_w \right)} \quad (3-2-24)$$

Constraint (3-2-24) implies that there exists a certain relation between injection parameter  $f_w$  and the concentrations of the injected gas both at the internal and external boundary layers, i.e. the quantities  $f_w$ ,  $W_w$  and  $W_\infty$  cannot be preassigned. It is derived from the fact that boundary conditions at the wall and at the external boundary layer do not coincide with each other.

The solutions of equations (3-2-19)-(3-2-21) are plotted in Figs. 3-17, 3-18 and 3-19.

In Fig. 3-17  $v_x/v_\infty = f(\xi)$  curves are plotted for injection parameters from 0.6 to 0.0 and for suction (a negative injection parameter). From this

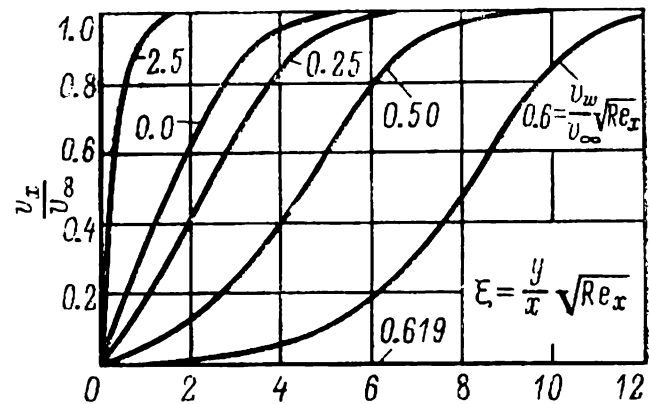


Fig. 3-17. Development of a dimensionless velocity profile in a laminar boundary layer depending on injection parameter

figure it is evident that for  $\xi = 0.619$  the longitudinal velocity  $v_x$  equals zero, i.e. the boundary layer is moved aside by injection.

In Fig. 3-18  $\theta = f(\xi)$  and  $\varphi(\xi)$  curves are plotted for injection and suction for the parameter  $(v_w/v_\infty)\sqrt{Re_x}$  ranging between 0.6 and 2.5. Here the concentration  $W_\infty$  is assumed to be equal to zero.

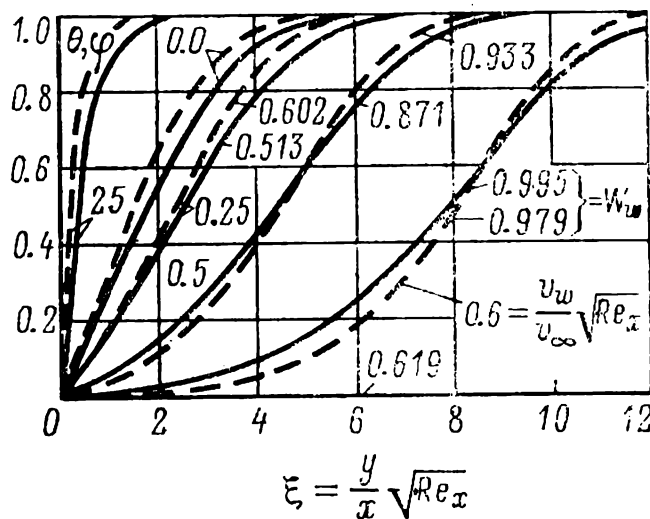


Fig. 3-18. Dimensionless profiles of temperature  $\theta$  and concentration  $\varphi$  in a laminar boundary layer of a flat plate at various injection parameters  $\frac{v_w}{v_\infty} \sqrt{Re_x}$  for Prandtl numbers 1.0 and 0.7 (solid line,  $Pr = 0.7$ ,  $Sc = 0.7$ ; dashed line,  $Pr = 1$ ,  $Sc = 1$ )

Plots of  $\varphi = f(\xi)$  for  $W_w = 0$  are presented in Fig. 3-19.

The following significant and important facts from the practical viewpoint should draw the readers' attention:

1. Injection destabilizes the boundary layer, as  $f_w$  increases the profiles become thicker, shorter and more curved, they become S-shaped. Emmons and Leigh [3-15] demonstrated that as  $f \rightarrow 0.619$  solution of the boundary

layer equations involves the conditions  $\left(\frac{\partial v_x}{\partial y}\right)_w = 0$  and  $v_x = 0$  at any finite  $y$ . In the process the boundary layer is moved aside from the plate surface, and therefore, the system of boundary-layer equations for strong injections does not take account of actual transfer in a laminar flow.

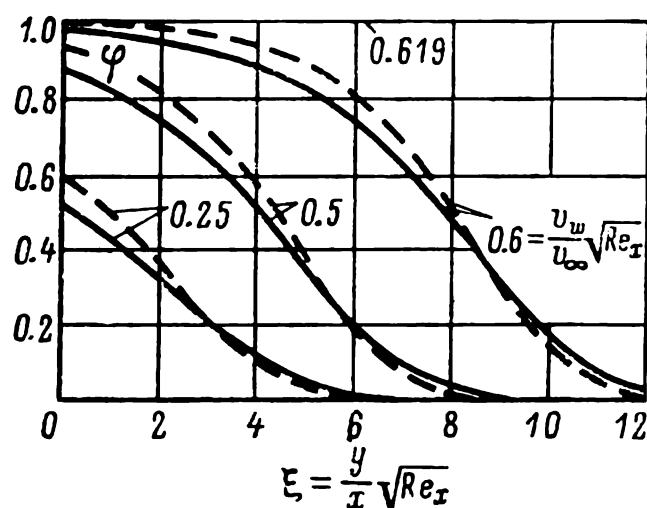


Fig. 3-19. Profiles of concentration  $\phi$  in a laminar boundary layer at various injection parameters  $\frac{v_w}{v_\infty} \sqrt{\text{Re}_x}$  when  $W_w = 0$ ,  $m = 0$  (solid line,  $\text{Sc} = 0.7$ ; dashed line,  $\text{Sc} = 1$ )

2. As the injection parameter increases the skin friction and heat-transfer rate decrease, i.e. injection may be used to protect surfaces from heating.

3. When  $\text{Pr} = \text{Sc}$  the transverse flow produces a similar effect on the development and deformation of temperature and concentration profiles, and when the condition  $\text{Pr} = \text{Sc} = 1$  is fulfilled a triple analogy is observed — this follows directly from equations (3-2-19)-(3-2-21).

Since a direct solution of boundary-layer equations is extremely difficult, one is forced to look for simpler ways of obtaining the desired results, but with lower accuracy.

One method of simplifying boundary-layer equations is the transition from their local satisfaction at each point to averaging over the boundary layer thickness, i.e. the use of integral boundary-layer equations.

Sastri and Hartnett [3-16] investigated the effect of starting length  $x_0$  without injection (Fig. 3-20).

The solutions of transfer equations are presented in Figs. 3-21 through 3-24. From figure 3-20 it is evident that the starting length effect is strongly manifested when the injection parameter is greater than 0.2. In the same figure a curve from Sparrow and Star's work [3-17] is shown. In this case the starting length equaled zero. Fig. 3-22 shows the effect of the starting length  $x_0$  on the Stanton number  $St$  for injection parameters 0.25 and 0.5. In the absence of injection the Stanton number is denoted by  $St_0$ . In Fig. 3-23 the dimensionless friction factor  $C_f/C_{f0}$  versus injection parameter  $\frac{v_w}{v_\infty} \sqrt{\text{Re}_x}$  curves are plotted for different values of  $x/x_0$  from 1.2 to  $\infty$ . In Fig. 3-24 the values calculated by the author are compared with the avail-

able data [3-17], [3-18]. The plots reveal the considerable effect of the starting length on all the integral properties of momentum and energy transfer.

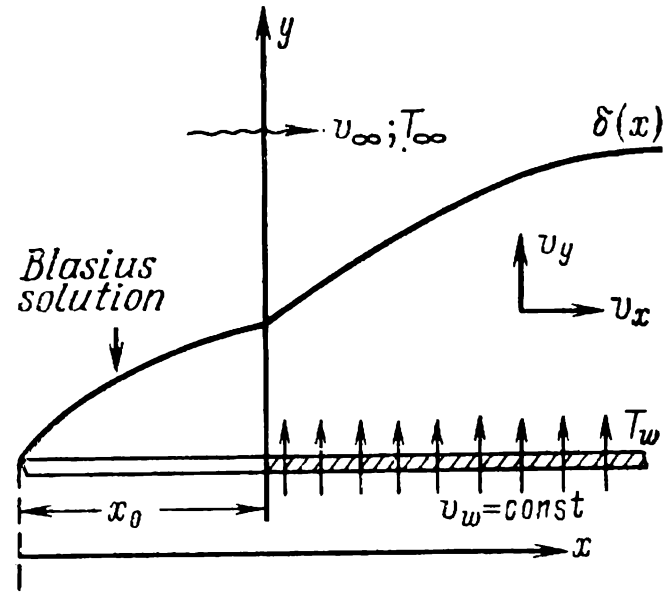


Fig. 3-20. Schematic showing injection into a boundary layer in the case of flow over a plate

For a gradientless flow over a flat porous plate with injection of a steady-state gas flow, the integral equations (3-2-13), (3-2-16) and (3-2-17) may be simplified (the second term on the left-hand side of equation (3-2-13) is neglected).

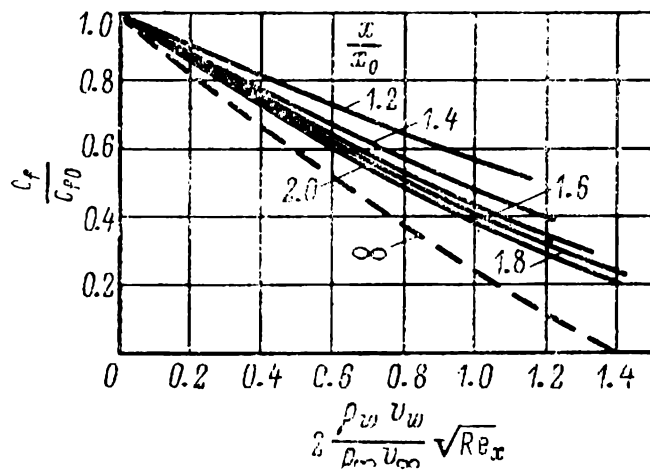


Fig. 3-21. Effect of starting length  $x_0$  on heat transfer at various injection parameters [3-16]. Dashed line shows the solution given in [3-17]

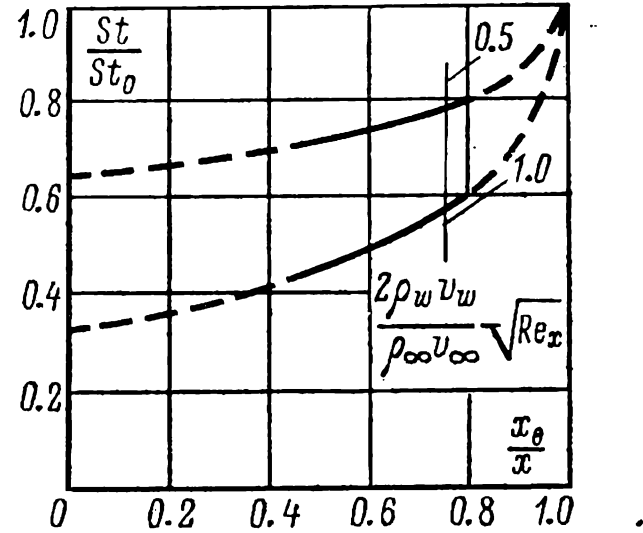


Fig. 3-22. Plot of  $St/St_0$  versus  $x/x_0$  for injection parameters 0.25 and 0.5 [3-16]

The momentum transfer equation

$$\frac{d\delta_{s,d}}{dx} - \frac{\rho_w v_w}{\rho_\infty v_\infty} = \frac{C_f}{2} \quad (3-2-25)$$

the enthalpy equation

$$\frac{d\delta_{e,d}}{dx} - \frac{c_{pw} \rho_w v_w}{c_{p\infty} \rho_\infty v_\infty} = St \quad (3-2-26)$$

the diffusion equation

$$\frac{d\delta_{m,d}}{dx} - \frac{\rho_w v_w}{\rho_\infty v_\infty} = St_m \frac{\rho_w}{\rho_\infty} \quad (3-2-27)$$

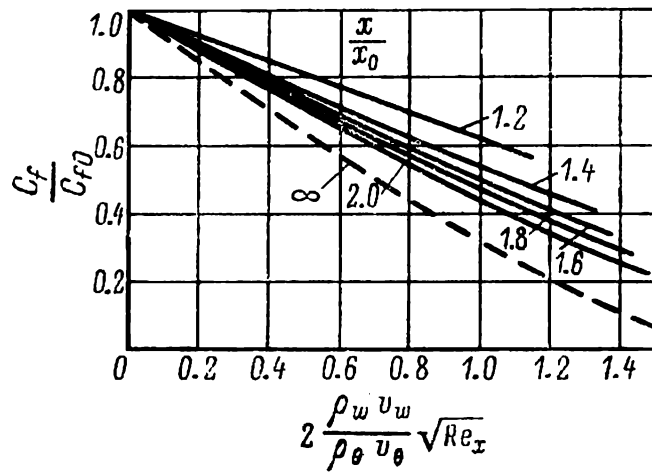


Fig. 3-23. Plot of relative friction factor  $C_f/C_{f0}$  versus injection parameter at various  $x/x_0$  (solid line—calculated data; dashed line shows Sparrow's data)

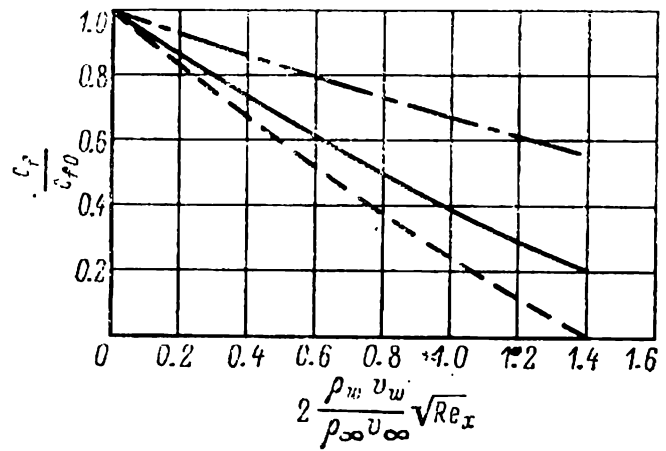


Fig. 3-24. Comparison of calculated data [3-16] (solid line) with the data of [3-17] when  $x_0 = 0$  (dashed line) and the data of [3-18] at  $x/x_0 = 2$  [3-18] (dot-and-dash line)

where  $\delta_{v,d}$ ,  $\delta_{e,d}$  and  $\delta_{m,d}$  are momentum, energy and mass displacement thicknesses, respectively, equal to

$$\delta_{v,d} = \int_0^\delta \frac{\rho v_x}{\rho_\infty v_\infty} \left( 1 - \frac{v_x}{v_\infty} \right) dy$$

$$\delta_{e,d} = \int_0^{\delta_H} \frac{\rho v_x}{\rho_\infty v_\infty} \left( 1 - \frac{H - H_w}{H_\infty - H_w} \right) dy$$

$$\delta_{m,d} = \int_0^{\delta_m} \frac{\rho v_x}{\rho_\infty v_\infty} \left( 1 - \frac{W - W_w}{W_\infty - W_w} \right) dy$$

For solving ordinary differential equations (3-2-25)-(3-2-27) it is necessary to know the relation between the boundary layer parameters and momentum, energy and mass flows at the wall.

Karman and Pohlhausen suggested to replace the unknown longitudinal velocity profile in a boundary layer with a certain interpolation (particularly, with a polynomial) which would satisfy certain prescribed boundary conditions at the wall and at the external boundary layer.

They proceeded from the theory of velocity boundary layer of finite thickness. The profile equation is expressed in dimensionless coordinates  $y/\delta$ , such that after its substitution in the integral momentum relation, it changes into an ordinary nonlinear differential equation with one unknown quantity  $\delta(x)$ . After solving the equation by any approximate method,  $\delta(x)$  and then all other unknown characteristics are found.

Thus, the calculation results will depend on how successful is the choice of the interpolation formula for the velocity profile and boundary conditions that determine the coefficients appearing in this formula. Such an approach to the problem solution is very convenient for analyzing experimental data when direct measurements are taken in a boundary layer and the conditions at the wall  $T_w(x)$ ,  $q_w(x)$  and  $j_w(x)$  are also known.

Systematization of all the available experimental data on injection of different gases into a laminar boundary layer at a flat porous plate in an air flow has resulted in generalized relations for local heat transfer and skin friction [3-19]

$$\frac{C_f}{C_{f0}} = 1 - 2.08 \left( \frac{M_2}{M_1} \right)^{1/2} \frac{\rho_w v_w}{\rho_\infty v_\infty} \sqrt{\text{Re}_x} \quad (3-2-28)$$

$$\frac{q}{q_0} = 1 - 1.82 \left( \frac{M_2}{M_1} \right)^{1/3} \frac{\rho_w v_w}{\rho_\infty v_\infty} \sqrt{\text{Re}_x} \quad (3-2-29)$$

where  $M$  is the molecular mass.

Quantities  $\rho$  and  $\text{Re}_x$  are calculated from the characteristic temperature  $T^*$

$$T^* = T_\infty + 0.5(T_w - T_\infty) + 0.22(T_{r0} - T_\infty) \quad (3-2-30)$$

where  $T_{r0}$  is the recovery temperature at the solid surface without mass injection but with identical boundary conditions. It follows from equations (3-2-28) and (3-2-29) that light gases are very effective coolants; this conclusion is physically reliable and has been verified experimentally.

Rapid development of computer engineering has offered almost infinite possibilities for very reliable and accurate numerical calculations of simultaneous heat- and mass-transfer processes by making a small number of assumptions. However, the disadvantage of numerical methods is their restricted validity. To obtain general quantitative relationships, additional treatment of a great number of particular solutions is required. But a definite problem can be solved numerically with any desired accuracy. This may be the probable reason for a limited number of experimental works on injection into a laminar boundary layer.

### (b) *Heat and Mass Transfer During Evaporation*

Theoretical results on injection of inert noncondensing gases are conventionally extended to evaporation from a solid surface into a forced gaseous

flow. In this case it is assumed that realization of the condition that  $T_w$  and  $W_w$  are independent of  $x$  is fulfilled automatically. Evaporation pertains to first kind phase transitions at constant temperature, and concentration of the generated vapour is unequivocally related to the evaporation temperature by the saturation curve equation. Therefore, it is not necessary to create porous materials with porosity varying according to a relation which would ensure injectant supply proportional to  $x^{-0.5}$ .

The available experimental data do not, however, confirm this assumption. Thus, for example, a comparison of the relations obtained by Z. P. Shulman [3-20] with the data listed in [3-21] on heat transfer of a circular cylinder has revealed that in a laminar boundary layer at equal parameters  $f_w$ , evaporation is essentially more effective than injection from a surface protection viewpoint. A similar situation may be found if the theoretical results [3-14] are compared with the results obtained in these experiments. The comparison revealed that in the same range of injection parameter  $f_w$ , the heat-transfer rate during evaporation is lower by a factor of 1.5-1.9 than during injection.

This discrepancy may be attributed to factors connected with the essential difference between injection and evaporation and also to the specific features of evaporation. The linear evaporation rate  $j_1/\rho_1$ , though being measured in velocity units, is not a velocity in the general sense of this word, but a quantity which characterizes macroscopic motion of the body per unit time (a molar motion velocity). Evaporation is a diffusion process induced by the concentration drop in a boundary layer. Without a concentration drop evaporation is impossible, whereas a gas homogeneous with the main one may be injected even into an isothermal gaseous flow since proper injection is a molar process of a fluid flow type due to a total pressure drop between the injectant and the main gas flows.

Diffusion is mixing of a material due to molecular (random) motion. A larger number of molecules move down the diffusional flow than in the reverse direction. This causes the molecular material flow characterized by conventional linear diffusion velocity.

The convective contribution to the transfer process is  $W_1/(1 - W_1)$  of the molecular transfer (the Stefan flow velocity  $v_{st} = j_1/\rho$ , where  $\rho$  is the mixture density. This very flow upon evaporation affects the hydrodynamics in a boundary layer. Thus, because of different transfer mechanisms of injection and evaporation, the transverse mass flow affects heat transfer in a different way.

Besides, evaporation has some other peculiarities.

In order to analyse the effect of vapour transfer during evaporation on the heat- and mass-transfer process, we shall return to Eckert and Hartnett's solution [3-14].



If we calculate the heat- and mass-transfer coefficients by formulas

$$\alpha = - \frac{\lambda \left( \frac{\partial T}{\partial y} \right)_w}{T_w - T_\infty} \quad (3-2-31)$$

$$\alpha_m = - \frac{\rho D (\partial W / \partial y)_w}{(W_w - W_\infty)} \quad (3-2-32)$$

and denote their values without injection by  $\alpha_0$  and  $\alpha_{m0}$ , respectively, then the results may be plotted as shown in Fig. 3-25. The figure shows that as  $v_w$  increases, the heat- and mass-transfer coefficients decrease and at  $(v_w/v_\infty) \sqrt{\text{Re}_x} = 0.64$  they are zero. Inversely, when the convective flow has the opposite direction, the heat- and mass-transfer coefficients increase; thus at  $(v_w/v_\infty) \sqrt{\text{Re}_x} = -4$  and  $\text{Pr}=\text{Sc}=1$ , they increase by a factor of 12.

It may be demonstrated that in the liquid evaporation processes the transverse mass flow factor  $f_w = (v_w/v_\infty) \sqrt{\text{Re}_x}$  is small and cannot affect heat- and mass-transfer coefficients.

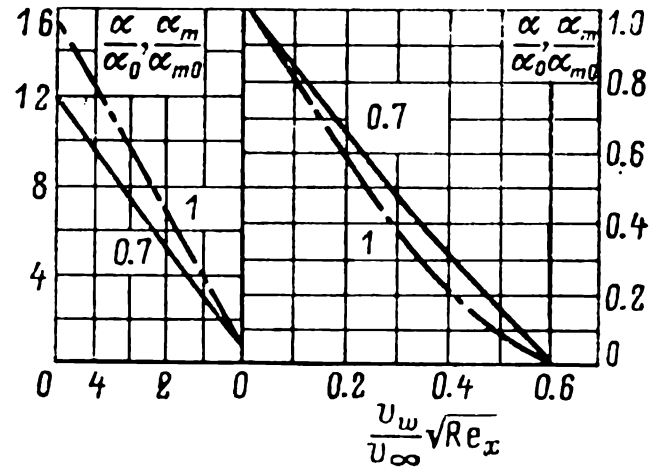


Fig. 3-25. Plot of heat and mass transfer ratios versus injection parameter  $\frac{v_w}{v_\infty} \sqrt{\text{Re}_x}$

In the process of evaporation from a free surface the transverse velocity  $v_w$  is

$$v_w = - \frac{D\rho}{1 - W_w} \left( \frac{\partial W}{\partial y} \right)_w = \frac{j_w}{(1 - W_w)\rho} \quad (3-2-33)$$

where  $j_w$  is the evaporation rate.

For simplification  $W_1 = \rho_{10} = 0$ ,  $v_w = \frac{j_w}{\rho}$  are assumed.

Then

$$f_w = \frac{j_w}{\rho v_\infty} \sqrt{\text{Re}_x} \quad (3-2-34)$$

Under steady-state conditions the evaporation rate

$$j_w = \frac{\alpha_x(T_\infty - T_w)}{r} = \frac{\lambda}{x} \text{Nu}_x \frac{T_\infty - T_w}{r} = \frac{\lambda}{rx} A_x \sqrt{\text{Re}_x} (T_\infty - T_w) \text{Pr}^{1/3} \quad (3-2-35)$$

Substitution of (3-2-35) into (3-2-34) yields

$$f_w = \frac{A_x c_p (T_\infty - T_w)}{r \text{Pr}^{2/3}} \quad (3-2-36)$$

Hence,  $f_w$  depends on the temperature driving force  $T_\infty - T_w$  and the thermal properties of the flow rather than on the flow velocity.

In order that coefficients  $\alpha$  and  $\alpha_m$  decrease by 10 per cent compared with coefficients  $\alpha_0$  and  $\alpha_{m0}$ , it is necessary that  $f_w \geq 0.05$  (see Fig. 3-25). If  $f_w = 0.05$ ,  $r = 579$  kcal/kg,  $A_x = 0.3$ ,  $\text{Pr} = 0.7$  are assumed, then the temperature driving force  $\Delta T = T_\infty - T_w$  should be above  $260^\circ\text{C}$ . Such temperatures are rarely met with during convective drying. For this reason the solution shown in Fig. 3-25 holds for a porous plate flow when inert gas is injected in the boundary layer at a sufficiently high velocity.

Bonding of the liquid with the wall material is one of the most essential effects of molecular interaction between the liquid and its vapour and the wall material. Therefore, if thermal means are used to remove liquid from a capillary-porous body, besides phase-transition heat the bond energy should be accounted for. The energy value depends on the type of the liquid bond with the capillary-porous material. Academician P. A. Rebinder [3-22] classified all bond forms into chemical, physico-chemical and physico-mechanical.

The passage pore system changes upon evaporation. Some passages may become impassable for vapour as they are choked by suspended liquid columns, liquid "plugs". The natural porous material structure is generally nonuniform. Most externally open pores intersect each other and the liquid level in them is determined not only by the value of the pressure head selected but also by the geometry of their structure. A certain amount of liquid spreads over the outer surface of porous bodies and fills the hollows between the pores and roughnesses. The surfaces of menisci and pore passages are curved, therefore, the effective mass-transfer surface turns out to be much greater than the total cross-sectional area of open pores.

Molar dispersion and evaporation of submicroscopic droplets in a boundary layer are characteristics inherent only in evaporation. The bulk evaporation hypothesis based on the dynamic character of sorption and desorption processes is formulated in [3-23] and reads: the flow effect (entrainment and condensation on the wall) results in the smallest droplets which enter the boundary layer. According to de Boer's adsorption theory

[3-24], evaporation is a dynamic process of sorption and desorption. Liquid molecules not only leave the surface (evaporation) but also continuously return (condensation). The evaporation rate is proportional to the molecular flow difference. Since condensation is nonuniform [3-25] and some areas are not fully wetted by adsorbed liquefied vapour, droplets appear which are not so closely bound with the liquid. These are carried away into the boundary layer by the gas flow and evaporate within the bulk. Bulk evaporation is a vapour source and a negative heat source in boundary layer equations. This hypothesis was confirmed by the observations of Malmquist and Meichsner [3-26] who in experiments on wood drying by superheated vapour found polymerized vapour molecules carried away into the boundary layer and their evaporation in its bulk. Three different locations of the evaporation surface are possible in the case of capillary-porous evaporation.

1. The surface of the body is constantly covered with a liquid layer of uniform thickness. In this case the process is similar to evaporation from a free surface.

2. Focus evaporation from capillary ends. The liquid level in capillaries coincides with the visible geometrical surface of the body, thus not the whole outer wall is an evaporation mirror. In the case of focus evaporation the liquid volume at the surface in contact with liquid flow changes quickly, which may even change the flow pattern in the boundary layer.

3. The evaporation surface may retract into the bulk of the material. Retraction of the evaporation zone is not uniform with respect to the wall surface and depends on capillary sizes. Because of the dry interlayer, additional thermal resistance to heat and mass transfer from the liquid to the ambient medium occurs. In this case the heat transfer agent circulates in emptied capillaries and the total pressure increases which favours droplet entrainment into the boundary layer. The process rate is greatly affected by the material structure, which allows the assumption to be made on a close relationship between external and internal heat and mass transfer problems.

Thus, for correct calculation of the evaporation rate in every specific case, all the effects involved should be identified and estimated. The earlier mentioned effects complicate the analytical solutions of boundary-value problems on simultaneous convection and evaporation in the boundary layer.

The analytical solution for a heat transfer problem when evaporation takes place from the bulk of the porous plate into the laminar boundary layer is presented in [3-27], where the differential heat transfer equation

$$v_x \frac{\partial T(x, y)}{\partial x} + v_y \frac{\partial T(x, y)}{\partial y} = a \frac{\partial^2 T(x, y)}{\partial y^2} - \frac{c_{p1} - c_{p2}}{c_p \rho} j_1 \frac{\partial T(x, y)}{\partial y} \quad (3-2-37)$$

is considered.

For qualitative analysis of the evaporation surface retardation effect on heat transfer, the solution of (3-2-37) may be used with the following simplifying assumptions. If  $Pr \ll 1$ , then it may be assumed that

$$v_x = \bar{v}_x = \text{const}; \quad v_y = \bar{v}_y = \text{const}; \quad v_1 = j/\rho = \text{const} \quad (3-2-38)$$

Equation (3-2-37) may now be written as

$$\bar{v}_x \frac{\partial T(x, y)}{\partial x} + \bar{v}_y^* \frac{\partial T(x, y)}{\partial y} = a \frac{\partial^2 T(x, y)}{\partial y^2} \quad (3-2-39)$$

where

$$\bar{v}_y^* = \bar{v}_y + \frac{c_{p1} - c_{p2}}{c_p} \bar{v}_1 = \text{const}$$

The temperature of the evaporation surface is assumed constant

$$T(x, -\xi) = T_e = \text{const}$$

The boundary conditions may thus be written as

$$\text{at } x = 0 \quad T(0, y) = T_\infty \quad (3-2-40)$$

$$\left. \begin{aligned} \text{at } y = -\xi \quad T(-\xi, x) &= T_e = \text{const} \\ \text{at } y = 0 \quad -\lambda_f \frac{\partial T(x, 0)}{\partial y} &= -\lambda_s \frac{\partial T(x, 0)}{\partial y} \\ \text{at } y = \infty \quad T(x, \infty) &= T_\infty \end{aligned} \right\} \quad (3-2-41)$$

where subscript  $s$  refers to a solid, and subscript  $f$ , to a liquid. It is assumed that the vapour flow enthalpy in the boundary layer of the solid  $(-\xi, 0)$  is the same as that in the air boundary layer  $(0, \delta)$ . The water enthalpy is spent on vaporization without additional heat supply.

For small  $\xi$  ( $\xi/l \ll 1$ ) it may be assumed that in a dry layer temperature distribution is linear. Then at  $y = 0$

$$-\frac{\partial T(0, x)}{\partial y} + H[T(0, x) - T_e] = 0 \quad (3-2-42)$$

where

$$H = \frac{\lambda_s}{\lambda_f \xi}$$

At large retractions  $\xi$  of the vaporization zone the temperature distribution within a dry interlayer may be approximated by the exponential law [3-28]

$$T(-y, x) = T_e + f(x) [1 - e^{-k(y+\xi)}] \quad (3-2-43)$$

where  $k$  is constant;  $f(x)$  is a function showing the surface temperature variation down the liquid flow. Constraint (3-2-42) still holds, but

$$H = \frac{\lambda_s k}{\lambda_f (\exp k\xi - 1)} \quad (3-2-44)$$

For the given boundary conditions the solution of equation (3-2-39) takes the form

$$\begin{aligned} \frac{T(x, y) - T_e}{T_\infty - T_e} = & \frac{H - \frac{\bar{v}_y^*}{2a}}{H - \frac{\bar{v}_y^*}{a}} \exp \left[ \left( H^2 - \frac{H\bar{v}_y^*}{a} \right) \frac{xa}{\bar{v}_x} + Hy \right] \operatorname{erfc} \left[ \left( H - \frac{\bar{v}_y^*}{2a} \right) \right] \\ & \times \left( \sqrt{\frac{xa}{\bar{v}_x}} + \frac{y}{2} \sqrt{\frac{\bar{v}_x}{ax}} \right) + \frac{1}{2} \operatorname{erfc} \left( \frac{\bar{v}_y^*}{2a} \sqrt{\frac{xa}{\bar{v}_x}} - \frac{y}{2} \sqrt{\frac{\bar{v}_x}{ax}} \right) \\ & - \frac{H}{2} \frac{\exp \left( \frac{\bar{v}_y^*}{a} y \right)}{H - \frac{\bar{v}_y^*}{a}} \operatorname{erfc} \left( \frac{\bar{v}_y^*}{2a} \sqrt{\frac{ax}{\bar{v}_x}} + \frac{y}{2} \sqrt{\frac{\bar{v}_x}{ax}} \right) \end{aligned} \quad (3-2-45)$$

From (3-2-45) the following particular cases may be obtained: (a) heat transfer without mass transfer; (b) heat transfer during liquid evaporation from a capillary porous surface ( $\xi = 0$ ,  $H \rightarrow \infty$ ).

Let us find the local Nusselt number

$$\operatorname{Nu}_x = \frac{-x}{T_\infty - T(0, x)} \frac{\partial T(0, x)}{\partial y}$$

and introduce the function  $N(K, B)$

$$\operatorname{Nu}_x = - \frac{x}{T_\infty - T(0, x)} \frac{\partial T(0, x)}{\partial y} = \frac{1}{\sqrt{\pi}} \sqrt{\operatorname{Re}_x \operatorname{Pr}} N(K, B) \quad (3-2-46)$$

where

$$N(K, B) = \frac{\varphi(K, B) - \frac{1}{2} \sqrt{\pi} B \operatorname{erfc} \frac{1}{2} B}{\left( 1 - \frac{B}{K} \right) - (K\sqrt{\pi})^{-1} \varphi(K, B) + \frac{1}{2} BK^{-1} \operatorname{erfc} \frac{1}{2} B} \quad (3-2-47)$$

Here

$$\left. \begin{aligned} \varphi(K, B) &= \left(1 - \frac{1}{2} \frac{B}{K}\right) \sqrt{\pi} K \exp(K^2 - BK) \operatorname{erfc}\left(K - \frac{1}{2} B\right) \\ K &= \frac{H_x}{\sqrt{\operatorname{Pe}_x}}; \quad B = \frac{\bar{v}_y^* \sqrt{x}}{\sqrt{a \bar{v}_x}} \end{aligned} \right\} \quad (3-2-48)$$

In Fig. 3-26 the functions  $N(K, B) = f(K)$  are plotted for various  $B$  (from 0 to 2). If  $\bar{v}_y^* = 0$  is assumed (small Prandtl numbers,  $\operatorname{Pr} \rightarrow 0$ ), then  $B = 0$ . In this case  $N(K, B) > 1$  (the curve  $N(K, 0)$  asymptotically approaches unity). The Nusselt number with a retracted evaporation surface will be greater than  $\operatorname{Nu}$  at a free evaporation surface ( $\xi = 0$ ,  $K = \infty$ ). This conclusion follows directly from solution (3-2-45), since the plate surface temperature increases down the flow (along the  $x$ -axis). From the general heat-transfer theory it is known that in this case the heat-transfer coefficient will increase.

Let us now demonstrate that coefficient  $B$  depends on  $\operatorname{Pr}$  and the Stefan flow velocity  $v_1$  ( $v_1 = j_w/\rho$ ). If the velocity distribution is assumed to

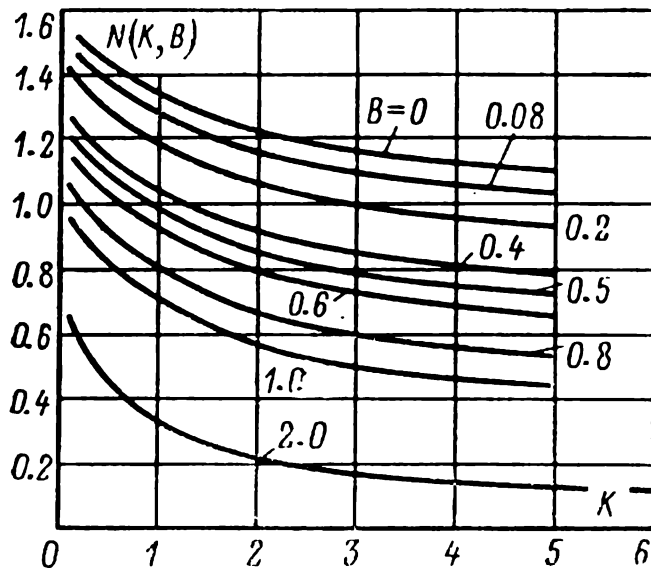


Fig. 3-26. Plot of  $N$  versus  $K$  for various values of  $B$

follow formula (3-1-39), then  $v_x = \frac{5}{8} v_\infty$ . If the Blasius distribution function  $f^1$  (Table 3-1) is taken, then  $\bar{v}_x = 0.66 v_\infty$

$$\left. \begin{aligned} B &= B_1 + B_2; \quad B_1 = 1.23 \sqrt{\operatorname{Re}_x \operatorname{Pr}} \bar{v}_y / v_\infty \\ B_2 &= 1.23 \sqrt{\operatorname{Re}_x \operatorname{Pr}} \left( \frac{c_{p1} - c_{p2}}{\rho c_p v_\infty} j_w \right) \end{aligned} \right\} \quad (3-2-49)$$

If approximate formula (3-1-20) is used, then from Fig. 3-27 we find  $\bar{v}_y \sqrt{\operatorname{Re}_x} / v_\infty = 0.43$ .

Coefficient  $B_1$  will then depend only on the Prandtl number

$$B_1 = 0.53 \sqrt{\text{Pr}} \quad (3-2-50)$$

For air ( $\text{Pr} = 0.7$ )  $B_1 = 0.44$ .

Coefficient  $B_2$  is proportional to the injection parameter  $f_w$  [see formula (3-2-34)]. For example, for humid air  $c_{p1} = 0.47$ ,  $c_{p2} = 0.24$ , then

$$B_2 = 0.427 \frac{j_w}{\rho v_\infty} \sqrt{\text{Re}_x} \quad (3-2-51)$$

If  $f_w$  from formula (3-2-34) is taken as the injection parameter, then

$$B_2 = 0.427 f_w \quad (3-2-52)$$

In convective drying of most materials the injection parameter is much less than 0.05. Hence, coefficient  $B_2$  is less than  $B_1$  ( $B_2/B_1 < 0.05$ ), i.e.  $B_2 \ll B_1$ .

Let us evaluate the solution procedure of equation (3-2-37) by averaging velocities  $\bar{v}_x$  and  $\bar{v}_y$  over the boundary layer. No evaporation is assumed ( $B_2 = 0$ ,  $\xi = 0$ ), and  $K = \infty$ .

In this case

$$\varphi(\infty, B) = \exp\left(-\frac{B^2}{4}\right) \quad (3-2-53)$$

$$N(\infty, B) = \exp\left(-\frac{B^2}{4}\right) - \frac{1}{2} \sqrt{\pi} B \operatorname{erfc} \frac{1}{2} B \quad (3-2-54)$$

where  $B = B_1$  [for air  $B = 0.44$ ;  $N(\infty, 0.44) = 0.659$ ].

Then from formula (3-2-46)

$$\text{Nu}_x = \frac{0.81}{\sqrt{\pi}} \sqrt{\text{Re}_x \text{Pr}} N(\infty, B) = 0.252 \sqrt{\text{Re}_x} \quad (3-2-55)$$

since

$$(\overline{\text{Re}_x})^{1/2} = 0.81 \text{Re}_x^{1/2}$$

According to formula (3-1-73), the proportionality factor between  $\text{Nu}_x$  and  $\text{Re}_x$  is

$$\frac{\text{Nu}_x}{\sqrt{\text{Re}_x}} = 0.332 \sqrt[3]{0.7} = 0.294 \quad (3-2-56)$$

and differs from that in formula (3-2-55) by about 10%. Thus the method of averaging velocities  $\bar{v}_x$  and  $\bar{v}_y$  gives satisfactory results.

From (3-2-54) it is evident that as  $B$  increases,  $N(\infty, B)$  decreases, or in other words, an increase in the evaporation rate (an increase in coefficient  $B_1$ ) results in a decrease of the Nusselt number  $Nu_x$ .

But the retraction of the evaporation surface (the decrease of coefficient  $K$ ) results in the increase of  $N(K, B)$ , and hence in  $Nu_x$  (see Fig. 3-26).

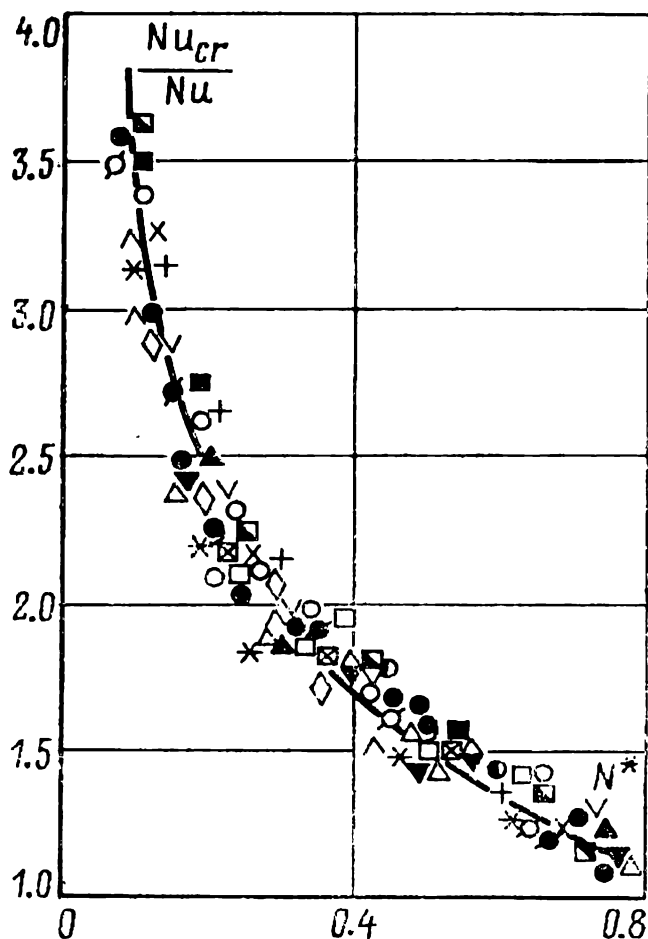


Fig. 3-27. Plot of dimensionless Nusselt number versus relative drying rate for various materials during convective drying

Therefore, depending on the values of coefficients  $K$  and  $B$ , when liquid evaporates from porous bodies the Nusselt number may be larger or smaller than  $Nu_{x0}$  with no evaporation  $N(K, B) = N(\infty, B_1)$ .

For example, in convective drying of different materials the mean Nusselt number  $Nu$  increases with the drying rate  $j_w$ . In Fig. 3-27 the ratio  $Nu_{cr}/Nu$  is plotted versus the drying rate  $j_w^*$  ( $j_w^* = j_w/j_{w\ cr}$ ), where subscript  $cr$  indicates critical moisture content. From Fig. 3-27

$$Nu = Nu_{cr} j_w^{*0.57} \quad (3-2-57)$$

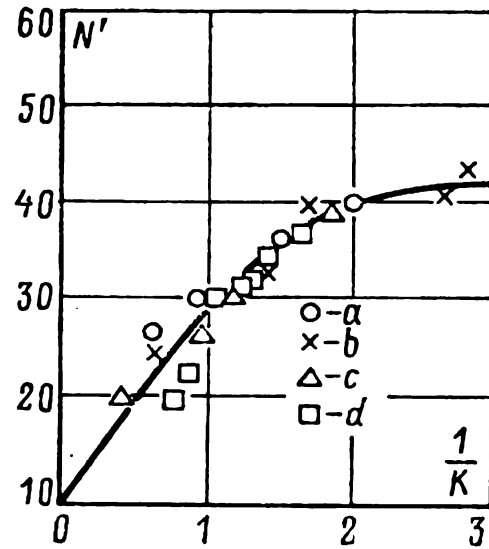
Hence, we may conclude that in a drying process liquid evaporation occurs at a certain depth from the surface of the body ( $\xi \neq 0$ ,  $K \neq \infty$ ).

The suggested physical model is verified experimentally [3-30]. Experiments were carried out on water evaporation in a laminar boundary layer of a porous plate at various depths  $\xi$  of the evaporation surface and



porosity  $\Pi$ . In Fig. 3-28  $N' = f(1/K)$  is plotted from the data obtained by using formula (3-2-47) for  $B = 0.2$  and the experimental points are shown. It is evident from Fig. 3-28 that the experimental points lie close to the predicted curve.

Fig. 3-28. Plot of  $N' \left[ N' = N\left(\frac{1}{K}, B\right) / N(\infty, B) \right]$  versus  $\frac{1}{K}$   
 $a - \text{porosity } \Pi = 42\%; b - \Pi = 38\%; c - \Pi = 36\%; d - \Pi = 35\%$



For convenience, equation (3-2-46) may be used in practical calculations to obtain a number of approximate formulas for  $N(K, B)$ .

Let us denote

$$N^* = \frac{N(K, B)}{N(\infty, B)} = \frac{\text{Nu}_x}{(\text{Nu}_x)_{\xi=0}} \quad (3-2-58)$$

The number  $N^*$  is the ratio of the Nusselt number at retracted surface to that without retraction (evaporation takes place at the surface of the body).

Then we shall have the following approximate formulas

$$N^* - 1 = N_0^* - \alpha K \quad \text{at} \quad 0.1 < K < 1.0 \quad (3-2-59)$$

$$N^* - 1 = \beta/K \quad \text{at} \quad 1 < K < \infty \quad (3-2-60)$$

Constant coefficients  $N_0^*, \alpha, \beta$  depend on coefficient  $B$  (see Table 3-5).

Table 3-5

Dependence of  $N_0^*, \alpha$  and  $\beta$  on coefficient  $B$

$B$	0	0.1	0.2	0.3	0.4	0.5	0.6
$N_0^*$	0.56	0.61	0.7	0.79	0.90	1.0	1.6
$\alpha$	0.21	0.25	0.29	0.33	0.37	0.41	0.44
$\beta$	—	0.46	0.54	0.62	0.74	0.83	0.92

Calculations have revealed [3-27] that when the surface is retracted even by 0.5 mm the heat-transfer rate increases by about 25% compared with a free surface.

Besides, for such small retractions it is very difficult to measure the actual surface temperature and evaporation temperature, and in the calculations of  $Nu_x$  the temperature drop  $T_\infty - T_e$  is introduced. Therefore, it would be more correct to express the coefficient of heat transfer from heated gas to the evaporation surface in terms of boundary layer and dehydrated porous layer

$$k = \left( \frac{1}{\alpha} + \frac{\xi}{\lambda_s} \right)^{-1} \quad (3-2-61)$$

If the heat-transfer coefficient is calculated from temperature difference ( $T_\infty - T_e$ ), the formula for the Nusselt number will take the form

$$Nu_{xe} = - \frac{x}{(T_\infty - T_e)} \frac{\partial T(0, x)}{\partial y} = \frac{1}{\sqrt{\pi}} \sqrt{Re_x Pr} N_e(K, B) \quad (3-2-62)$$

where

$$N_e(K, B) = \left( 1 - \frac{B}{K} \right)^{-1} \left[ \varphi(K, B) - \frac{1}{2} \sqrt{\pi} B \operatorname{erfc} \frac{1}{2} B \right] \quad (3-2-63)$$

One can see from (3-2-62) that  $N_e(K, B)$  increases with  $K$ , i.e. as the evaporation surface retracts farther to the center ( $\xi$  increases, and the coefficient  $K$  decreases), the Nusselt  $Nu_{xe}$  number decreases. In this case retraction of the evaporation surface has a similar effect as gas injection into the boundary layer. This effect may physically be explained as follows. Since the temperature driving force ( $T_\infty - T_e$ ) is constant, then with the retraction of the evaporation surface to the center, the heat-transfer coefficient decreases due to an increase in the heat resistance of the boundary layer. In this case  $Nu_{xe}$  should be defined in terms of the heat-transfer factor  $k$  rather than the heat-transfer coefficient  $\alpha$ .

For approximate calculations the following formula may be used

$$N_e^* = \frac{N_e(K, B)}{N_e(\infty, B)} = \frac{Nu_{xe}}{(Nu_{xe})_{\xi=0}} = CK^m \quad (3-2-64)$$

where  $C$  and  $m$  are constants.

In [3-37] experimental data are presented on water evaporation from a porous plate ( $20 \times 8.7 \times 1.4$  cm) with porosity  $\Pi = 40\%$ . The air flow velocity varies between 1 and 10 m/sec, and the temperature from

50° to 120°C. The experimental results are presented in Fig. 3-29 which shows that

$$Nu_e = 0.05 Re^{0.6} K^{0.22} \text{ at } 10^4 < Re < 10^5 \quad (3-2-65)$$

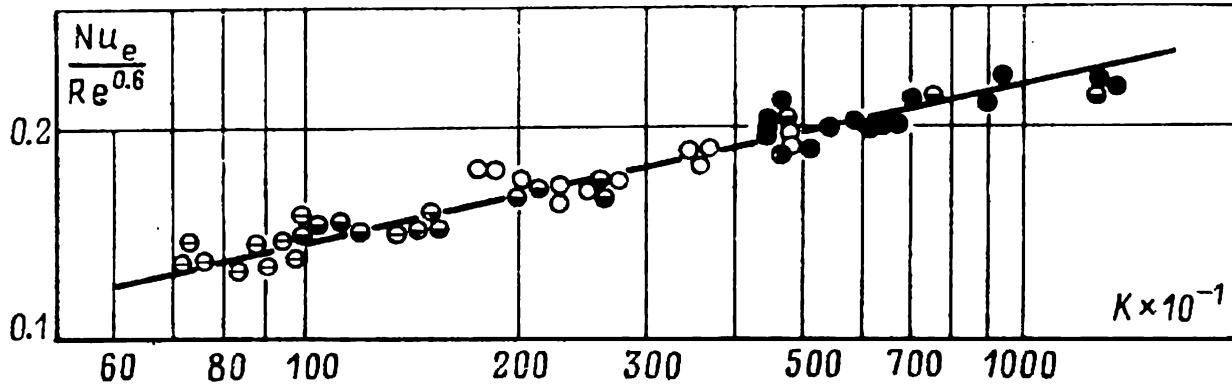


Fig. 3-29. Plot of  $Nu_e/Re^{0.6}$  versus  $K$   
Depth  $\xi = 0.7$  mm;  $\circ$  — 1.8 mm;  $\ominus$  — 3.2 mm;  $\odot$  — 4.5 mm

The authors of a number of works [3-13] to [3-20] used in formula (3-2-65) the exponent  $m$  equal to 0.3 or 0.28.

The differential diffusion equation may be solved in a similar way

$$v_x \frac{\partial \rho_{k0}}{\partial x} + v_y \frac{\partial \rho_{k0}}{\partial y} = D \frac{\partial^2 \rho_{k0}}{\partial y^2} \quad (3-2-66)$$

But the variation of the mass-transfer (diffusion) Nusselt number  $(Nu_x)_m$  does not agree with that of the heat-transfer Nusselt number  $Nu_x$  for coefficients  $K$  and  $B$ , as the inequality

$$\frac{\lambda_s}{\lambda_f} \neq \frac{D_s}{D_f} \quad (3-2-67)$$

holds.

In all the experimental studies it was established that the diffusion Nusselt number increased with the evaporation rate. Thus, in [3-37] an empirical formula for water evaporation into an air flow is given based on the experimental data of a number of researchers

$$Nu_l \equiv Sh = 0.078 Re^{0.6} K^{-0.6} \text{ at } 1.5 \times 10^4 < Re < 2 \times 10^7 \quad (3-2-68)$$

where  $Sh$  is the Sherwood number.

D. B. Spalding [3-31] suggested a simplified calculation procedure for convective heat transfer under conditions of steady-state nonadiabatic evaporation from the surface in a flow. The method is based on the assumption that specific convective heat flux  $q_\alpha$  during nonadiabatic evaporation is proportional to the deviation of actual surface temperature  $T_w$  from  $T_e$  (equilibrium temperature) which the surface would attain in the adiabatic process, i.e.

$$q_\alpha = \alpha_0 (T_e - T_w) \frac{c_{p\text{ eff}}}{c_{p\infty}} \quad (3-2-69)$$

where  $\alpha_0$  is the heat-transfer coefficient (without mass transfer) of an impermeable wall located with a moist porous surface under similar heat and flow conditions;  $c_{p\,eff}$  is the effective heat capacity of a vapour-gas mixture equal to

$$c_{p\,eff} = \frac{h_\infty - h_w}{T_e - T_w} \quad (3-2-70)$$

where  $h$  is the enthalpy.

If the heat-transfer coefficient is defined in an ordinary way, then

$$\frac{\alpha}{\alpha_0} = \frac{T_e - T_w}{T_\infty - T_w} \frac{c_{p\,eff}}{c_{p\,\infty}} = \frac{h_\infty - h_w}{c_{p\,\infty}T_\infty - c_{p\,\infty}T_w} \quad (3-2-71)$$

In case of a forced flow over a body, a zero vapour concentration in the main flow may be assumed, hence  $c_{p\,\infty}T_\infty = h_\infty$ . If the heat-capacity variations in the temperature range between  $T_\infty$  and  $T_w$  are neglected (the calculations have revealed that for air the heat capacity variations do not exceed 3% when temperature varies by 50°C), then  $c_{p\,\infty}T_w$  is equal to the air enthalpy at the plate surface without evaporation. Then

$$\frac{\alpha_u}{\alpha_0} = \frac{h_\infty - h_w}{h_\infty - h_{w0}} \quad (3-2-72)$$

Hence, the coefficients for heat transfer during evaporation and in a process without mass transfer, under similar conditions in the flow and at the wall, are proportional to the enthalpy drop in the boundary layer. The enthalpy of a vapour-gas mixture is calculated from the additivity rule for gaseous mixtures depending on heat capacities and concentrations of the species. From formula (3-2-72) it follows that  $(\alpha_u/\alpha_0) < 1$  if  $c_{p1} > c_{p2}$  (the heat capacity of liquid vapour is greater than that of the main gas flow) and  $(\alpha_u/\alpha_0) > 1$  when  $c_{p1} < c_{p2}$ .

Experiments have confirmed the possibility of decreasing the heat-transfer rate during evaporation of water, ethyl alcohol and a number of other liquids, the heat capacities of whose vapours are higher than that of air compared with "dry" heat transfer. The second inequality, however, requires additional verification. No experimental works on evaporation of liquids with  $c_{p1} < c_{p2}$  are known to the author.

To conclude the Section we shall discuss the works of A. V. Nesterenko [3-32], who demonstrated that experimental data of many works fall on one curve if the Gukhman number  $Gu$  is used.

From an analysis of numerous experimental data on heat and mass transfer during forced flow of moist gas, A. V. Nesterenko established the following relation

$$Nu = 2 + A Pr^{0.33} Re^n Gu^m \quad (3-2-73)$$

A similar relation for mass transfer is as follows

$$\text{Sh} = \text{Nu}_m = 2 + A'(\text{Sc})^{0.33} \text{Re}^{n'} \text{Gu}^{m'} \quad (3-2-74)$$

Constants  $A, n, m, A', n', m'$  are given in Table 3-6.

Table 3-6

Values of Constants Appearing in Nesterenko's Formula

Re number	$A$	$n$	$m$	$A'$	$n'$	$m'$
$1-2 \times 10^2$	1.07	0.48	0.175	0.83	0.53	0.135
$3.15 \times 10^3-2.2 \times 10^4$	0.51	0.61	0.175	0.49	0.61	0.135
$2.2 \times 10^4-3.15 \times 10^5$	0.027	0.90	0.175	0.0248	0.90	0.135

Upon making a rigorous analysis of heat and mass transfer as a single process it became necessary to define the physical properties of the medium in terms of  $\text{Pr}$  and  $\text{Sc}$  simultaneously or by their product. But an analysis of available experimental data reveals that the effect of  $\text{Sc}$  on heat transfer is small (the exponent of  $\text{Sc}$  in the relation for  $\text{Nu}$  is very small). The effect of  $\text{Pr}$  on mass transfer is also minor.

No strict physical justification is yet available for the introduction of  $\text{Gu}$  into the expressions for  $\text{Nu}$  and  $\text{Nu}_m$ . One of the hypotheses is that in the case of air flow over a liquid surface the finest liquid droplets enter the boundary layer\*.

Bulk evaporation of liquid particles occurs under adiabatic conditions, when the particle temperature is close to the temperature of adiabatic saturation of air  $t_M$ . The heat-transfer equation should, therefore, be supplemented with a term representing negative heat source equal to the product of specific evaporation heat  $r$  by vapour source  $\dot{I}_V(r\dot{I}_V)$ . The differential equation of mass diffusion should involve the mass source  $\dot{I}_V$ .

An approximate calculation demonstrates that the mass source  $\dot{I}_V$  is directly proportional to  $\text{Gu}$

$$\dot{I}_V \sim \text{Gu} = \frac{T_c - T_M}{T_c} \quad (3-2-75)$$

With radiative heat transfer the bulk evaporation will be intensified due to the absorption of infrared rays by liquid particles. Then the dimensionless relation  $\text{Nu} = f(\text{Re}, \text{Pr}, \text{Gu})$  should include an additional Lebedev parametric number  $\text{Lb}$  ( $\text{Lb} = T_r/T_c$ ), where  $T_r$  is the temperature of the radiator. Thus, the  $\text{Gu}$  number, according to the given hypothesis, characterizes bulk evaporation of fine liquid droplets in a boundary layer.

\* The mechanism of droplet entrainment into a boundary layer has been discussed earlier.

It should be pointed out that the Nusselt number increases as  $Gu$  is increased (a decrease of relative humidity). The heat-transfer rate in the presence of mass transfer is thus larger than the rate of pure heat transfer ("dry" heat transfer). This statement holds not only for a laminar, but also for a turbulent flow over a liquid surface.

Thus, in the drying and free evaporation processes the effect of transverse vapour-gas mixture flow (the Stefan flow) on heat transfer is small. Unlike gas injection into a boundary layer, an increase in the evaporation rate results in an increase in the Nusselt numbers. However, these conclusions cannot be extended to ice evaporation (sublimation). If during convective drying the injection parameter  $f_w$ , according to formulas (3-2-36), is less than 0.05, then during ice sublimation it will be much higher, since ice melting heat  $r$  is considerably less than the vaporization heat.

In [3-32] the problem of heat and mass transfer in a heated gas flow over solidified liquid has been solved. Initial differential equations (3-2-6)-(3-2-7) and the following boundary conditions were assumed

$$\text{at } y = 0 \quad v_x = 0; \quad v_y = v_w; \quad \lambda \partial T / \partial y = r \rho v_w \quad (3-2-76)$$

$$\text{at } y \rightarrow \infty \quad v = v_\infty; \quad T = T_\infty \quad (3-2-77)$$

The solution was obtained by iteration procedure. In Fig. 3-30 the velocity and temperature profiles for different values of the temperature driving force are presented.

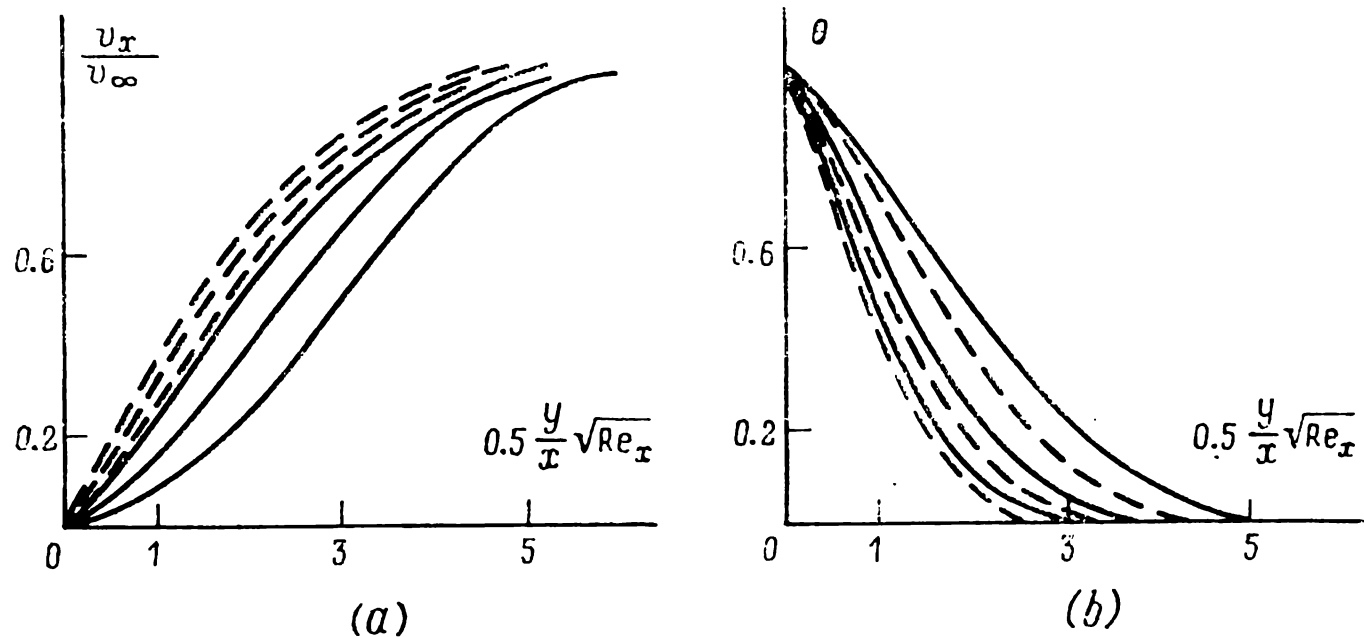


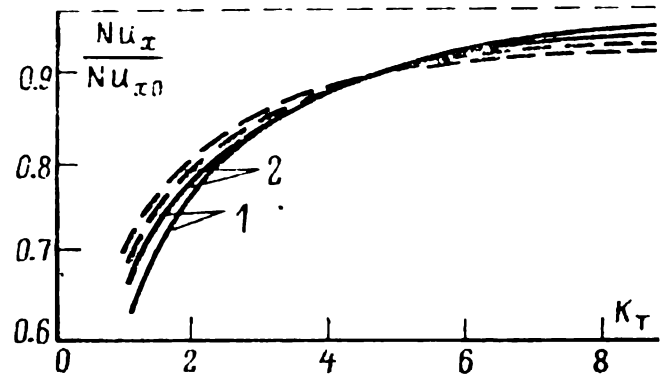
Fig. 3-30. Profiles of relative velocity (a) and temperature (b) in a boundary layer at a melting surface for various values of temperature drop

A graph of the local Nusselt number as a function of parameter  $K_T$  is presented in Fig. 3-31. Parameter  $K_T$  is inversely proportional to injection parameter  $f_w$

$$K_T = \frac{r}{c_p(T_\infty - T_w)} = A_x \text{Pr}^{-2/3} f_w^{-1} \quad (3-2-78)$$

From figure 3-31 it may be seen that with a decrease in parameter  $K_T$ , i.e. as injection parameter  $f_w$  increases, the Nusselt number decreases.

Fig. 3-31. Plot of relative Nusselt number versus Kutateladze number (solid line — calculated, dashed line — data)  
1 — for water; 2 — for ethylene glycol



During convective drying ( $r \approx 580$ ,  $\Delta T \approx 200^\circ\text{C}$ ,  $c_p = 0.25$ )  $K_T \approx 12^*$ . Then according to the figure the decrease of the Nusselt number does not exceed 2%. With ice sublimation ( $r = 80$ ,  $\Delta T = 50^\circ\text{C}$ ,  $c_p = 0.25$ ) parameter  $K_T = 6.4$ . In this case the decrease of the Nusselt number is about 10%.

### 3-3. HEAT AND MASS TRANSFER IN PIPE FLOWS AND IN FLOWS PAST COMPLEX GEOMETRIES

#### (a) Pipe Flows

In a laminar incompressible flow in a cylindrical tube, the velocity and temperature distributions follow parabolic law (see Fig. 3-32). The length of the portion of hydrodynamic stabilization  $X_v$  and of the thermal stabilization  $X_T$  are

$$X_v = 0.05 \text{ Re}_D D; \quad X_T = 0.05 \text{ Re}_D \text{ Pr } D \quad (3-3-1)$$

where pipe diameter  $D$  is taken as the characteristic dimension in the Reynolds number expression.

The local Nusselt number is maximum at the beginning of pipe length (at the fluid inlet), it then gradually decreases approaching a constant

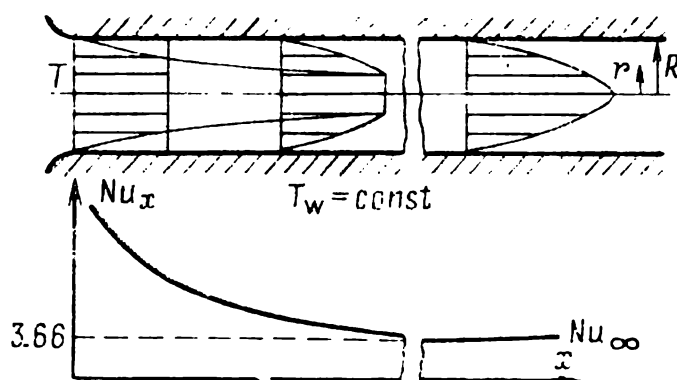


Fig. 3-32. Distribution of temperatures and velocities in the starting length of a tube with fluid being cooled

value  $Nu_\infty$  (see Fig. 3-32). At the first kind boundary conditions wall temperature is constant ( $T_w = \text{const}$ ),  $Nu_\infty = 3.66$ ; at the second kind

\* The specific heat capacity of humid air is  $(0.24-0.47) d$ , where  $d$  is the air moisture content.

boundary conditions wall heat flux is constant ( $q_w = \text{const}$ ),  $\text{Nu}_\infty = 4.36$ . If in the Nusselt number expression the pipe diameter is taken as the characteristic dimension ( $\text{Nu} = \frac{\alpha D}{\lambda_f}$ ), we may write the following formula for the local Nusselt number [3-13]

$$\text{Nu}_x = \text{Nu}_\infty + a \left[ \left( \frac{D}{x} \right) \text{Re}_D \text{Pr} \right] \left\{ 1 + b \left[ \frac{D}{x} \text{Re}_D \text{Pr} \right]^m \right\}^{-1} \quad (3-3-2)$$

Constants  $a$ ,  $b$  and  $m$  are presented in Table 3-7.

Table 3-7

Constants  $a$ ,  $b$  and  $m$  used in Equation (3-3-2)

Wall conditions	Velocity profile	Pr	$\text{Nu}_x$	$\text{Nu}_\infty$	$a$	$b$	$m$
$T_w = \text{const}$	parabolic	any	av.	3.66	0.0668	0.040	2/3
$T_w = \text{const}$	developed	0.7	av.	3.66	0.1040	0.016	0.8
$q_w = \text{const}$	parabolic	any	local	4.36	0.023	0.0012	1.0
$q_w = \text{const}$	developed	0.7	local	4.36	0.036	0.0011	1.0

Instead of equation (3-3-2), the formula suggested by S. Whitaker [3-34] may be used

$$\text{Nu}_x = 1.86 \text{Re}_D^{1/3} \text{Pr}^{1/3} \left( \frac{x}{D} \right)^{-1/3} \left( \frac{\eta_b}{\eta_w} \right)^{0.14} \quad (3-3-3)$$

where  $\text{Nu}_x > 3.66$  when  $\text{Re}_D \text{Pr} > 100$  and  $\eta_b$  is the coefficient of viscosity at the mean temperature.

For different duct geometries, Nusselt numbers  $\text{Nu}_\infty$  over the developed length may be taken from Table 3-8, compiled by B. K. Shah and A. L. London [3-33] on the basis of a great number of experimental data.

If  $x$  is taken as the flow direction, then the cross-sectional area of the pipe lies within the plane  $xy$ .

There are three types of boundary conditions; the first one is classical:

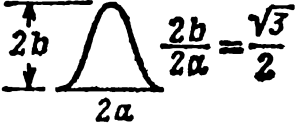
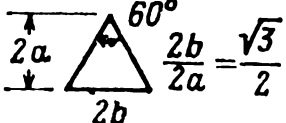
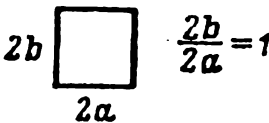

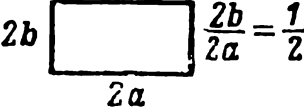

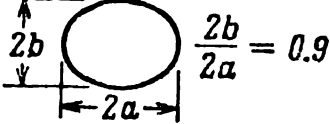
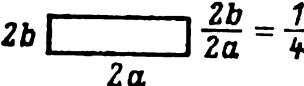
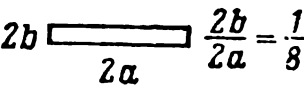
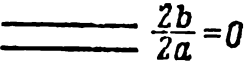

1.  $T_w = \text{const}$  irrespective of  $x, y, z$  ( $\text{Nu}_\infty \equiv \text{Nu}_I$ );
2.  $T_w = \text{const}$  irrespective of  $y, z$  ( $\text{Nu}_\infty \equiv \text{Nu}_{II}$ );

$$q'(x) = c_p v_M \frac{dT_m}{dx} = \text{const} \text{ irrespective of } x, \quad (3-3-4)$$



Table 3-8

 Nu<sub>∞</sub> for Tubes of Different Cross Sections (Three Types of Boundary Conditions)

Geometry $L/D_h > 100$	Nu <sub>I</sub>	Nu <sub>II</sub>	Nu <sub>III</sub>	$f Re$
 $\frac{2b}{2a} = \frac{\sqrt{3}}{2}$	2.39	3.014	1.474	12.630
 $\frac{2b}{2a} = \frac{\sqrt{3}}{2}$	2.47	3.111	1.892	13.333
 $\frac{2b}{2a} = 1$	2.976	3.608	3.091	14.227
	3.33	4.002	3.862	15.054
 $\frac{2b}{2a} = \frac{1}{2}$	3.391	4.123	3.017	15.548
	3.657	4.364	4.364	16.000
 $\frac{2b}{2a} = 0.9$	3.66	5.099	—	18.700
 $\frac{2b}{2a} = \frac{1}{4}$	4.439	5.331	2.930	18.233
 $\frac{2b}{2a} = \frac{1}{8}$	5.597	6.490	2.904	20.585
 $\frac{2b}{2a} = 0$	7.541	8.235	8.235	24.000
 $\frac{b}{a} = 0$	4.861	5.385	—	24.000

 Note:  $D_h$  is the hydraulic diameter.

where  $T_m$  is the mean temperature and  $v_M$  is the fluid mass velocity;

$$3. q_w = \text{const irrespective of } y, z \text{ (Nu}_\infty \equiv \text{Nu}_{III}); \quad (3-3-5)$$

$$q'(x) = c_p v_M \frac{dT_m}{dx} = \text{const, irrespective of } x.$$

For a turbulent flow through a cylindrical pipe, the following expression may be used for calculation of the Nusselt number

$$\text{Nu}_D = \frac{\bar{\alpha} D}{\lambda_f} = 0.023 \text{Pr}^{\frac{1}{3}} \text{Re}_D^{0.8} \text{ at } 10^4 < \text{Re}_D < 10^5 \quad (3-3-6)$$

over the Prandtl number range ( $0.5\text{Pr} < 100$ ),  $L/D > 60$ .

Some researchers are of the opinion that the exponent of  $\text{Pr}$  should necessarily be different: for a cooling process it should be taken equal to 0.3 instead of  $1/3$ , and for a heating process — 0.4.

For large  $\text{Pr}$  numbers the empirical expression

$$\text{Nu}_D = 0.023 \text{Pr}^{1/3} \text{Re}_D^{0.8} \left( \frac{\eta_\infty}{\eta_w} \right)^{0.14} \quad (3-3-7)$$

is often used for the following ranges of  $\text{Re}_D$  and  $\text{Pr}$ :

$$10^4 < \text{Re}_D < 10^5; \quad 0.7 < \text{Pr} < 16700 \text{ at } L/D > 60.$$

### (b) *Flows over Bodies*

Treatment of experimental data reported by different authors on flows over spheres and cylinders enabled S. Whitaker [3-34] to suggest the following approximate formulas:

for spheres

$$\text{Nu} - 2 = (0.4 \text{Re}^{0.5} + 0.06 \text{Re}^{2/3}) \text{Pr}^{0.4} \left( \frac{\eta_b}{\eta_w} \right)^{0.25} \quad (3-3-8)$$

$$\text{at } 3.5 < \text{Re} < 7.6 \times 10^4; \quad 0.7 < \text{Pr} < 380;$$

for cylinders

$$\text{Nu} = (0.4 \text{Re}^{0.5} + 0.06 \text{Re}^{2/3}) \text{Pr}^{0.4} \left( \frac{\eta_b}{\eta_w} \right)^{0.25} \quad (3-3-9)$$

$$\text{at } 1 < \text{Re} < 10^5; \quad 0.7 < \text{Pr} < 300.$$

It is clearly seen that expression (3-3-9) differs from equation (3-3-8) only by a factor of 2, since for a cylinder  $Nu \rightarrow 0$  when  $Re \rightarrow 0$ .

One of the first attempts to obtain an approximate expression for the Nusselt number in a flow over complicated geometries was made by O. Krischer and G. Loos [3-54]. Instead of ordinary characteristic dimensions, a universal characteristic dimension  $l'$  ( $l'$  is the flow length over a body) is used. For spheres and cylinders (in a cross-flow)  $l' = 1/2 \pi D$ , for a rhomb  $l' = a + b$ , where  $a$  and  $b$  are the rhomb sides, for a triangular prism  $l' = 3/2 l$ , where  $l$  is the prism side length, etc. The ordinary expression

$$Nu_{l'} = 0.662 Pr^{1/3} Re_{l'}^{0.5} \quad (3-3-10)$$

may then be used.

The diffusive Nusselt number or the Sherwood number may be calculated in terms of the thermal Nusselt number from the formula

$$Sh_{l'} = Nu_{l'}/N_p \quad (3-3-11)$$

where  $N_p$  is the correction factor equal to

$$N_p = (p_{10w} - p_{10\infty}) / \log_e \frac{1 - p_{10\infty}}{1 - p_{10w}} \quad (3-3-12)$$

where  $p_{10w}$  and  $p_{10\infty}$  are relative partial vapour pressures at the solid surface and in the air flow, respectively.

When a heated gas is blown through a layer of particles the Reynolds and Nusselt numbers are determined by

$$Re = \frac{v_M D_p}{\eta_f(1 - \Pi)}; \quad Nu = \frac{\alpha D_p}{\lambda_f} \left( \frac{\Pi}{1 - \Pi} \right) \quad (3-3-13)$$

where  $\Pi$  is the porosity,  $\eta_f$  is the coefficient of fluid viscosity,  $\lambda_f$  is the fluid thermal conductivity,  $v_M$  is the mass velocity,  $D_p$  is the reduced diameter of a particle equal to the sixfold volume  $V_s$  of the particles divided by the particle surface area  $S_s$  ( $D_p = 6 V_s / S_s$ ).

When air is blown through a pipe ( $Pr = 0.7$ ), the Nusselt number expression takes the form

$$Nu = (0.5 Re^{0.5} + 0.2 Re^{2/3}) Pr^{1/3} \left( \frac{\eta_b}{\eta_w} \right)^{0.14} \quad (3-3-14)$$

at  $22 < Re < 8 \times 10^3$ .

### 3-4. SIMULTANEOUS TURBULENT HEAT AND MASS TRANSFER

At present no strict mathematical solution is available for the transfer problem in a turbulent boundary layer, despite the fact that a great number

of studies have been made on the subject \*. The nature of wall turbulence is still not clear, and this does not enable one to provide a complete analytical description of molar transfer of momentum, energy and mass. The available calculation procedures are either based on very approximate and simplified models of the phenomenon which are essentially extensions of Prandtl and Karman's concepts of mixing length, laminar and turbulent sublayers, etc., or are based on integral relations between momentum, energy and diffusion involving empirical relations. The latter are called semi-empirical theories since experiments play an important part.

The interest in turbulent transfer problems is not surprising as most of the boundary layers which may be encountered in engineering practice and in nature (geophysical applications) are turbulent. Besides, a transverse mass flow from the surface destabilizes the boundary layer and may even change the flow pattern in it.

Patankar and Spalding [3-35] classify the available boundary layer theories as follows:

Physical basis	Mathematics
<i>1. Complete theories</i>	
Based on empirical data, expressions for local properties such as effective viscosity, effective Prandtl number $Pr$ , effective Schmidt number $Sc$	Calculation methods for partial differential equations of transfer of mass, energy, chemical and other quantities
<i>2. Integral theories</i>	
Empirical equations relating integral quantities such as skin friction factor, momentum thickness and displacement thickness	Calculation method for ordinary differential equations describing integral properties of boundary layers

Compared with the physical basis listed under item 1, the integral theories are based on less general equations and require more experimental data.

### *(a) Differential Equations*

Let us consider a flow over rounded bodies when heat and mass transfer takes place. In the general case chemical or phase conversions (mass and

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\* A brief survey of the turbulence theory is given in Chapter 1.

heat sources) may occur in the boundary layer. The boundary layer is assumed to be in thermodynamic equilibrium. The thermodynamic properties may then be expressed as follows:

The specific enthalpy of a gaseous mixture (fluid flow)

$$h = \sum_i \rho_{i0} h_i \quad (3-4-1)$$

where  $h_i$  is the specific enthalpy of the  $i$ -th species, equal to

$$h_i = \int_0^T c_{pi} dT + h_0 \quad (3-4-2)$$

Here  $h_0$  is the specific heat of chemical or phase conversions;  $\rho_{i0}$  is the relative concentration of the  $i$ -th species ( $\rho_{i0} = \rho_i/\rho$ );  $\rho$  is the mixture density

$$\rho = \sum_i \rho_i; \quad \sum_i \rho_{i0} = 1 \quad (3-4-3)$$

If  $h_0$  is independent of temperature, the specific heat capacity at constant pressure of the  $i$ -th species will be

$$c_{pi} = \frac{dh_i}{dT} \quad (3-4-4)$$

In accordance with the adopted coordinate system (Fig. 3-33), the continuity equation will take the form

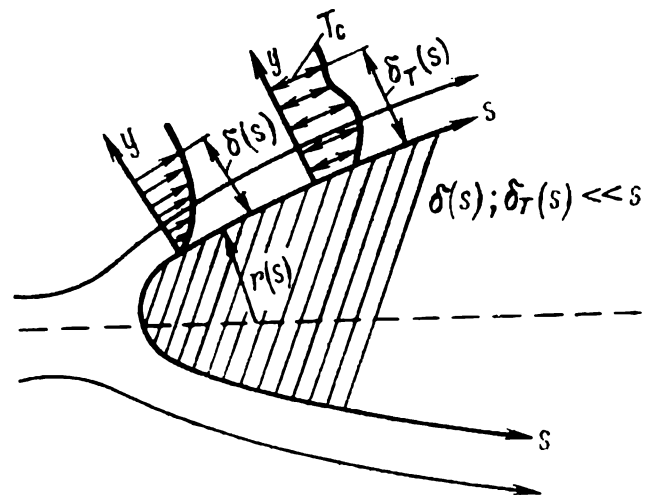


Fig. 3-33. Coordinate system of a boundary layer for a rounded body

$$\frac{\partial \rho v_x r^k}{\partial s} + \frac{\partial \rho v_y r^k}{\partial y} = 0 \quad (3-4-5)$$

where  $r(s)$  is the radius of curvature of the body; when  $k = 0$ , the body degenerates to a flat plate.

The momentum transfer equation of a boundary layer, in the case of a compressible liquid, will have the form

$$\rho v_s \frac{\partial v_s}{\partial s} + \rho v_y \frac{\partial v_s}{\partial y} = - \frac{\partial p}{\partial s} + \frac{\partial}{\partial y} \left( \eta \frac{\partial v_s}{\partial y} \right) \quad (3-4-6)$$

Here, it is assumed that  $\partial p / \partial y = 0$ ,  $p = p_c = \text{const.}$

The specific heat flux

$$q = - \lambda \vec{\nabla} t + \sum_i h_i j_i = - \lambda \vec{\nabla} t - \sum_i h_i D \rho \vec{\nabla} \rho_{i0} \quad (3-4-7)$$

where  $j_i$  is the mass flow of the  $i$ -th species.

Using the relations

$$dh = \sum_i \rho_{i0} dh_i + \sum_i h_i d\rho_{i0}; \quad c_p = \sum_i \rho_{i0} c_{pi} \quad (3-4-8)$$

and equation (3-4-4) we arrive at

$$q = - a \rho \vec{\nabla} h + D \rho \left( \frac{1}{\text{Le}} - 1 \right) \sum_i h_i \vec{\nabla} \rho_{i0} \quad (3-4-9)$$

where  $\text{Le}$  is the Lewis number ( $\text{Le} = D/a$ ).

If  $H$  is used to denote the stagnation enthalpy

$$H = h + \frac{1}{2} v^2 \quad (3-4-10)$$

the energy transfer equation becomes

$$\begin{aligned} \rho v_s \frac{\partial H}{\partial s} + \rho v_y \frac{\partial H}{\partial y} = \frac{\partial}{\partial y} \left[ a \rho \frac{\partial H}{\partial y} + \eta \left( 1 - \frac{1}{\text{Pr}} \right) \frac{1}{2} \frac{\partial v_s^2}{\partial y} \right] \\ - \frac{\partial}{\partial y} \left[ \left( \frac{1}{\text{Le}} - 1 \right) \rho D \sum_i h_i \frac{\partial \rho_{i0}}{\partial y} \right] \end{aligned} \quad (3-4-11)$$

Let us transform equation (3-4-11). Use of (3-4-6) and (3-4-10) gives

$$\left. \begin{aligned} \rho v_s \frac{\partial H}{\partial s} + \rho v_y \frac{\partial H}{\partial y} &= \rho v_s \frac{\partial h}{\partial s} + \rho v_y \frac{\partial h}{\partial y} + v_s \left( \rho v_s \frac{\partial v_s}{\partial s} + \rho v_y \frac{\partial v_s}{\partial y} \right) \\ \rho v_s \frac{\partial H}{\partial s} + \rho v_y \frac{\partial H}{\partial y} &= \rho v_s \frac{\partial h}{\partial s} + \rho v_y \frac{\partial h}{\partial y} + v_s \frac{\partial}{\partial y} \left( \eta \frac{\partial v_s}{\partial y} \right) - v_s \frac{\partial p}{\partial s} \end{aligned} \right\} \quad (3-4-12)$$

$$v_s \frac{\partial}{\partial y} \left( \eta \frac{\partial v_s}{\partial y} \right) = \frac{\partial}{\partial y} \left( \eta \frac{1}{2} \frac{\partial v_s^2}{\partial y} \right) - \eta \left( \frac{\partial v_s}{\partial y} \right)^2 \quad (3-4-13)$$

From equation (3-4-10) we get

$$\frac{\eta}{\text{Pr}} \frac{\partial h}{\partial y} = \frac{\eta}{\text{Pr}} \sum_i h_i \frac{\partial \rho_{i0}}{\partial y} + \frac{c_p \eta}{\text{Pr}} \frac{\partial T}{\partial y} \quad (3-4-14)$$

Hence, equation (3-4-11) takes the form

$$\rho v_s \frac{\partial h}{\partial s} + \rho v_y \frac{\partial h}{\partial y} = v_s \frac{\partial p}{\partial s} + \frac{\partial}{\partial y} \left( \lambda \frac{\partial T}{\partial y} + \rho D \sum_i h_i \frac{\partial \rho_{i0}}{\partial y} \right) + \eta \left( \frac{\partial v_s}{\partial y} \right)^2 \quad (3-4-15)$$

The differential mass transfer equation then takes the form

$$\rho v_s \frac{\partial \rho_{i0}}{\partial s} + \rho v_y \frac{\partial \rho_{i0}}{\partial y} = \frac{\partial}{\partial y} \left( \rho D \frac{\partial \rho_{i0}}{\partial y} \right) + \dot{I}_v \quad (3-4-16)$$

The problem is usually solved with the following boundary conditions

$$\left. \begin{array}{l} \text{at } y = 0 \quad \rho = \rho_w; \quad T = T_w; \quad h = h_w; \quad H = H_w; \\ \quad \quad \quad v_s = 0; \quad \rho_{i0} = (\rho_{i0})_w \\ \text{at } y \rightarrow \infty \quad \rho \rightarrow \rho_c; \quad T \rightarrow T_c; \quad h \rightarrow h_c; \quad H \rightarrow H_c; \\ \quad \quad \quad v_s \rightarrow v_a; \quad v_y \rightarrow 0; \quad \rho_{i0} \rightarrow (\rho_{i0})_c \end{array} \right\} \quad (3-4-17)$$

Transfer properties  $[\eta(\rho_{i0}, T), \lambda(\rho_{i0}, T), D(T)]$  and pressure  $p(s)$  are generally variables. The system of equations (3-4-6), (3-4-15) and (3-4-16) may be solved numerically on a computer.

When the Soret and Dufuor effects cannot be neglected, the energy and mass transfer equations are

$$\rho v_s \frac{\partial h}{\partial s} + \rho v_y \frac{\partial h}{\partial y} = v_s \frac{\partial p}{\partial s} + \frac{\partial}{\partial y} \left[ \lambda \frac{\partial T}{\partial y} + \rho D \sum_i (h_i + Q_i^*) \frac{\partial \rho_{i0}}{\partial y} \right] + \eta \left( \frac{\partial v_s}{\partial y} \right)^2 \quad (3-4-18)$$

$$\rho v_s \frac{\partial \rho_{i0}}{\partial s} + \rho v_y \frac{\partial \rho_{i0}}{\partial y} = \frac{\partial}{\partial y} \left[ \rho D \left( \frac{\partial \rho_{i0}}{\partial y} + \frac{K_T}{T} \frac{\partial T}{\partial y} \right) \right] + \dot{I}_v \quad (3-4-19)$$

where  $Q^*$  is the isothermal heat of transfer and  $K_T$  is the relative thermal diffusion factor.

Factor  $K_T$  depends on the following: (a) mass and diameter ratios of the two types of molecules ( $K_T$  increases with the ratios); (b) the nature of intermolecular forces; (c) relative concentration of species  $\rho_{i0}$ . For a binary mixture  $K_T$  depends on the thermodiffusion constant  $\alpha^*$  and concentrations  $\rho_{10}$  and  $\rho_{20}$

$$K_T = \alpha^* \rho_{10} \rho_{20} \quad (3-4-20)$$

$\alpha^*$  and  $K_T$  are determined experimentally.

The diffusive mass flow density is

$$j_1 = -\rho D_{12} \left( \frac{\partial \rho_{10}}{\partial y} + \frac{\alpha^* \rho_{10}(1 - \rho_{10})}{T} \frac{\partial T}{\partial y} \right) \quad (3-4-21)$$

The first term in equation (3-4-21) determines the mass flow due to concentrational diffusion, the second term — the flow caused by the temperature gradient effect. The sign of the second term depends on the sign of  $\partial T/\partial y$  and  $\alpha^*$ .

If species 1 is lighter, then  $\alpha^* < 0$ ; if it is heavier, then  $\alpha^* > 0$ . The quantity  $\alpha^*$  decreases with the amount of the heavier species.

In the case of a turbulent boundary layer, diffusivity  $D$ , viscosity  $\eta$  and thermal conductivity  $\lambda$  should be considered as the sum of molecular and turbulent transfer coefficients.

The integral theories of a turbulent boundary layer employ the following equations [3-41], [3-42]

$$\frac{d \text{Re}'}{dx} = \text{Re}_l(1 + h)f = \text{Re}_l \left( \frac{1}{2} c_f + j_w^* \right) \quad (3-4-22)$$

$$\frac{d \text{Re}'_h}{dx} + \text{Re}_l f_h = \text{Re}_l (\text{St} + j_w^*) \quad (3-4-23)$$

$$\frac{d \text{Re}'_D}{dx} + \text{Re}_l f_D = \text{Re}_l (\text{St}_D + j_w^*) \quad (3-4-24)$$

where  $\text{Re}'$ ,  $\text{Re}'_h$ ,  $\text{Re}'_D$  are the Reynolds numbers calculated for momentum displacement thickness, energy displacement thickness and mass displacement thickness;  $c_f$  is the friction factor;  $f, f_h, f_D$  are the form-parameters

$$f = \frac{\delta'}{v_\infty} \frac{dv_\infty}{dx}; \quad f_h = \frac{\delta'_h}{\Delta H} \frac{d(\Delta H)}{dx}, \quad f_D = \frac{\delta'_D d(\Delta \rho_k)}{\Delta_k dx} \quad (3-4-25)$$

and  $j_w^* = j_w / \rho_\infty v_\infty$  is the injection parameter.

### (b) Non-Condensable Gas Injection

V. P. Motulevich [3-36] considered the problem of heat transfer and friction for a plate in a gas flow, when a turbulent boundary layer is formed upon injecting an alien gas through the pores. A set of equations for a binary turbulent boundary layer was solved by Motulevich (see Chapter 1);  $\eta, \lambda, D$  were assumed to be the sum of molecular and molar coefficients. The equation of state was written as

$$\frac{\rho}{\rho_\infty} = \frac{H_\infty}{H} \left[ \frac{1 - \rho_{10}(1 - c_{p2}/c_{p1})}{1 - \rho_{10}(1 - m_1/m_2)} \right] \quad (3-4-26)$$



where  $H$  is the stagnation enthalpy and  $m$  is the molecular mass.

The following boundary conditions were adopted:

$$\left. \begin{array}{l} \text{at } y = 0 \quad v_x = 0; \quad v_y = v_w(x); \quad T_w = \text{const}; \quad \rho_{10w} = \text{const} \\ \text{at } y = \delta \quad v_x = v_\infty; \quad T = T_\infty; \quad \rho_{10\infty} = 0 \end{array} \right\} \quad (3-4-27)$$

Moreover, the following simplifying assumptions were used:

1. The boundary layer is turbulent from the very beginning of the plate;
2. The heat capacity of each species is independent of temperature;
3. The partial density of each species is defined by the Clapeyron equation;

4. Total (molecular and molar) Pr and Sc numbers are equal to 1. The latter assumption implies similarity of velocity fields, stagnation enthalpy, and concentration of the injectant

$$\frac{v_x}{v_\infty} = \frac{H_0 - H_{0w}}{H_{0\infty} - H_{0w}} = \frac{\rho_{10} - \rho_{10w}}{\rho_{10w}} \quad (3-4-28)$$

The relation between temperature and velocity distributions in the boundary layer follows directly from equation (3-4-28)

$$T^* = \frac{1}{1 - \rho_{10w}(1 - v_x^*)(1 - c_p^*)} \left\{ [1 - \rho_{10w}(1 - c_p^*)] T_w^* \right. \quad (3-4-29)$$

$$\left. + \left\{ 1 - [1 - \rho_{10w}(1 - c_p^*)] T_w^* + \frac{k-1}{2} M_\infty^2 \right\} v_x^* - \frac{k-1}{2} M_\infty^2 v_x^{*2} \right\}$$

Here

$$T_w^* = \frac{T}{T_\infty}; \quad v_x^* = \frac{v_x}{v_\infty}; \quad c_p^* = \frac{c_{p1}}{c_{p2}}$$

and  $M$  is the Mach number.

The relation between the distributions of density and velocity in the boundary layer may be obtained in a similar manner.

If (3-4-29) is differentiated with respect to  $y$  and  $(\partial T / \partial y)_w = 0$  is assumed, we obtain an expression for the equilibrium temperature of the wall

$$T_e = T_\infty \left( 1 + \frac{k-1}{2} M_\infty^2 \right) \quad (3-4-30)$$

Equation (3-4-30) shows that the equilibrium temperature is affected neither by the type of the injectant nor by the injection itself. It follows from relations (3-4-29) and (3-4-30) that

$$\text{Nu} = \frac{1}{2} C_f \text{Re} \quad (3-4-31)$$

i.e. the injection of an alien gas produces the same effect on heat transfer and friction. Therefore, for solving the problem in this formulation it is necessary and sufficient to study the dynamic properties of the boundary layer.

Close to the wall the equality

$$\sigma_w = \sigma - \rho_w v_w v_x \quad (3-4-32)$$

approximately holds.

Further analysis is based on Prandtl's concept of mixing length  $l$  and its relation with fluctuation velocities and on the Van Driest relation [3-38] for shear stress in a compressible turbulent flow.

Equation (3-4-32) may be reduced to

$$\sigma_w = \rho l^2 \left( \frac{\partial v_x}{\partial y} \right)^2 - \rho_w \bar{v}_w v_x + \eta \frac{\partial v_x}{\partial y} \quad (3-4-33)$$

Molecular friction is generally neglected. Proceeding from the relation between temperature, concentration and velocity expressed by (3-4-28) through (3-4-33), a similar relation for density may be obtained, which is used for integration of equation (3-4-33). The velocity distribution in the boundary layer is described by

$$\begin{aligned} \frac{\log_e y}{x} = & \int \left\{ T_w^* \frac{C_f}{2} [1 - \rho_{10w}(1 - c_p^*)] \left( 1 + 2v_x^* \frac{\omega^*}{c_p^*} \right) \left( 1 + 2v_x^* \frac{\omega^*}{c_p^*} \right) \right. \\ & \times \left. \frac{1 - \rho_{10w}(1 - v_x^*) \left( 1 - \frac{m_2}{m_1} \right)}{1 - \rho_{10w}(1 - v_x^*)(1 - c_p^*)} - (1 - Bv_x^* - A^2 v_x^{*2}) \right\}^{-\frac{1}{2}} dv_x + \text{const} \end{aligned} \quad (3-4-34)$$

The constant in equation (3-4-34) is determined from the condition which defines the Karman logarithmic velocity profile at  $\rho_w v_w = \rho_{10w} = M_\infty = 0$  and  $T_w^* = 1$ .

Dorrans and Dor [3-39] demonstrated that at moderate mass velocities

$$-\frac{1}{\kappa} \log_e \left[ \frac{\rho_w v_\infty}{\eta_w} \sqrt{\frac{\rho_\infty}{\rho_w} \frac{C_f}{2}} - D_1 \right] = \text{const} \quad (3-4-35)$$

where  $D_1$  is a constant.

Substitution of (3-4-35) into (3-4-34) yields the final expression for velocity profile in a boundary layer

$$\frac{\rho_w v_\infty}{\eta_w} \sqrt{\frac{\rho_\infty}{\rho_w} \frac{C_f}{2}} y = \exp [I_1(v_x^*) - D_1(\kappa)]$$

where

$$I_1(v_x^*) = \int_0^{v_x^*} E \left[ \left( 1 + 2v_x^* \frac{\omega^*}{C_f} \right) \frac{1 - \rho_{10w}(1 - v_x^*) \left( 1 - \frac{m_2}{m_1} \right)}{1 - \rho_{10w}(1 - v_x^*)(1 - c_p^*)} \right. \\ \left. \times (1 + Bv_x^* - A^2 v_x^{*2}) \right]^{-\frac{1}{2}} dv_x^* \quad (3-4-36)$$

$$A = \frac{(k-1) M_\infty^2}{2T_w^*[1 - \rho_{10w}(1 - c_p^*)]}$$

$$B = \{T_w^*[1 - \rho_{10w}(1 - c_p^*)]\}^{-1} - 1 + A^2$$

$$E = \kappa \left\{ T_w^* \frac{C_f}{2} [1 - \rho_{10w}(1 - c_p^*)] \right\}^{1/2}$$

$$\omega^* = \frac{\rho_w v_w}{\rho_\infty v_\infty}$$

Constants  $D_1$  and  $\kappa$  are found from equation

$$\frac{d\delta_{m,d}}{dx} = \frac{C_f}{2} + \omega^* \quad (3-4-37)$$

which is solved for the external boundary layer. The numerical values of  $\kappa$  and  $D_1$  proved to be  $\kappa \approx 0.4$  and  $D_1 \approx 6.38$ . The final relations used for the calculation of skin friction and the boundary layer thickness are as follows:

the skin friction equation

$$(C_f + 2\omega^*) \text{Re}_\infty = 0.388 \frac{\eta_w}{\eta_\infty} I_2 \quad (3-4-38)$$

the boundary-layer thickness equation

$$\frac{\delta}{x} = 0.194 [1 - \rho_{10w}(1 - c_p^*)] T_w^* \frac{\eta_w}{\eta_\infty} \frac{E}{\text{Re}_\infty} \exp I_1(1) \sqrt{\frac{1 - \rho_{10w} \left( 1 - \frac{m_1}{m_2} \right)}{1 - \rho_{10w}(1 - c_p^*)}}$$

where

$$I_2 = E^2 \sqrt{\frac{1 - \rho_{10w} \left(1 - \frac{m_2}{m_1}\right)}{1 - \rho_{10w}(1 - c_p^*)}} \int \left[ \frac{1 - \rho_{10w}(1 - v_x^*)(1 - c_p^*)}{1 - \rho_{10w}(1 - v_x^*) \left(1 - \frac{m_2}{m_1}\right)} \right]^{3/2} \\ \times \frac{v_x^*(1 - v_x^*) \exp I_1(v_x^*)}{\sqrt{\left(1 + 2v_x^* \frac{\omega_w^*}{C_f}\right) (1 + Bv_x^* - A^2 v_x^{*2})^3}} dv_x^* \quad (3-4-39)$$

Since it was assumed that  $Pr = Sc = 1$ , the obtained results may be used for the calculation of heat and mass transfer rates.

Yu. V. Lapin [3-40] solved a set of transfer equations with the same assumptions as in [3-36], but he used Karman's hypothesis for calculating the properties of the turbulent boundary layer

$$\sigma_{xy} = \rho \kappa^2 \left( \frac{dv_x^*}{dy} \right)^4 \frac{1}{\left( \frac{d^2 v_x^*}{dy^2} \right)^2} \quad (3-4-40)$$

Moreover it is assumed that  $\kappa$  is constant and it remains universal and equal to 0.40 when a substance is injected.

For the calculation of friction factor  $C_f$  for a flat porous plate, through which substance is injected, a relation is obtained

$$\frac{C_f}{C_{f0}} = \left( \frac{FK}{2N} \right)^2 \quad (3-4-41)$$

in which  $C_{f0}$  is the friction factor of the plate in a flow of isothermal incompressible fluid when there is no cross mass flow

$$C_{f0} = 0.0263 Re^{-1/7}; \quad F = 0.25 C_{f0}^{-1/2} = 1.55 Re^{1/14} \quad (3-4-42)$$

$$K = \int_0^1 \frac{\rho/\rho_\infty}{1 + Bv_x^*} dv_x^*; \quad v_x^* = \frac{v_x}{v_\infty} \quad (3-4-43)$$

$$B = \frac{C_q}{C_f} = \frac{\rho_w v_w v_\infty}{\sigma_w}; \quad C_q = \frac{2\rho_w v_w}{\rho_\infty v_\infty} \quad (3-4-44)$$

$$C_f = \frac{2\sigma_w}{\rho_\infty v_\infty^2} \quad (3-4-45)$$

$N$  in equation (3-4-41) is found from the solution of the transcendental equation:

$$N + \log_e N = \log_e \frac{FK}{2} + \frac{F+G}{2} \quad (3-4-46)$$

where

$$G = \log_e \frac{\eta_\infty}{\eta_w}$$

Expression (3-4-41) for friction factor is general in the sense that it is independent of a specific form of the density versus velocity in the boundary layer function and viscosity formula. It may, therefore, be used for diverse assumptions on the nature of these relations. From (3-4-41) it follows that the consumption coefficient  $C_q$  has a certain limit value at which skin friction at the wall equals zero. If  $Re \rightarrow \infty$  or  $F \rightarrow \infty$ , then

$$\frac{C_f}{C_{f0}} = K^2 \quad (3-4-47)$$

or

$$\frac{C_f}{C_{f0}} = \left[ \int_0^1 \sqrt{\frac{(C_f/C_{f0})(\rho/\rho_\infty)}{(C_f/C_{f0}) + (C_q/C_{f0})v_x^*}} dv_x^* \right]^2 \quad (3-4-48)$$

Assuming  $C_f/C_{f0} = 0$ , we get the limit value of the consumption coefficient\*

$$C_{q \text{ lim}} = C_{f0} \left[ \int_0^1 \sqrt{\frac{\rho/\rho_\infty}{v_x^*}} dv_x^* \right]^2 \quad (3-4-49)$$

When a gas having the same viscosity as that of the main flow is injected ( $\rho = \rho_\infty = \text{const}$ ), then  $C_{q \text{ lim}} = 4C_{f0}$ .

The relation between density and velocity in the boundary layer  $\rho/\rho_\infty = f(v_x^*)$ , necessary for calculation, may be different for different assumptions.

If

$$Pr_c = Sc_c = 1; \quad c_{pi} = \text{const}; \quad T_w(x) = \text{const}$$

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\* Formulas (3-4-47)-(3-4-49) were first obtained in 1962 by S. S. Kutateladze and A. I. Leontiev.

then

$$\frac{\rho}{\rho_{\infty}} = \frac{\left\{1 + B \left[1 - \left(1 - \frac{c_{p1}}{c_{p2}}\right)(1 - v_x^*)\right]\right\}(1 - \omega - \gamma)}{\left\{1 + B \left[1 - \left(1 - \frac{m_2}{m_1}\right)(1 - v_x^*)\right]\right\} \times \left\{1 - \omega v_x^* - \gamma v_x^{*2} - \frac{B}{1 + B} \left(1 - \frac{c_{p2}}{c_{p1}}\right)(1 - v_x^*)\right\}} \quad (3-4-50)$$

where

$$\omega = 1 - \left(1 - \frac{k-1}{2} M_{\infty}^2\right) \frac{T}{T_w}; \quad \gamma = \frac{k-1}{2} M_{\infty}^2 \frac{T_{\infty}}{T_w}$$

With  $C_f$  found, we may find the heat-transfer rate, using the Reynolds analogy, from the relation  $St = C_f/2$ .

Actually in the boundary layers of gaseous mixtures,  $Pr$  and  $Sc$  are not only different from unity but they will not be constant across the boundary layer. The deviations of  $Pr$  and  $Sc$  from unity are significant when highly effective coolants such as  $H_2$  and  $He$  are injected. For example, for  $H_2$ -air mixture at  $T = 273^\circ K$  and  $p = 1 \text{ kgf/cm}^2$ ,  $Sc$  varies from 0.2 to 1.7 for various injection rates. Similar variation of  $Sc$  is observed when  $He$  is injected into air.

Since  $Pr$  and  $Sc$  are not constant across the laminar sublayer and differ from unity, the Reynolds analogy does not hold and the analysis is considerably complicated. In [3-40] the relation between density and velocity in a boundary layer is presented for  $Pr \neq Sc \neq 1$

$$\frac{\rho}{\rho_{\infty}} = \frac{\left\{1 + B \left[1 - \left(1 - \frac{c_{p1}}{c_{p2}}\right)(1 - v_x^*)\right]\right\}(1 - \omega - \gamma)}{\left\{1 + B \left[1 - \left(1 - \frac{m_2}{m_1}\right)(1 - v_x^*)\right]\right\} \times \left\{1 + \Omega(1 - v_x^*) - \gamma v_x^{*2} - \omega - \frac{B}{1 + B} \left(1 - \frac{c_{p2}}{c_{p1}}\right)(1 - v_x^*)\right\}} \quad (3-4-51)$$

where

$$\Omega = \frac{2St}{C_f} \left(1 - \frac{H_t}{H_w}\right)$$

where  $H_t$  is the equilibrium gas enthalpy at the wall in the absence of

heat transfer between the gas and the wall equal to

$$\frac{H_t}{H_\infty} = \frac{1 - (1 - \text{Pr})r(v_x^*)_{vt}}{1 - \text{Sc}B^2(\text{Sc} - \text{Pr})\left(\frac{c_{p1}}{c_{p2}} - 1\right)(1 - \rho_{10w})\frac{c_{p1}(v_x^*)_v}{c_{p2}2\beta^2}} \quad (3-4-52)$$

where  $(v_x^*)_{vt}$  is the dimensionless velocity at the laminar sublayer boundary

$$(v_x^*)_{vt} = \frac{1}{\sqrt{\text{Pr}}} \frac{\text{Re}}{h} + \frac{\beta}{2\text{Pr}} \left(\frac{\text{Re}}{h}\right)^2 \quad (3-4-53)$$

Here,  $\beta = 1 + \rho_{10w}(c_p^* - 1)$  and  $\text{Re}$  is the Reynolds number defined in terms of laminar sublayer thickness  $\delta_v$  and dynamic velocity

$$\text{Re} = \frac{\delta_v v_*}{v_w} = \frac{\delta_v v_0}{v_w} = \frac{\delta_v (\delta_w / \rho_w)^{1/2}}{v_w} \quad (3-4-54)$$

$\text{Re}$  is assumed equal to the Reynolds number for incompressible gas without injection  $\text{Re}_0 = 11.5$ ; a constant dynamic viscosity  $\eta_w$  in a laminar sublayer is assumed. The other quantities are

$$h = v_\infty \sqrt{\frac{\rho_w}{\sigma_w}}; \quad r = \frac{k-1}{2} M_\infty^2 \left(1 + \frac{k-1}{2} M_\infty^2\right)^{-1}; \quad (3-4-55)$$

$$\rho_{10w} = 1 - (1 + B)^{-1}[1 + B(v_x^*)_D]^{(1-\text{Sc})}$$

In the last expression  $(v_x^*)_D$  is the dimensionless velocity at the diffusive laminar sublayer boundary.

Besides an expression is obtained relating friction and heat transfer (a parameter of the Reynolds analogy)

$$\frac{\text{St}}{C_f/2} = \frac{1 - \text{Sc}B^2(\text{Sc} - \text{Pr})\left(\frac{c_{p1}}{c_{p2}} - 1\right)(1 - \rho_{10w})\frac{(v_x^*)_{vt}c_{p1}}{2\beta^2c_{p2}}}{1 - (1 - \text{Pr})(v_x^*)_{vt} - B\text{Pr}\left[B(1 - \text{Pr}) + (\text{Sc} - \text{Pr})\left(\frac{c_{p2}}{c_{p1}} - 1\right)(1 - \rho_{10w})\frac{v_{xt}^{*2}}{2\beta}\right]} \quad (3-4-56)$$

The calculations are made by the successive approximation method. An analysis reveals the following:

1. The difference of  $Pr$  and  $Sc$  from 1 in a laminar sublayer has a minor effect on the friction factor. Therefore,  $C_f$  may be calculated using the techniques based on the assumption  $Pr = Sc = 1$ .

2. When calculating heat transfer between the gas and the wall and the recovery coefficient

$$K = \frac{H_t - c_{p\infty}T_\infty}{H_\infty - c_{p\infty}T_\infty} \approx \frac{H_t}{H_\infty} \quad (3-4-57)$$

it is necessary to take into account the difference of  $Pr$  and  $Sc$  from unity.

3. Injection into a boundary layer decreases  $St$  ( $C_f/2$ ).

4. The enthalpy recovery coefficient  $K \approx H_t/H_\infty$  increases with the injection rate and may be greater than unity\*.

Integral equations of momentum and energy for a boundary layer may be solved if the laws of resistance and heat transfer in a turbulent boundary layer are known.

In the general case  $f_1$  and  $f_2$  depend on temperature factors and dimensionless parameters which account for the effects of longitudinal pressure gradient, transverse mass flow at the heat transfer surface, degree of dissociation of the main flow, etc.

S.S. Kutateladze and A.I. Leontiev [3-41], [3-42] developed a theory for calculating turbulent boundary layer properties based on the so-called limiting resistance and heat-transfer laws for a turbulent boundary layer, which are involved in processes with an infinitesimal viscosity, i.e. when  $Re \rightarrow \infty$ . For an isothermal flow over a permeable plate, the limiting resistance law is of the form

$$\int_0^1 \frac{d\omega}{\sqrt{(\psi + b\omega) \frac{\rho}{\rho_\infty}}} = 1 \quad (3-4-58)$$

and the critical injection parameter, corresponding to separation of the boundary layer ( $\psi = 0$ ), equals

$$b_{cr} = \left( \int_0^1 \frac{d\omega}{\sqrt{(\rho_\infty/\rho)\omega}} \right)^2 \quad (3-4-59)$$

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\* This conclusion is not consistent with the available experimental data (*Editor's remark*).



It is assumed that  $\Psi = \Psi_T$ . In relations (3-4-58) and (3-4-59) the notations used by Kutateladze and Leontiev [3-41] have been employed:

$\omega = \frac{v}{v_\infty}$  is the dimensionless velocity;

$b = \frac{2\rho_w v_w}{C_{f0}\rho_\infty v_\infty}$  is the wall permeability factor;

$\Psi = \left(\frac{C_f}{C_{f0}}\right)_{Re^{**}}$  is the friction factor ratio at  $Re^{**} = idem$ ;

$\Psi_T = \left(\frac{St}{St_0}\right)_{Re_T^{**}}$  is the Stanton number ratio at  $Re_T^{**} = idem$ .

Function  $\Psi$  in expression (3-4-58) may be calculated from

$$\Psi = \left\{ \int_0^1 \frac{V \frac{\psi}{\psi_1} + \left(1 - \frac{\psi}{\psi_1}\right) \omega}{V[\psi + (1 - \psi)\omega](1 + b_1\omega)} d\omega \right\}^{1/2} \quad (3-4-60)$$

and the critical injection parameter is found as

$$b_{cr} = \left\{ \int \frac{V \frac{R_\infty}{R_1} + \left(1 - \frac{R_\infty}{R_1}\right) \omega}{V[\psi + (1 - \psi)\omega] \omega} d\omega \right\}^2 \quad (3-4-61)$$

In expressions (3-4-60) and (3-4-61) the following notations have been used:  $\psi = T_\infty/T_w$  is the temperature factor;  $R$  is the gas constant;  $b_1 = 2\rho_{1w}v_w/C_f\rho_\infty v_\infty$  is the ratio of the wall permeability factor to the actual friction factor  $\psi_1 = \rho_\infty/\rho_w$ .

In the derivation of (3-4-61) it was assumed that at the separation point  $\rho_w = \rho_{1w}$  and  $\psi_1 = \psi \frac{R_1}{R_\infty}$ .

Formula (3-4-61) may be approximated by simple relations:

(a) when gases heavier than the main gas are injected ( $m_1 > m_2$ )

$$b_{cr} \approx b_{cr1} \left( 0.37 + 0.67 \frac{m_1}{m_2} \right) \quad (3-4-62)$$

(b) when  $m_1 < m_2$

$$b_{cr} \approx b_{cr1} \left( 0.25 + 0.75 \frac{m_1}{m_2} \right) \quad (3-4-62a)$$

where  $b_{cr1}$  corresponds to the critical injection parameter for a homogeneous nonisothermal boundary layer and may be calculated from the following expressions:

1. When  $\psi < 1$

$$b_{cr1} = \frac{1}{1 - \psi} \left( \log_e \frac{1 + \sqrt{1 - \psi}}{1 - \sqrt{1 - \psi}} \right)^2 \quad (3-4-63)$$

2. when  $\psi > 1$

$$b_{cr} = \frac{1}{\psi - 1} \left( \cos^{-1} \frac{2 - \psi}{\psi} \right)^2 \quad (3-4-64)$$

For very light gases

$$b_{cr} \left| \frac{m_1}{m_2} \rightarrow 0 \right. = \left( \frac{2}{\sqrt{\psi} + 1} \right)^2 \quad (3-4-65)$$

Equation (3-4-60) may be solved by numerical methods, With  $\psi$  known, we may solve the momentum equation, and knowing  $\psi_T$  the energy equation may be solved.

Patankar and Spalding [3-35] suggested a turbulent boundary layer calculation method based on integration of equations for a one-dimensional boundary layer in the immediate vicinity of the wall\*. This resulted in expressions relating momentum, mass and enthalpy fluxes at the wall with velocity, concentration and temperature at the external one-dimensional boundary layer, which may be used for a wide range of problems.

There are many other methods for solving turbulent transfer problems. The difference between  $Nu$  and  $C_f$ , predicted by different theories, amounts to 100% and more.

In Fig. 3-34 a curve representing  $St/St_0$  as a function of  $b$ , calculated by different methods, is plotted for various coolants.

It is evident that at moderate injection rates the correlation between predicted and experimental values is much better than at larger injection rates. Besides, the deviations from the predicted values are considerable when light gases are injected and when evaporation takes place (as a rule, all the solutions of equations for boundary conditions  $T_w = \text{const}$  and  $\rho_{10w} = \text{const}$  are extended to phase conversions at the external boundary layer of the body).

The discrepancy between the calculated and experimental values, when light gases are injected, may be attributed to secondary effects (thermodiffusion and diffusive thermal conductivity). A more detailed analysis of these effects is presented in the following section.

The discrepancy between the calculated and measured skin-friction factors and heat-transfer coefficients in the range of high injection rates

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\* This method is essentially the Kutateladze and Leontiev method [3-42] (*Editor's remark*).

may primarily be attributed to the fact that some theories are based on the assumption that  $C_f/C_{f0} \rightarrow 0$ , when  $b \rightarrow \infty$ , in others this result is achieved for finite injection rates.

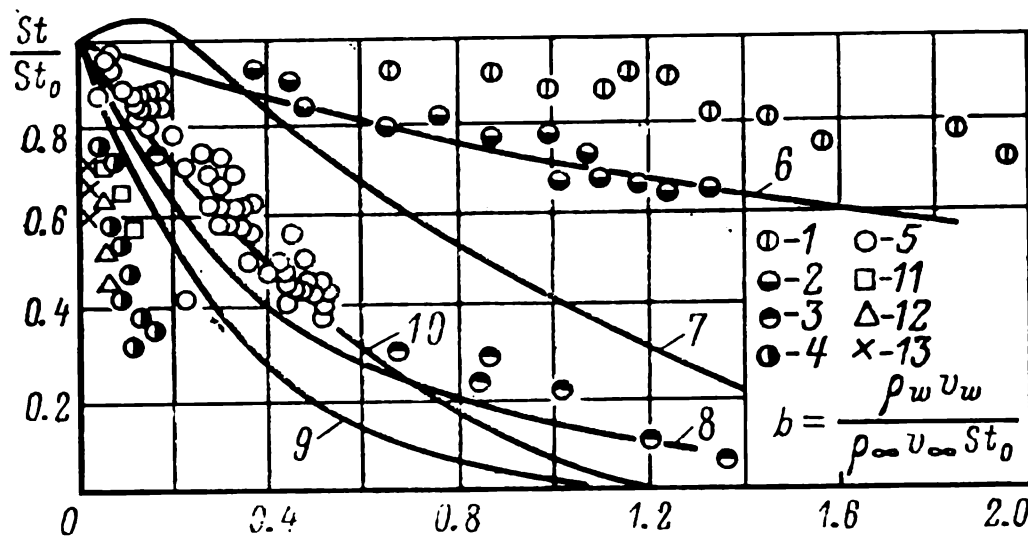


Fig. 3-34. Plot of  $St/St_0$  versus injection parameter  $b$  ( $b = \rho_w v_w / \rho_{\infty} v_{\infty} St_0$ ) in the case of injection through a porous plate (after various authors)

1 — Freon-12 [3-61]; 2 —  $Co_2$  [3-61]; 3 — helium [3-43]; 4 — helium [3-61]; 5 — helium [3-44]; 6 — Freon-12 [3-36]; 7 — helium [3-45]; 8 — helium [3-36]; 9 — helium [3-46]; 10 — helium [3-47]; 11 — benzene [3-48]; 12 — acetone [3-48]; 13 — ethyl alcohol [3-48]

The question of existence of critical injection velocities at moderate flow rates of an injectant is at present disputable, as measurement of small friction factors and small heat fluxes involves significant errors.

Interest in such problems is aroused by the necessity of achieving 100% concentration of the injectant at the material surface layer, of protecting the wall from chemical or mechanical erosion, and of creating a protective shield, etc.

The very scanty data presently available demonstrate that in principle it is possible to create conditions under which the injectant concentration will amount to about 100%.

In [3-49] experimental data on the study of a turbulent boundary layer at the permeable surface in the interval of high injection velocities and finite Reynolds numbers are presented. The experiments were based on the chemical interaction between the main flow (acidic medium) and the injectant (basic medium). At certain injection rates a layer of the injectant was formed on the porous surface. This was clearly indicated by phenolphthalein present therein. Besides, it should be mentioned that a film with  $\rho_{10w} = 100\%$  appeared at the same critical injection parameters, even when the injectant concentration fell to 0.0015% NaOH (very dilute solution) and the concentration of hydrochloric acid in the main flow amounted to 30%. From the solution neutralization conditions  $\rho_{10w}$  may be evaluated, calculations showed that the value of  $\rho_{10w}$  exceeded 99.994%.

In [3-50] interaction of a gas flow with solids at a high rate of injection through pores is also discussed. Here, it is noted that whereas for a laminar flow exact numerical solutions may, in principle, be obtained on the basis

of the simultaneous analysis of the Navier-Stokes, energy and diffusion equations, for turbulent transfer no exact solutions exist even for much simpler cases. Experiments, employing interferometric and thermoanemometric metering circuits that produce minimum distortion in the general

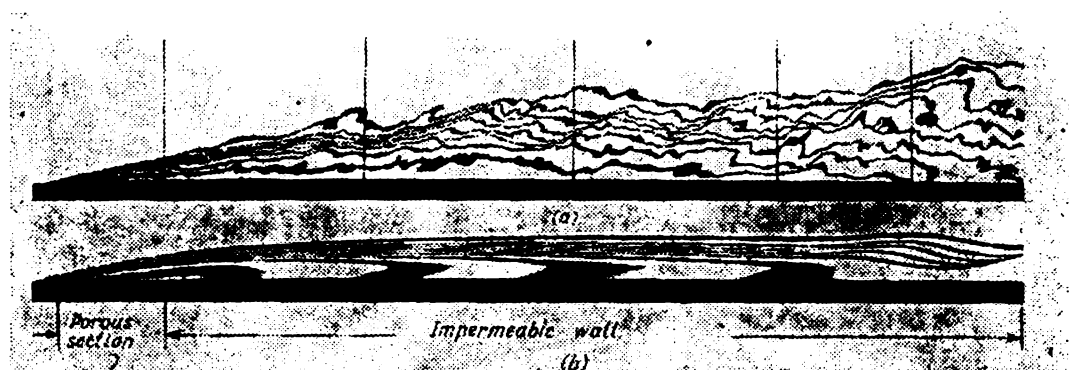


Fig. 3-35. Interferograms of a boundary layer when carbon dioxide is supplied through a porous plate

picture of flow, enable one to correctly choose the model of the phenomenon and to select a method for its calculation. Measurements were taken for laminar and turbulent flows over porous permeable bodies.

Figure 3-35 shows interferograms of a boundary layer when carbon dioxide is injected through a porous plate with various densities of transverse mass flow; it may be clearly seen that the boundary layer has been detached from the plate surface. A schematic diagram of the process is shown in Fig. 3-36. In the zone where the layer has been detached the lon-

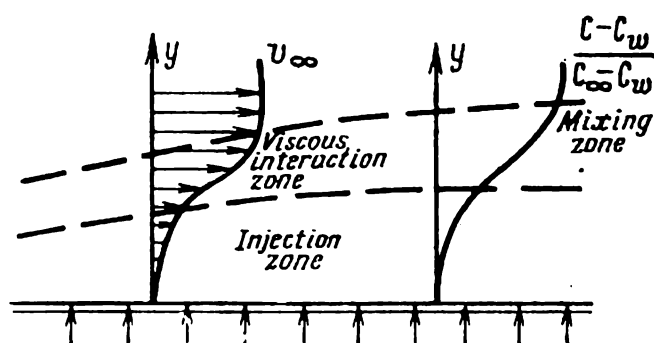


Fig. 3-36. Schematic of a gas flow in the boundary layer during injection

gitudinal velocity is small and the concentration is nearly constant, which is verified experimentally (Fig. 3-37). Similar experimental data for a turbulent boundary layer are presented in Figs. 3-38 and 3-39.

Figure 3-37 shows that the velocity profiles are S-shaped with an inflection point in the middle of the mixing zone; the concentration profiles are also S-shaped; concentration at the wall increases with the injection rate and at certain values of the transverse mass flow rates it is 100%. Analysis of the available data shows that the boundary layer in this case consists of two different regions of the detached layer, where gradients of all the parameters are small and the injectant concentration approaches

100%, and a relatively narrow zone of viscous interaction, in which all the values change quickly from their values in the detached zone to the parameters of the undisturbed flow.

The plate is no longer a solid with  $p = \text{const}$ ; a favourable pressure gradient appears which should be accounted for when developing the calcu-

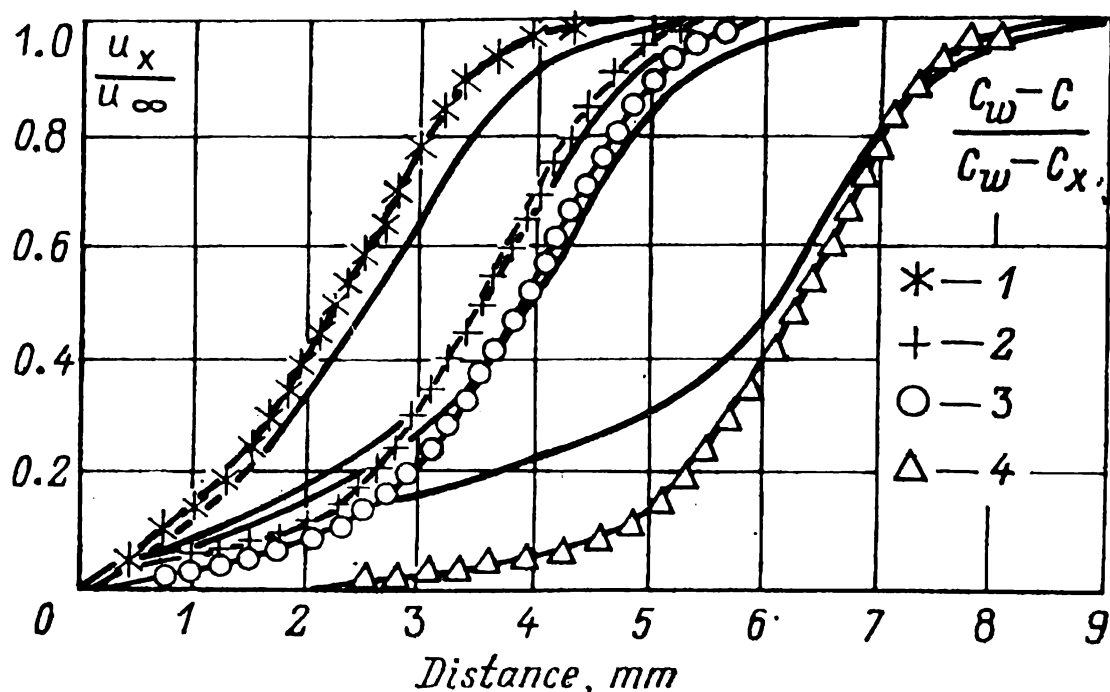


Fig. 3-37. Profiles of velocity and concentration when  $\text{CO}_2$  is injected through a porous plate

$v_\infty = 3.8$  m/sec;  $x = 79.6$  mm; 1 —  $qv_y = 0.009$ ; 2 — 0.014; 3 — 0.016; 4 — 0.027

lation method. The constraint  $dp/dv < 0$  is a probable explanation of the fact that in a laminar flow detachment occurred with injection parameter  $f_w = 1.53-1.84$  (at  $\text{Re} = 4 \times 10^3 - 20 \times 10^3$ ) whereas Eckert and Hartnett's theory [3-14] predicts  $(f_w)_{cr} = 0.619$ .

When a permeable porous plate interacts with a turbulent gas flow, the general picture is similar to that of a laminar flow. A detachment zone with small gradients of parameters is also observed, which is externally bounded by the intensive interaction region. When light gases are injected, detachment takes place at significantly low injectant flow rates.

A monotonous decrease of the transfer rate as the relative injectant flow rate increases is the main result of all the available theoretical works on the transverse mass flow effects on heat transfer in a boundary layer. Besides, certain data are available which reveal that heat transfer rate increases at low injectant flow rates [3-51], [3-52]. It is possible that at low injectant flow rates this effect is caused by the absence of a solid film of the injectant gas at the very exit from the porous wall. The gas in this case, flows out of the porous wall in the form of separate jets (because of the material's inhomogeneity) which destroy the viscous sublayer and intensify the mixing of gas moles, thus increasing the heat transfer rate.

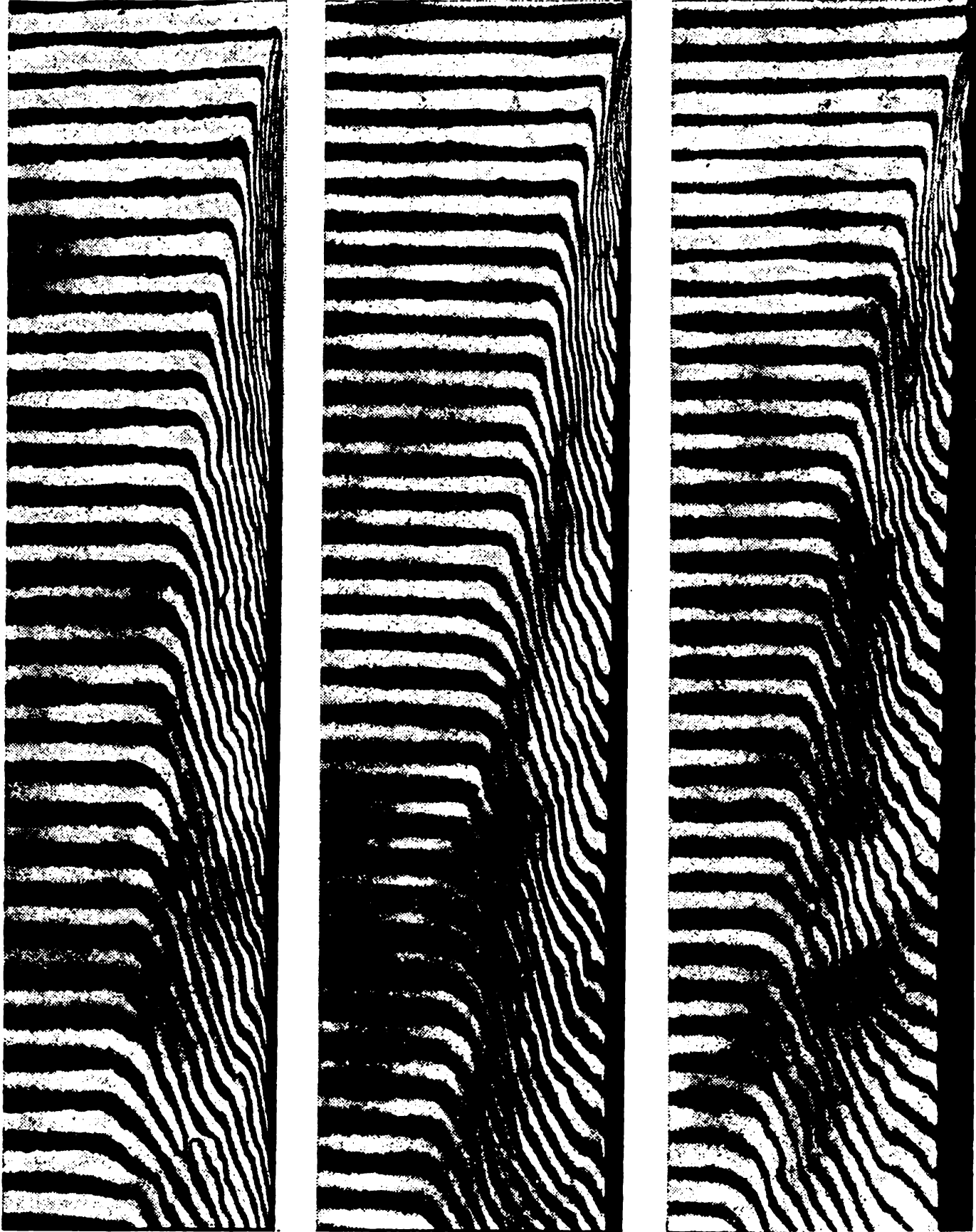


Fig. 3-38. Interferograms of a turbulent boundary layer when He is injected at  $v_\infty = 28$  m/sec  
 Top photo shows  $\rho_w v_w / \rho_\infty v_\infty = 0.002$ ; centre - 0.004; bottom - 0.008

For the calculation of local heat and mass transfer at moderate injection rates ( $b_0 = 0.1-4$ ), the following empirical relations were obtained in [3-53]

$$\text{Nu}_x = 0.0289 \text{Re}_x^{0.8} \text{Pr}^{0.4} \exp(-0.36 b_H) \quad (3-4-66)$$

$$(\text{Nu}_x)_m = 0.0195 \text{Re}_x^{0.8} \text{Sc}^{-0.6} b_0 \kappa_w^{-1} \quad (3-4-67)$$

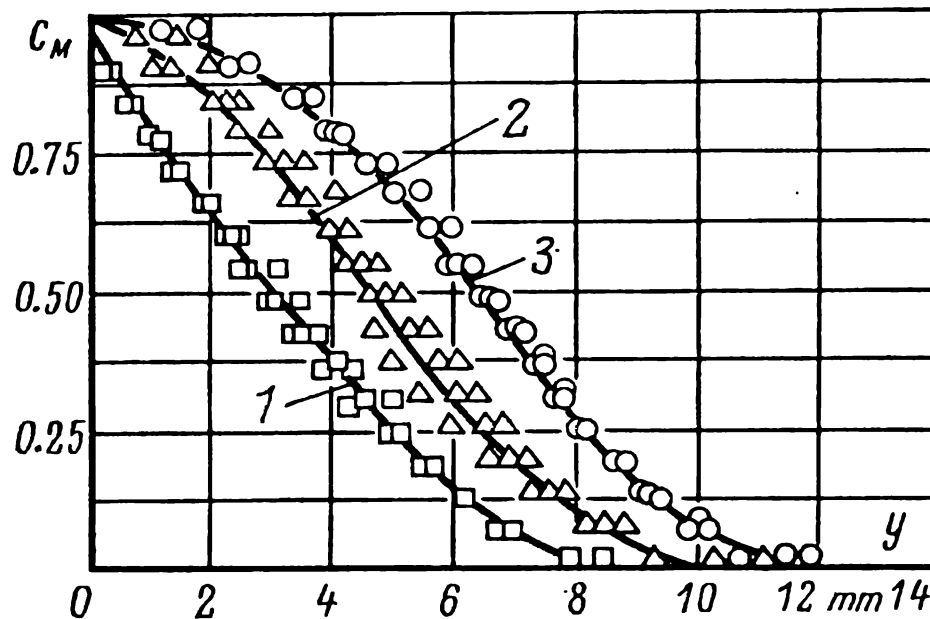


Fig. 3-39. Profiles of molar concentration  $C_M$  for carbon dioxide when it is injected through a porous plate

$v_\infty = 28$  m/sec;  $x = 60$  mm; 1 -  $\rho v_y = 0.031$ ; 2 -  $\rho v_y = 0.042$ ; 3 -  $\rho v_y = 0.053$

where  $b_H = \frac{c_{pw}\rho_w v_w}{c_{p\infty}\rho_\infty v_\infty \text{St}_0}$  is the dimensionless enthalpy mass transfer parameter;

$b_0 = \frac{\rho_w v_w}{\rho_\infty v_\infty H_0}$  is the relative dynamic mass transfer parameter;

$\kappa$  is the mole or volume fraction of the coolant.

Expressions (3-4-66) and (3-4-67) indicate that a transverse mass flow produces different effects on heat and mass transfer; in this range of injection parameters the heat transfer rate decreases and the mass transfer rate increases.

### 3-5. FREE CONVECTION

Fluid flow due to density differences in the external force field is generally called free convection. Such external forces are gravity forces, and the density difference, in a very simple case, is the result of the temperature drop between the solid surface and the fluid. Under conditions of

free convection the boundary layer thickness is much greater than during forced motion, since the velocities of free motion are small.

(a) *Free Convection in an Infinite Volume*

Let us consider heat transfer between a vertical plate and fluid. The  $x$ -axis is directed along the plate (vertical direction) and the  $y$ -axis is normal to it (horizontal direction).

The momentum and energy transfer equations for the present case may be written as

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2} + \beta g(T - T_\infty) \quad (3-5-1)$$

$$v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} = a \frac{\partial^2 T}{\partial y^2} \quad (3-5-2)$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \quad (3-5-3)$$

where  $\beta$  is the volume expansion coefficient. The boundary conditions are

$$\text{at } y = 0 \quad v_x = 0; \quad v_y = 0; \quad T = T_w \quad (3-5-4)$$

$$\text{at } y = \infty \quad v_x = 0; \quad T = T_\infty \quad (3-5-5)$$

Let us introduce the following notations

$$\text{Gr}_x = \frac{\beta g x^3 (T_w - T_\infty)}{\nu^2}; \quad \xi = \frac{y}{x} \left( \frac{1}{4} \text{Gr}_x \right)^{1/4} \quad (3-5-6)$$

where  $\text{Gr}_x$  is the local Grashof number.

The stream function  $\psi$  will then be

$$\psi = 4\nu \left( \frac{\text{Gr}_x}{4} \right)^{1/4} \varphi(\xi) \quad (3-5-7)$$

$$v_x = \left[ \frac{\beta g (T_w - T_\infty)}{4\nu^2} \right]^{1/4} 4\nu x^{1/2} \varphi' \quad (3-5-8)$$

$$v_y = \nu x^{-1/4} \left[ \frac{\beta g (T_w - T_\infty)}{4\nu^2} \right]^{1/4} (\xi \varphi' - 3\varphi) \quad (3-5-9)$$

Equations (3-5-1) and (3-5-2) take the form

$$\varphi''' + 3\varphi\varphi' - 2\varphi'^2 + \theta = 0 \quad (3-5-10)$$

$$\theta'' + 3 \text{Pr} \varphi\theta' = 0 \quad (3-5-11)$$



where  $\theta = \frac{T - T_\infty}{T_w - T_\infty}$  is the dimensionless temperature.

The boundary conditions are as follows

$$\text{at } \xi = 0 \quad \varphi = \varphi' = 0; \quad \theta = 1 \quad (3-5-12)$$

$$\text{at } \xi = \infty \quad \varphi' = 0; \quad \theta = 0 \quad (3-5-13)$$

Equations (3-5-10) and (3-5-11) were solved by Ostrach who used a numerical method. Distribution of the relative velocity  $\varphi'$  [ $\varphi' =$

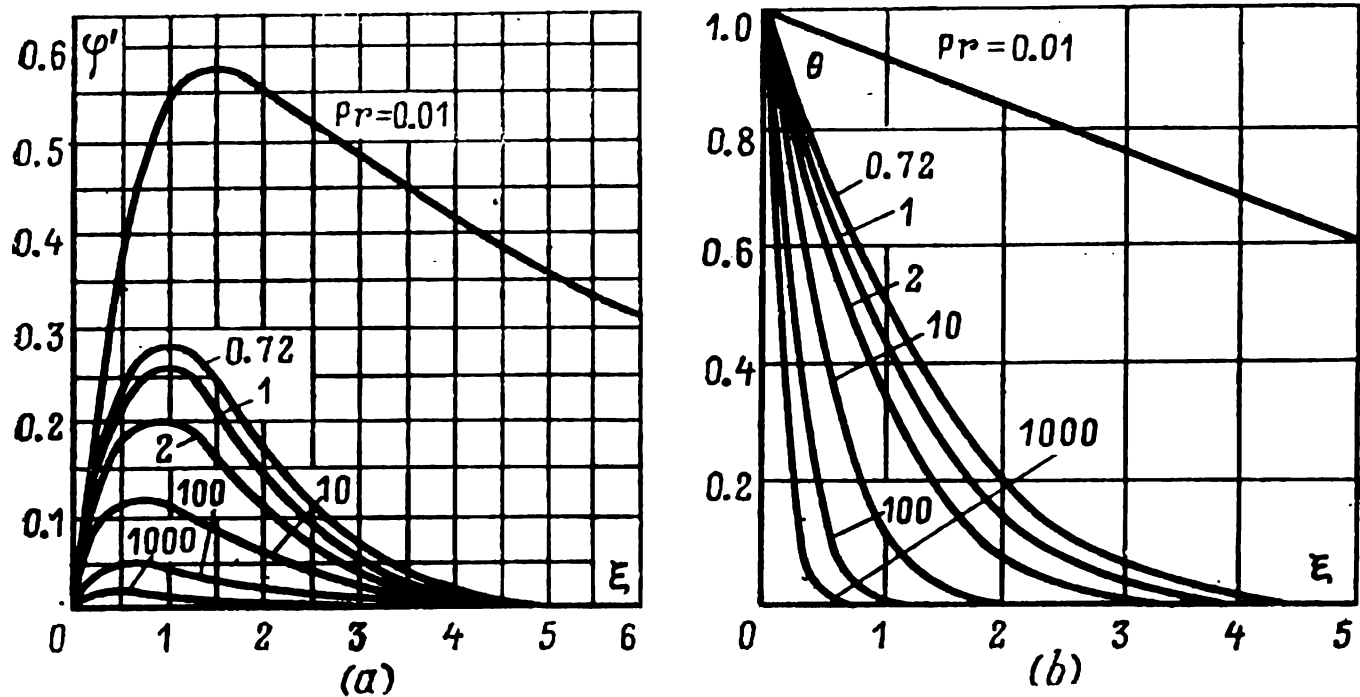


Fig. 3-40. Plot of relative velocity  $\varphi'$  (a) and temperature  $\theta$  (b) versus dimensionless complex  $\xi = \frac{y}{x} \left( \frac{1}{4} Gr_x \right)$

$= v_x x / 2\nu(Gr_x^{1/2})$  versus the dimensionless coordinate  $\xi[\varphi(\xi)]$  is presented in Fig. 3-40 a, and the temperature distribution  $\theta(\xi)$  for different Pr numbers is plotted in Fig. 3-40 b.

The local Nusselt number

$$Nu_x = \frac{\alpha_x x}{\lambda} = \frac{q_x x}{\lambda(T_w - T_\infty)} = - \left( \frac{Gr_x}{4} \right)^{1/4} [-\theta'(0)] \quad (3-5-14)$$

Values of  $\theta'(0)$  and  $\varphi''(0)$  are given in Table 3-9.

Table 3-9

Values of  $\theta'(0)$  and  $\varphi''(0)$

Pr	0.01	0.733	1	2	10	100	1000
$-\theta(0)$	0.0812	0.5080	0.5671	0.7165	1.1694	2.191	3.966
$\varphi''(0)$	0.9862	0.6741	0.6421	0.5713	0.4192	0.2517	0.1450

The given problem may be solved approximately. A parabolic temperature distribution along the normal to the surface is assumed

$$\vartheta = \vartheta_n \left(1 - \frac{y}{\delta}\right)^2 \quad (3-5-15)$$

where  $\vartheta_n = t_n - t_c = \text{const}$  is the temperature difference at  $y = 0$ . The velocity distribution is given as

$$v_x = v_1 \frac{y}{\delta} \left(1 - \frac{y}{\delta}\right)^2 \quad (3-5-16)$$

where  $v_1$  is a certain coefficient independent of  $y$  measured in units of velocity.

From equation (3-5-16) it follows that the fluid velocity along the plate is maximum at a distance  $y = \frac{1}{3} \delta$  from the surface. Then  $v_1$  will be  $27/4 (v_x)_{\text{max}}$ .

Using Karman's method we may get the relation for boundary layer thickness  $\delta$  as

$$\frac{\delta}{x} = 3.93 \text{Pr}^{-0.5} + (0.952 + \text{Pr})^{0.25} (\text{Gr}_x)^{0.25} \quad (3-5-17)$$

where  $\text{Gr}_x$  is the local Grashof number

$$\text{Gr}_x = \frac{g\beta\vartheta_n x^3}{\nu^2} \quad (3-5-18)$$

Here  $\beta$  is the volume expansion coefficient of the fluid and  $g$  is the acceleration due to gravity.

The heat flux from the plate surface

$$q = \alpha\vartheta_n = -\lambda \left(\frac{\partial\vartheta}{\partial y}\right)_n = \frac{2\lambda\vartheta_n}{\delta} \quad (3-5-19)$$

Hence, the local Nusselt number is

$$\text{Nu}_x = \frac{\alpha x}{\lambda} = 2 \frac{x}{\delta} \quad (3-5-20)$$

and the local heat transfer coefficient is inversely proportional to  $\sqrt[4]{x}$ . Substitution of an appropriate expression for  $\delta$  gives

$$\text{Nu}_x = 0.508 \text{Pr}^{0.5} (0.952 + \text{Pr})^{-0.25} (\text{Gr}_x)^{0.25} \quad (3-5-21)$$

The mean heat transfer coefficient is  $\bar{\alpha} = \frac{4}{3} \alpha$ .

For ideal gases the volume expansion coefficient  $\beta$  equals  $1/T$ ; for a small temperature difference  $\beta = 1/T_c$  may be assumed. For air  $Pr = 0.714$ , then

$$Nu_x = 0.378 (Gr_x)^{0.25} \quad (3-5-22)$$

More accurate calculations result in the same relation between  $Nu_x$  and  $Gr_x$ , but with a factor of 0.360 instead of 0.378. Thus, approximated calculations give results close to the exact, though time-consuming, calculations.

Experimental data corroborate expressions (3-5-20) -(3-5-22) and equations (3-5-15) and (3-5-16) for temperature and velocity profiles in a boundary layer. The agreement between calculated and experimental values is about 10% for  $Pr$  numbers in the range from 0.01 to 1000.

Expression (3-5-21) gives better results than the widely used formula

$$Nu_x = B (Gr_x Pr)^{0.25} \quad (3-5-23)$$

obtained from the generalized experimental data for air.

During free convection a boundary layer far from the edge becomes turbulent when its thickness attains a certain value. For air this transition occurs at critical value  $(Gr_x)_k = 10^9$ , that corresponds to  $(Re_\delta)_k = 550$ , where  $Re_\delta = v_{max} \delta/\nu$ .

Fluid motion close to heated horizontal plates is of a different pattern and depends on the plate size. Plates with their heated surfaces facing upwards give rise to ascending flows as per the scheme shown in Fig. 3-41a (small size plates) and in Fig. 3-41b (large size plates). In the latter case the middle part of the plate is isolated by fluid flows ascending from the plate

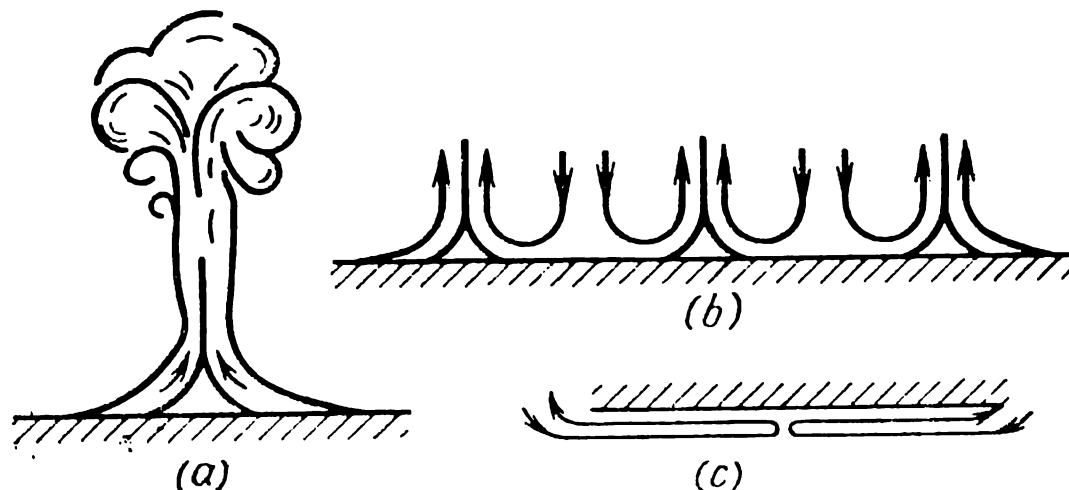


Fig. 3-41. Schematic of free motion of fluid close to heated horizontal plates

edges. The middle part of the plate is cooled due to the inflow of cold fluid from above.

When the heated surface faces downward, fluid motion in the boundary layer follows the pattern shown in Fig. 3-41c while fluid below this layer remains stationary.

For thin wires ( $d = 0.2\text{--}1.0\text{ mm}$ ) the flow remains laminar even at high temperature driving forces. At small temperature driving forces a fixed film of the heated fluid is formed at the wire which corresponds to film heat transfer.

Free convection in a turbulent fluid flow along a heated vertical plate is governed by the following equations for temperature and velocity distributions in a boundary layer

$$\vartheta = \vartheta_n \left[ 1 - \left( \frac{y}{\delta} \right)^{1/7} \right]; v_x = v_1 \left( 1 - \frac{y}{\delta} \right)^4 \left( \frac{y}{\delta} \right)^{1/7} \quad (3-5-24)$$

In the velocity distribution equation the factor  $\left( \frac{y}{\delta} \right)^{1/7}$  is especially introduced to calculate shear stress similar to turbulent stress during forced motion.

Calculations similar to those considered above give the following dimensionless equation

$$\text{Nu}_x = 0.0295 (\text{Gr}_x)^{2/5} \text{Pr}^{7/5} [1 + 0.494 \text{Pr}^{2/3}]^{-2/5} \quad (3-5-25)$$

Most popular is the empirical formula obtained by generalization of the experimental data on heat transfer from different types of bodies (plates, wires, tubes, spheres) during free convection [3-62]. For tubes and spheres their diameter is taken as the characteristic dimension, for plates — their height. Coefficients and thermodynamic parameters have been calculated for mean temperature of the boundary layer.

The dimensionless equation takes the form

$$\text{Nu} = B(\text{Gr Pr})^n \quad (3-5-26)$$

the values of constants  $B$  and  $n$  are given in Table 3-10.

Table 3-10

Constants  $B$  and  $n$

Gr Pr	$B$	$n$
$1 \times 10^{-3}\text{--}5 \times 10^3$	1.18	1/8
$1 \times 10^2\text{--}2 \times 10^7$	0.54	1/4
$2 \times 10^7\text{--}1 \times 10^{13}$	0.135	1/3

At  $(\text{Gr Pr}) < 1$  the Nusselt number remains practically invariable and equals 0.5. Thus, heat is transferred by conduction ( $\alpha = 0.5 \lambda/l$ ). From an analysis of the values of  $B$  and  $n$  the following conclusion, important for practical purposes, may be drawn: in the range of  $(\text{Gr Pr})$  from  $1 \cdot 10^7$  to  $1 \cdot 10^{13}$  the exponent  $n = 1/3$ . Hence, in this range heat transfer is independent of the geometrical sizes since the characteristic dimension enters

into  $Nu$  at the 1st power and into  $Gr$  at the 3rd power. Thus, in this range the heat-transfer process is a similarity process, and this fact enables one to study heat transfer on small-size models. This is of great practical importance for building heating systems.

A review of published works on free convection is presented in [3-55]. In [3-56] the problem has been solved for a plate inclined at an angle to the horizon. The plate temperature was maintained at a constant level.

### (b) Free Convection in a Finite Volume

First we shall consider heat transfer between two parallel plates at different temperatures ( $t_1 \neq t_2$ ). If the plates are located horizontally and the upper has a higher temperature than the lower one ( $t_1 > t_2$ ), then heat from one plate to the other will be transferred through a fluid layer by conduction; the only exception to this being heat transfer close to the plate edges.

If the upper plate is at a much lower temperature than the lower one, then the heated particles will gather at the upper plate. This leads to an unsteady state which at  $(Gr \ Pr) \geq 1700$  transforms into a state of closed convective fluid flows circulating within the space between the plates.

In the case of vertical walls the motion pattern depends on the ratio of the plate height  $l$  to the distance  $L$  between the plates. Studies have revealed that at

$$Gr_L < 124 \ Pr^{-2} \left( \frac{20}{21} + Pr \right) \frac{l}{L} \quad (3-5-27)$$

temperature distribution in a fluid layer along the normal to the plates is linear. Only at the lower and upper edges of the plates the distribution is not linear; the deviation propagates within the layer over a distance  $L$ . Heat is transferred by conduction.

An analysis of interferometric patterns has revealed the location of boundary layers in the space between the plates as shown in Fig. 3-42. The boundary layer thickness  $\delta_1$  at the hotter plate increases in the upward direction; at the colder plate the thickness  $\delta_2$  increases in the opposite direction.

Temperature in the middle part of the fluid volume is constant. Temperature distribution in the boundary layer is similar to that in the boundary layer of a single plate. Heat-transfer coefficients, in the case of two plates, differ by not more than 20% from those obtained from expressions for single plates. In this case the temperature of the fluid at rest is assumed equal to the temperature  $\vartheta_c$  of the core of the fluid between the plates. This situation maintains until the two boundary layers merge. This happens when  $GrPr$  decreases. But, as experiments have demonstrated, temperature in the fluid core is not uniform and increases from bottom to top.

Schmidt [3-57] has established that heat transfer from a hot plate to a cold one through a vertical fluid layer is intensified if the space between the plates is separated by thin horizontal walls. This may be attributed to

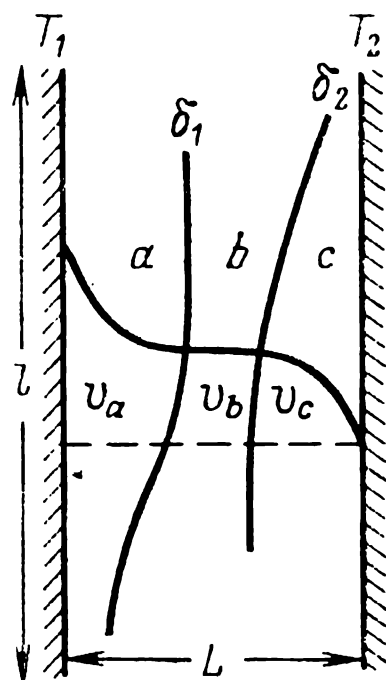


Fig. 3-42. Schematic showing development of boundary layer and distribution of temperature between vertical plates

the development of boundary layers at large values of  $Gr_L Pr$ . The boundary layers at the heated and cold plates begin at each of the horizontal walls, they are thinner than in the absence of walls. This results in intense heat transfer.

Schematic diagrams of fluid circulation in slots of various widths are shown in Fig. 3-43. If the slot width  $L$  is sufficiently large, the ascending and descending flows do not affect each other (Fig. 3-43a). With small  $L$  the flows affect each other which may result in internal circulation as if partitions were present (Fig. 3-43b). As mentioned earlier, in the case of horizontal plates, no convection will occur if the hot surface is at the top (Fig. 3-43c). In the opposite case cellular circulation flows may arise (Fig. 3-43d).

Most cases of two-dimensional problems of free convection have been solved numerically or approximately. Readers interested in the details of solutions may refer to the original works [3-57].

With such a complicated hydrodynamic situation it is extremely difficult to calculate heat-transfer coefficients for fluids in a finite volume. Therefore, in engineering calculations heat transfer between two plates through a fluid layer in a slot is considered as heat transfer by conduction, and an equivalent heat-conduction coefficient is then used.

$$\lambda_l = \frac{q}{(t_{n1} - t_{n2})} \quad (3-5-28)$$

The ratio of the equivalent heat-conduction coefficient to the true one is called a convection factor  $\varepsilon_k$ ,  $\varepsilon_k = \lambda_l/\lambda_c$ ; fluid circulation within a slot depends on  $Gr\ Pr$ , therefore

$$\varepsilon_k = f(Gr\ Pr)$$

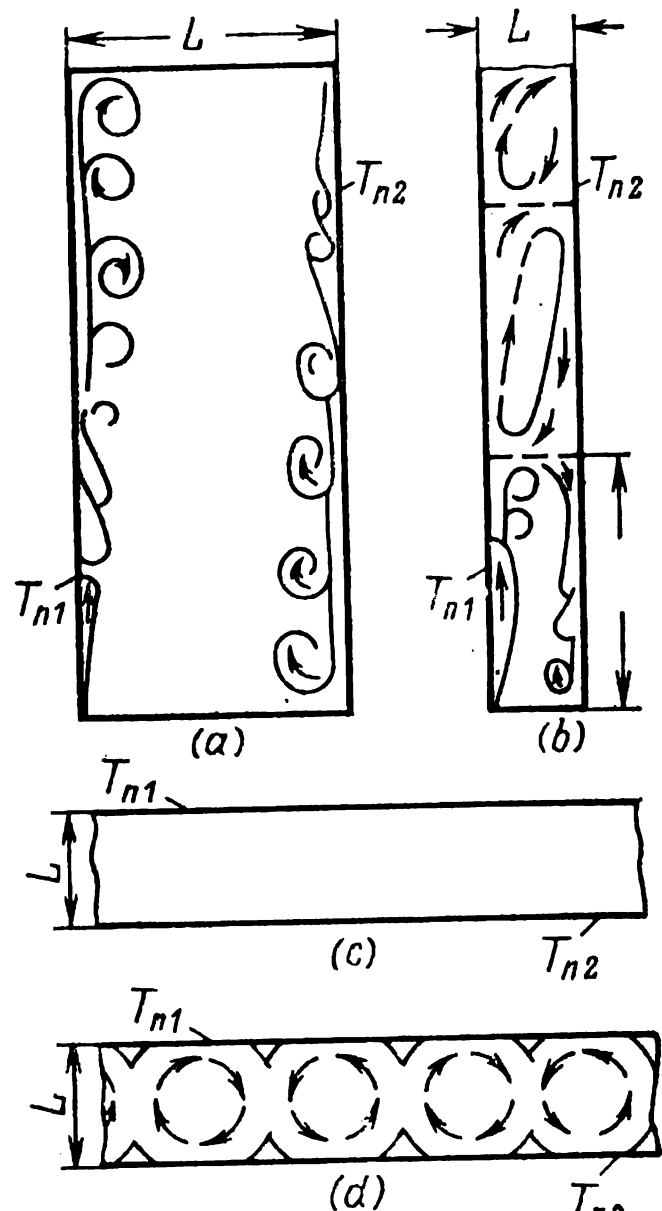


Fig. 3-43. Schematic showing circulation of fluid in vertical and horizontal slots

Figure 3-44 shows the plot of  $\varepsilon_k = f(Gr\ Pr)$  taken from [3-58]. In the calculation of  $Pr\ Gr$  the slot width  $L$  was taken as the characteristic dimension (irrespective of the slot geometry); the parameters were calculated for a mean temperature of  $0.5(T_{n1} + T_{n2})$ . The figure shows that when  $Gr\ Pr < 1000$  the convection factor equals unity ( $\varepsilon_k = 1$ ). Therefore, for  $Gr\ Pr < 1000$  heat through a fluid layer in a slot is transferred by conduction. For a first approximation certain sections of the curve  $\log \varepsilon_k = f[\log(Gr\ Pr)]$  may be considered as straight lines. We may then write

$$\varepsilon_k = 0.105 (Gr\ Pr)^{0.3} \text{ at } 10^3 < Gr\ Pr < 10^6 \quad (3-5-29)$$

$$\varepsilon_k = 0.40 (Gr\ Pr)^{0.2} \text{ at } 10^6 < Gr\ Pr < 10^{10} \quad (3-5-30)$$

The decrease in heat-transfer rate at large values of  $(Gr Pr)$  may be attributed to mutual influence of heated and cooled fluid flows within the slot.

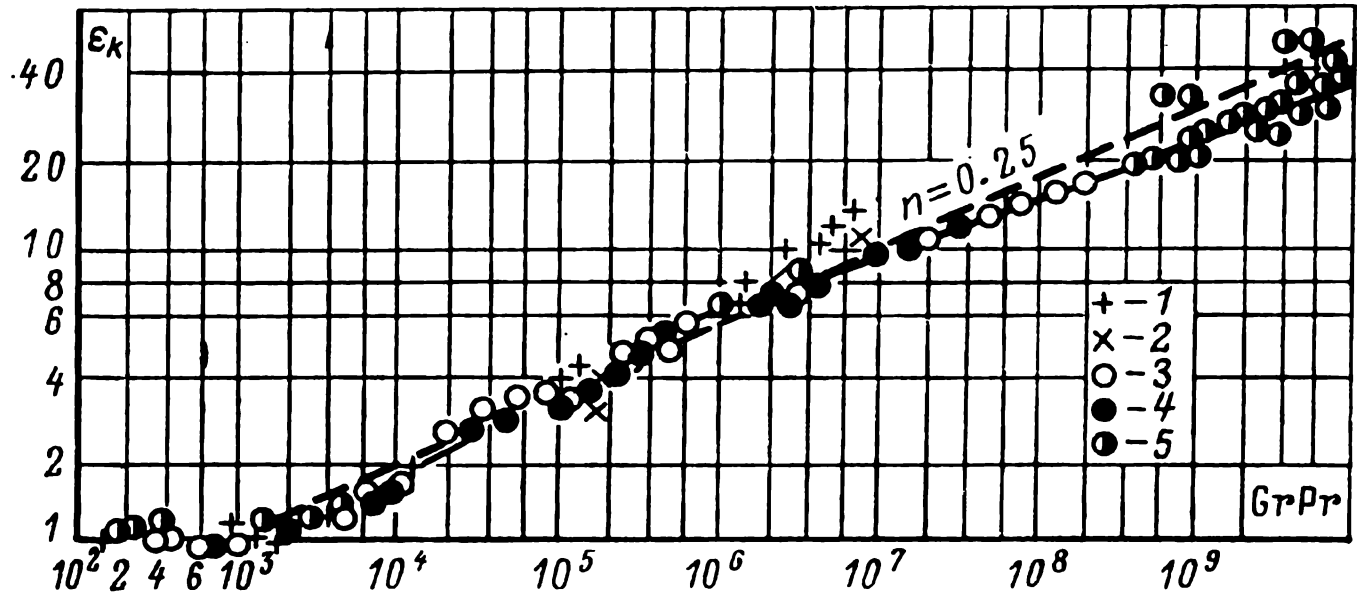


Fig. 3-44. Plot of free convection coefficient versus  $GrPr$  during heat transfer in a closed space

1 — horizontal flat gaseous interlayer; 2 — vertical flat gaseous interlayer; 3 — cylindrical gaseous interlayer; 4 — cylindrical liquid interlayer; 5 — spherical gaseous interlayer

In approximate calculations at  $(Gr Pr) > 10^3$ , instead of expressions (3-5-29) and (3-5-30), the relation

$$\varepsilon_k = 0.18 (\beta g Pr)^{1/4} \sqrt[4]{L^3 \Delta t} \quad (3-5-31)$$

may be used, where  $\Delta T = T_{n1} - T_{n2}$  is the temperature difference between the slot walls.

### (c) Free Convection when Heating from Above

One of the interesting free convection problems involves inducing convection by heating from above. A detailed experimental and theoretical analysis of this phenomenon in a strictly horizontal cavity was carried out at the Heat and Mass Transfer Institute, Byelorussian Academy of Sciences.

Numerical study of free convection excited by nonuniform heating from above was conducted in rectangular cavities.

The problem was mathematically formulated as follows [3-59]:

$$\frac{\partial \varphi}{\partial \tau} = \nabla^2 \varphi + \frac{\partial \psi}{\partial x} \frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \varphi}{\partial x} + Gr \frac{\partial T}{\partial x} \quad (3-5-32)$$

$$\frac{\partial T}{\partial \tau} = \frac{1}{Pr} \nabla^2 T + \frac{\partial \psi}{\partial x} \frac{\partial T}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial T}{\partial x} \quad (3-5-33)$$

$$\varphi = - \nabla^2 \psi \quad (3-5-34)$$



where  $T$  is the temperature;  $\psi$  and  $\phi$  are the dimensionless stream function and velocity vortex, respectively. The dimensionless relations for stream function, temperature, coordinates  $x$ ,  $y$  and time involve kinematic viscosity  $\nu$ , characteristic temperature difference  $\theta$  at the region boundary, length  $l$  of a square side and ratio  $l^2/\nu$ .

Numbers  $Gr$  and  $Pr$  are found in the normal fashion from the expressions:

$$Gr = g\beta\theta l^3/\nu, \quad Pr = \nu/a$$

At the region boundary the velocity components and the stream function are assumed to be zero.

Several types of boundary conditions for temperature are considered.

1. "Pure" heating from above. When  $0 \leq x \leq 1$ ,  $y = 0$  the temperature equals zero. The boundary conditions at the upper wall are as follows:

Calculation No.	Boundary Conditions
1	$T = \sin \pi x \quad 0 \leq x \leq 1$
2	$T = \begin{cases} \sin 2\pi x & 0 \leq x \leq 0.5 \\ 0 & 0.5 \leq x \leq 1 \end{cases}$
3	$T = \begin{cases} \sin 4\pi x & 0 \leq x \leq 0.5 \\ 0 & 0.5 \leq x \leq 1 \end{cases}$
4	$T = 1 - (2x - 1)^n \quad 0 \leq x \leq 1$
5	$T = [1 - (2x - 1)^2]^{1/2} \quad 0 \leq x \leq 1$

2. Simultaneous heating from above and from one side. One of the sides and the lower face of the square are maintained at a constant temperature equal to zero; at the upper face and the other side a linear distribution of the dimensionless temperatures is prescribed

$$y = 1, \quad T = x; \quad x = 1, \quad T = y$$

3. Heating from above, from the side and from below was done for the following temperature distribution. The lower and both lateral faces were maintained at a constant temperature equal to zero, and at the upper face the sinusoidal distribution was given by

$$T = \sin n\pi x, \quad n = 2, 3, 4$$

For the given boundary conditions the differential equations were solved numerically on a computer. The obtained solutions were analysed and the relations were plotted. The relations were verified directly by expe-

riments. These were carried out in a cavity formed by two horizontal brass plates, by side walls made of optical glass and by end faces made of organic glass. Visual observations were made and photographs of the flow pattern were taken through side walls. In the horizontal and end walls 30 copper-constantan thermocouples were embedded to register temperature distribution over the walls.

Different temperature distribution patterns were created on the upper wall by pumping water through a sectioned jacket on the external plate wall from three thermostats.

Temperature uniformity at the lower cold plate was maintained by intensive circulation of thermostatted water at ambient temperature through a multi-sectioned jacket. In all runs deviations from the mean wall temperature did not exceed  $0.15^{\circ}\text{C}$ .

Visual observations of flow patterns and temperature measurements with a movable copper-constantan thermocouple have demonstrated that in the middle part, which makes up about 11% of the cavity width, there exists a two-dimensional flow with very similar velocities and cell sizes. As the distances to the front and back walls are decreased, the cell sizes decrease and the velocity of travel falls to zero.

All further measurements were made in the middle plane  $z/2$ . Two series of experiments were carried out: hydrodynamic and thermal. Air and distilled water were used as working fluids.

Convection rates were measured in the following manner. The cavity filled with air containing suspended particles of zinc stearate or with water containing particles of powder aluminium was illuminated from the end face. An optical quantum generator ЛТ-55 was used as a light source. The flow pattern was photographed using intermittent illumination. Thus, the track lengths were used to calculate the flow velocity at a certain point. Photographs were taken in reflected light. The obturator was a 1.1 mm thick aluminium disc having 60 radial slits and was rotated by a synchronous electric motor.

In the thermal experiments for exact temperature measurements the whole installation had additional insulation of glass wool. The temperature field in each cell was measured by a movable copper-constantan thermocouple (wire diameter of 0.05 mm) with a compensating heater. The thermocouple was moved along the cavity height by a special device with an accuracy of 0.02 mm.

Heat-transfer coefficients were calculated from experimentally measured vertical temperature gradients at the hot wall surface.

Periodic, delta-shaped and nearly linear temperature distributions were found among others.

To study convection during periodic heating the scientists succeeded in obtaining experimentally the distribution, which may be assumed cosinusoidal with an accuracy of 10%.

In the case of delta-shaped distribution the temperature distribution was accomplished experimentally which with an accuracy of 7% is approximated by the function

$$T = Ae^{-c(x - 0.07)^2} + B(0.07 - x)^2$$

where  $x$  is the coordinate along the cavity length:  $0 \leq x \leq 0.14$ ;  $A, B, C$  are factors:  $30 \leq A \leq 50$ ,  $2700 \leq B \leq 3700$ ,  $300 \leq C \leq 400$ .

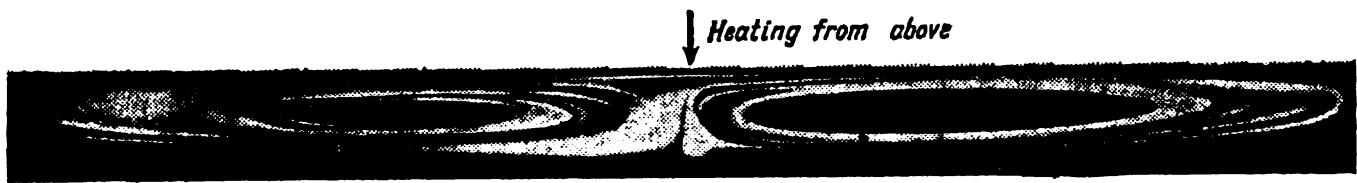


Fig. 3-45. Convective motion of air for delta-shaped heating from above

A convective air flow for a delta-shape distribution with heating from above is shown in Fig. 3-45. Cigarette smoke was used for visualization.

Convection at nearly linear temperature change was studied over a distribution, which with an accuracy of 3.6% is described by the function

$$T = -Ath \frac{x - 0.065}{c} + B$$

where  $2 \leq A \leq 20$ ;  $21.4 \leq B \leq 42.4$ ;  $0.03 \leq C \leq 0.05$ .

The considered cases enable one to draw the following conclusions on convection structure when heating from above:

- (a) the flow is of a pronounced cellular type;
- (b) each extreme point in the temperature distribution at the upper wall forms the cell interface;
- (c) the shapes of cell interfaces depend on the shapes of temperature profiles near the extreme point; an experiment was carried out with changing slopes of the temperature curve for a delta-shape heating from above; an insignificant increase of the temperature gradient resulted in deviation of the interface towards larger gradient (Fig. 3-46);
- (d) the shape of temperature nonuniformity at the upper wall affects the flow structure, the values of nonuniformity being the same  $\Delta T_L = T_{\max} - T_{\min}$  (Fig. 3-47); as the approximately linear heating pattern changes, the shape of a convective cell also changes;
- (e) the fluid flow direction in each cell near the upper wall is opposite to the temperature gradient within the cell;
- (f) the cell occupies the whole cavity along its height and the cell length is equal to the characteristic dimension of the temperature nonuniformity.

In Fig. 3-48 a graph of dimensionless temperature  $\theta = (T - T_0) / (T_h - T_0)$ , is plotted, where  $T_0$  is the cold wall temperature;  $T_h$  is the temperature of the upper heated wall in the considered cross-section in upward

and downward flows of convective cells for similar conditions of periodic heating from above. From Figure 3-48 it is clear that heat through the ca-

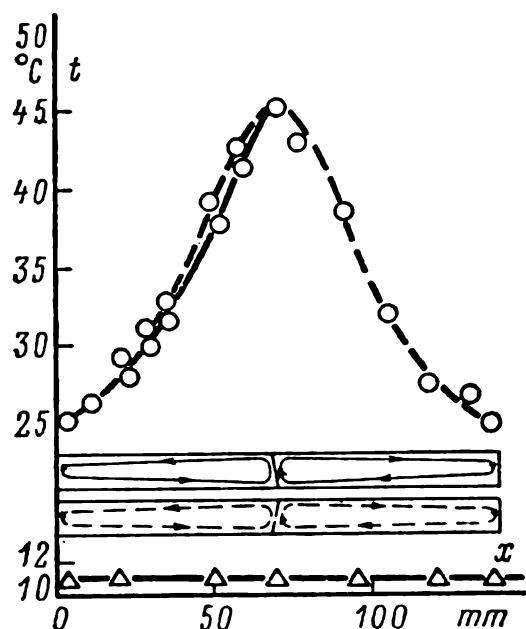


Fig. 3-46. Structure of the flow in a cavity for delta-shaped distribution of temperature at the upper wall of the cavity

Lower wall temperature is constant

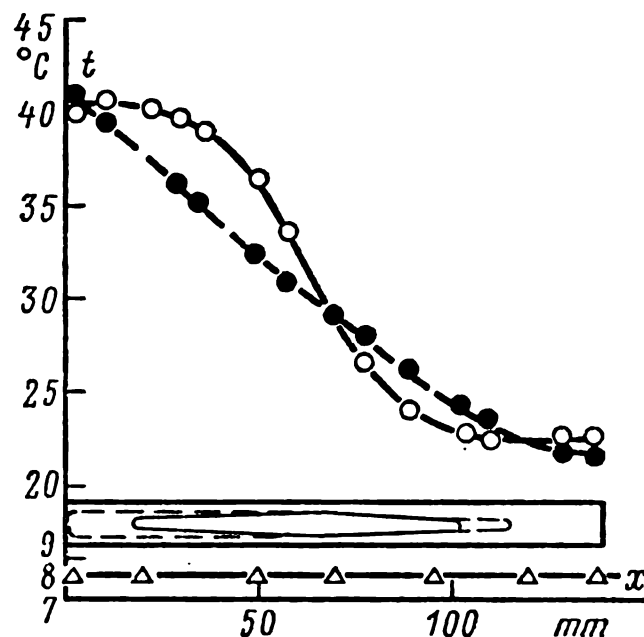


Fig. 3-47. Structure of the flow for approximate distribution of temperature at the upper wall of the cavity

vity for the given heating regime is largely transferred by conduction with insignificant convective contributions in downward and upward flows.

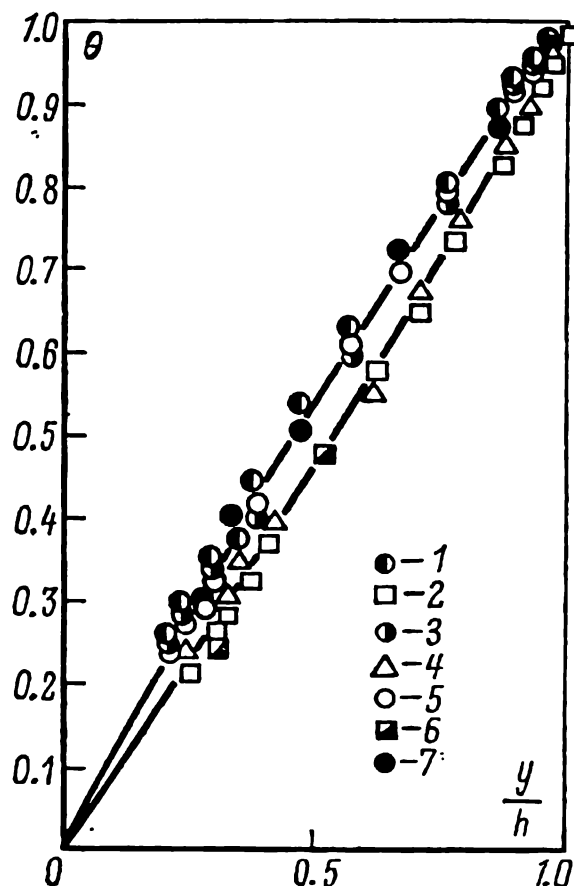


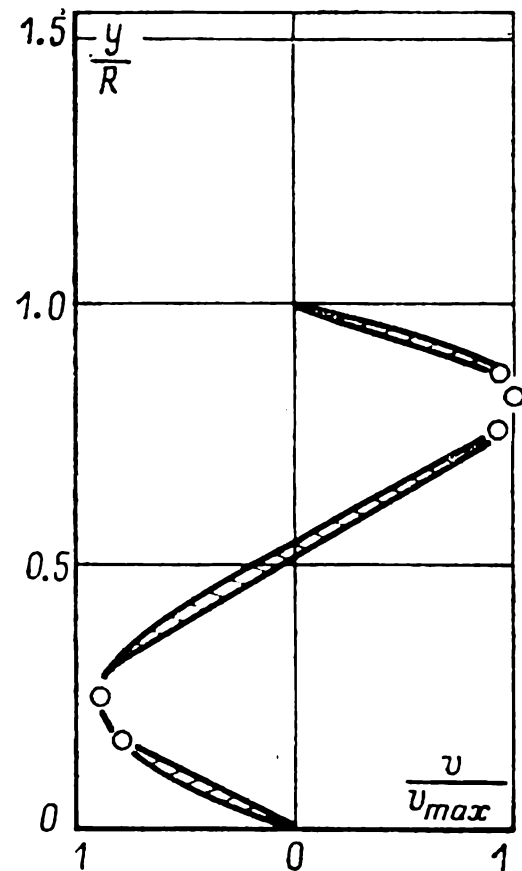
Fig. 3-48. Dimensionless profile of temperature  $\theta = (T - T_0)/(T_h - T_0)$  in downward and upward cell flows during periodic heating of the upper wall

1 - wall temperature at a distance of 5 mm; 2 - at 35 mm; 3 - at 75 mm; 4 - at 110 mm; 5 - at 135 mm; 6 - coincidence of measured values in upward layers; 7 - coincidence of measured values in downward layers

The horizontal velocity profiles in the middle part of each cell for these heating conditions are shown in Fig. 3-49. The velocity profiles in different cells practically coincide and have a pronounced asymmetry.

Fig. 3-49. Flow velocity in a cell during nonuniform heating from above

A profile of velocities in central sections of all four cells during periodic heating from above is shown; open circles show coincidence of measured values; shaded sections — experimental points



A plot of  $Nu$  versus  $Ra = Gr Pr$ , where  $Gr$  is the Grashof number, drawn for the cavity height  $h$  and the maximum temperature gradient at the upper wall  $\Delta T_L = T_{\max} - T_{\min}$ , is shown in Fig. 3-50. For this plot  $Pr = 0.707$  and the temperature ratio  $\Delta T_L^{\max}/\Delta T_h^{\min} \approx 1$ .

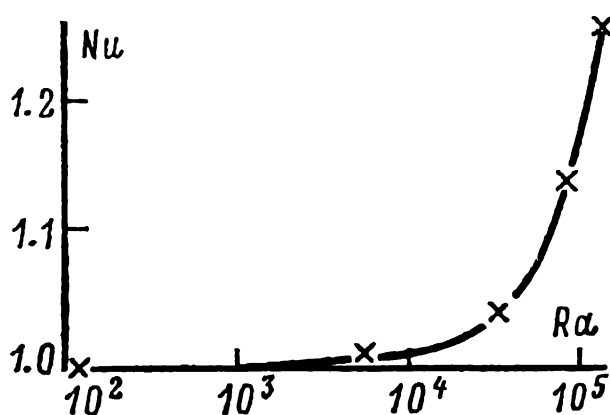


Fig. 3-50. Plot of Nusselt number versus Rayleigh number ( $Ra$ ) at  $\Delta T_{\max}/\Delta T_{\min} \approx 1$

It is obvious that the effect of convection on heat transfer becomes pronounced at  $Ra \approx 10^4$ .

### 3-6. THERMOCONVECTIVE WAVES

Oscillatory motions with small amplitudes in compressible fluids are called sound waves.

As the oscillations in a sound wave are small, the velocity in it is also small. Therefore, in the Euler equation

$$\frac{\partial \vec{v}}{\partial \tau} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\vec{\nabla} p / \rho \quad (3-6-1)$$

the term  $(\vec{v} \cdot \vec{\nabla}) \vec{v}$  may be neglected.

Let

$$p = p_0 + p' \quad (3-6-2)$$

$$\rho = \rho_0 + \rho' \quad (3-6-3)$$

where  $p_0$  and  $\rho_0$  are constant equilibrium pressure and density, and  $p'$  and  $\rho'$  are their variations ( $\rho' \ll \rho_0$ ;  $p' \ll p_0$ ).

After substitution of (3-6-3) the continuity equation

$$\frac{\partial \rho}{\partial \tau} + \text{div } \rho \vec{v} = 0 \quad (3-6-4)$$

takes the form

$$\frac{\partial \rho'}{\partial \tau} + \rho_0 \text{div } \vec{v} = 0 \quad (3-6-5)$$

Since  $p'$ ,  $\rho'$ ,  $\vec{v}$  are the quantities of the first order of smallness, and their products are the quantities of the second order of smallness, they may be neglected. Equation (3-6-1) then takes the form

$$\frac{\partial \vec{v}}{\partial \tau} + \frac{\vec{\nabla} p'}{\rho_0} = 0 \quad (3-6-6)$$

It is assumed that the particle velocity  $\vec{v}$  is considerably less than the sound velocity  $c$  ( $v \ll c$ ). This is fulfilled using (3-6-3) as the base. Since  $\rho'/\rho \ll \ll 1$ , then  $v/c \ll 1$ . In an ideal fluid a sound wave under adiabatic conditions and for small variations of  $p'$  and  $\rho'$  is governed by the relation

$$\frac{p'}{\rho'} = \left( \frac{\partial p}{\partial \rho_0} \right)_s \quad (3-6-7)$$

Let us find  $\rho'$  from (3-6-7) and substitute it into (3-6-5)

$$\frac{\partial p'}{\partial \tau} + \rho_0 \left( \frac{\partial p}{\partial \rho_0} \right)_s \text{div } \vec{v} = 0 \quad (3-6-8)$$

Equations (3-6-8) and (3-6-6) completely describe a sound wave motion. The number of variables  $v$  and  $p'$  may be reduced to one if the velocity potential

$$\vec{v} = \vec{\nabla} \varphi \quad (3-6-9)$$

is introduced.

From the Euler equation (3-6-6) we get

$$p' = -\rho_0 \frac{\partial \varphi}{\partial \tau} = -\rho \frac{\partial \varphi}{\partial \tau} \quad (3-6-10)$$

(the subscript at  $p_0$  and  $\rho_0$  is omitted).

Equation (3-6-8) now takes the form

$$\frac{\partial^2 \varphi}{\partial \tau^2} - c^2 \nabla^2 \varphi = 0 \quad (3-6-11)$$

where  $c$  is a certain quantity equal to

$$c = \sqrt{\left( \frac{\partial p}{\partial \rho} \right)_s} \quad (3-6-12)$$

Equation (3-6-11) is a hyperbolic or wave equation. As far as oscillatory motions in fluids are concerned,  $c$  is called the sound velocity.

If we take operation grad in the equation for  $\varphi$ , it may be seen that each of the components of velocity  $\vec{v}$  satisfies the wave equation. Taking the derivative of (3-6-11) with respect to time, we also obtain a hyperbolic equation for  $p'$ , and hence for  $\rho'$ . For a one-dimensional problem it may be written as

$$\frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial \tau^2} = 0 \quad (3-6-13)$$

With new variables defined by

$$\xi = x + a\tau; \quad \eta = x - a\tau \quad (3-6-14)$$

equation (3-6-11) takes the form

$$\frac{\partial^2 \varphi}{\partial \xi \partial \eta} = 0 \quad (3-6-15)$$

Integration of the above equation with respect to  $\xi$  gives

$$\frac{\partial \varphi(\xi, \eta)}{\partial \eta} = F(\eta) \quad (3-6-16)$$

where  $F(\eta)$  is a function of variable  $\eta$  alone.

Integration of (3-6-16) at fixed  $\xi$  yields

$$\varphi(\xi, \eta) = \int F(\eta) d\eta + f_1(\xi) = f_1(\xi) + f_2(\eta) \quad (3-6-17)$$

Hence

$$\varphi(x, \tau) = f_1(x + c\tau) + f_2(x - c\tau) \quad (3-6-18)$$

Distributions of  $p'$ ,  $\rho'$  and  $\bar{v}$  in a plane wave are described by the same kind of functions.

Let the function  $f_1$  for density  $\rho'$  be equal to zero. Then

$$\rho' = f_2(x - c\tau) \quad (3-6-19)$$

In each plane  $x = \text{const}$  the density varies with time.

At any moment the density is different for different  $x$ . The density is the same provided

$$x - c\tau = \text{const}$$

If at a certain moment  $\tau = 0$  at point  $x_1$  the density has a definite value, then after time  $\tau$  the density will have the same value at  $x = c\tau$ .

Thus,  $f_2(x - c\tau)$  is a running plane wave, which propagates in the positive direction of  $x$ . It is obvious that  $f_1(x + c\tau)$  is a wave propagating in the opposite (negative) direction of  $x$ . The velocity in a sound wave

$$v_x = \frac{\partial \varphi}{\partial x} \quad (3-6-20)$$

is directed along  $x$ , i.e. in the direction of wave propagation. Sound waves in the fluid are, therefore, longitudinal. The relation between  $\bar{v}$  and  $c$  may be found.

We have

$$\varphi = f(x - c\tau); \quad v_x = \frac{\partial \varphi}{\partial x} = v = f'(x - c\tau) \quad (3-6-21)$$

$$p' = -\rho \frac{\partial \varphi}{\partial \tau} = \rho c f'(x - c\tau) \quad (3-6-22)$$

Hence

$$v = p' / \rho c \quad (3-6-23)$$

Since according to (3-6-7) and (3-6-12)  $p' = c^2 \rho'$ , then

$$v = \frac{c \rho'}{\rho} \quad (3-6-24)$$

The temperature oscillation  $T$  is found from

$$\frac{T'}{p'} = \left( \frac{\partial T}{\partial p} \right)_s \quad (3-6-25)$$

It is known that

$$\left( \frac{\partial T}{\partial p} \right)_s = \frac{T}{c_p} \left( \frac{\partial V}{\partial T} \right)_p \quad (3-6-26)$$



where  $V$  is the volume;  $c_p$  is the fluid heat capacity at constant pressure. If the thermal expansion coefficient is denoted by  $\beta$

$$\beta = \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_p \quad (3-6-27)$$

then

$$T' = \frac{c\beta T}{c_p} v \quad (3-6-28)$$

Here it is desirable to find sound velocity in an ideal gas. From thermodynamics it is known that

$$\left( \frac{\partial p}{\partial \rho} \right)_s = \frac{c_p}{c_v} \left( \frac{\partial p}{\partial \rho} \right)_T \quad (3-6-29)$$

Using the ideal gas equation

$$p = \rho RT/M$$

we find

$$c = \sqrt{\left( \frac{\partial p}{\partial \rho} \right)_s} = \sqrt{\frac{c_p RT}{c_v M}} = \sqrt{\gamma \frac{RT}{M}} \quad (3-6-30)$$

where  $\gamma = c_p/c_v$  is the ratio between isobaric and isochoric heat capacities;  $M$  is the molecular mass and  $R$  is the universal gas constant.

The sound velocity in gases is almost of the same order as the mean thermal velocity of molecules.

Let us now demonstrate that in a viscous heat conducting fluid in a constant gravity field, weakly attenuating thermal convective waves may propagate.

In an isothermal fluid, in which gravity forces are not accounted for, transverse and temperature waves are strongly attenuated [3-59]. At a distance equal to wave length, the amplitude of these waves decreases by  $\exp(2\pi) \approx 540$  times. This is, essentially, an aperiodic motion which should be called oscillatory rather than wave motion.

The situation with nonisothermal fluids is quite different. Let a temperature gradient exist in the direction of the gravity field. At mechanical equilibrium the gradient should have a constant value. If  $\nabla T$  has the same direction as the gravity field, then there exists a critical absolute value of the gradient above which convection takes place [3-60]. The conditions necessary for mechanical equilibrium are assumed to be fulfilled.

We shall examine how small disturbances of temperature, velocity, pressure and density propagate against the background of this mechanical equilibrium with a constant temperature gradient. The following typical example reveals the basic particularities.

Let us assume that fluid occupies the right-hand half-space  $x \geq 0$  and is bounded by plane  $x = 0$ . The gravity field  $\vec{g}$  has the direction antiparallel to  $y$ . Axes  $x$  and  $y$  are assumed to be mutually perpendicular. Along the  $y$ -direction within the whole half-space  $x \geq 0$  there exists a constant temperature gradient  $\partial T_0 / \partial y = \gamma$ . Let the boundary surface oscillate in its own plane along the  $y$ -axis with frequency  $\omega$ , and the surface temperature vary with time following a harmonic law. Find the resulting steady motion and the temperature distribution in the fluid. This problem is a typical two-dimensional problem of simultaneous free and forced convection and is described by the following constitutive equations

$$\left. \begin{aligned} \rho \left[ \frac{\partial \vec{v}}{\partial \tau} + (\vec{v} \cdot \nabla) \vec{v} \right] &= - \nabla p + \eta \nabla^2 \vec{v} + \rho \vec{g} \\ \rho c_p \left[ \frac{\partial T}{\partial \tau} + \vec{v} \cdot \nabla T \right] &= \lambda \nabla^2 T + \eta \Phi; \quad \frac{\partial \rho}{\partial t} + \text{div} (\rho \vec{v}) = 0 \\ f(\rho, p, T) &= 0 \end{aligned} \right\} \quad (3-6-31)$$

for the boundary conditions

$$x = 0 \left\{ \begin{aligned} T &= \theta_0 \cos \omega t, \quad v_i = v_{i0} \cos \omega t \\ \rho &= m_0 \cos \omega t, \quad p = \Pi_0 \cos \omega t \end{aligned} \right. \quad (3-6-32)$$

$$\lim_{x \rightarrow \infty} (T, v_i, \rho, p) = 0 \quad (3-6-33)$$

Boundary conditions (3-6-32) include possible periodic oscillations of density, pressure and all the velocity components. If the amplitudes of temperature  $T$  and velocity  $\vec{v}$  oscillations are small, the solution of equations (3-6-31) through (3-6-33) may naturally be sought as

$$\left. \begin{aligned} T &= T_0 + T'(x, t); \quad v_i = v'_i(x, t) \\ \rho &= \rho_0 + \rho'(x, t); \quad p = p_0 + p'(x, t) \end{aligned} \right\} \quad (3-6-34)$$

The subscript 0 indicates equilibrium distributions of temperature, density and pressure;  $T'$ ,  $\rho'$ ,  $p'$  are small disturbances of these parameters. The velocity  $v'_i$  is assumed to be of the same order of smallness. By substituting the series (3-6-34) into equations (3-6-31) through (3-6-33), neglec-

ting small quantities of higher than the first order, and omitting the superscript (prime), we get the following linear system of equations

$$\frac{\partial T}{\partial \tau} + \gamma v_y = a \frac{\partial^2 T}{\partial x^2}; \quad \gamma = \frac{\partial T_0}{\partial y}; \quad a = \frac{\lambda}{\rho_0 c_p} \quad (3-6-35)$$

$$\frac{\partial v_y}{\partial \tau} = \nu \frac{\partial^2 v_y}{\partial x^2} - \beta T; \quad \beta = \rho_0^{-1} g_y \left( \frac{\partial \rho}{\partial T} \right)_{T=T_0} = \text{const} \quad (3-6-36)$$

$$\rho = T \left( \frac{\partial \rho}{\partial T} \right)_{T=T_0} \quad (3-6-37)$$

$$\frac{\partial \rho}{\partial \tau} + \rho_0 \frac{\partial v_x}{\partial x} = 0 \quad (3-6-38)$$

$$\frac{\partial v_x}{\partial \tau} = - \frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 v_x}{\partial x^2} \quad (3-6-39)$$

$$x = 0 \begin{cases} T = \theta_0 \exp(i\omega\tau); & v_y = v_{y0} \exp(i\omega\tau) \\ v_x = v_{x0} \exp(i\omega\tau); & p = \Pi_0 \exp(i\omega\tau) \end{cases} \quad (3-6-40)$$

$$\lim_{x \rightarrow \infty} (T, v_x, v_y, p) = 0 \quad (3-6-41)$$

where  $v_x, v_y$  are the velocity components along the  $x$ - and  $y$ -axes.

In the derivation of equations (3-6-37) through (3-6-41) density variations with pressure were neglected. A constant equilibrium density distribution  $\rho_0$  was also assumed. For a negative temperature gradient  $\gamma < 0$  this restriction is probably unessential. The density variation due to temperature is compensated for, in this case, by the change in density due to pressure. In the reverse case when both fields have the same direction the equations (3-6-35)-(3-6-39) are valid for relatively large regions along  $y$ -axis.

A steady periodic solution of equations (3-6-35)-(3-6-39) will be sought in the form of plane waves

$$\begin{cases} T = \theta \exp i(\omega\tau - Kx); & v_y = v_{y0} \exp i(\omega\tau - Kx) \\ v_x = v_{x0} \exp i(\omega\tau - Kx); & p = \Pi \exp i(\omega\tau - Kx) \end{cases} \quad (3-6-42)$$

The first three equations may be integrated independently, which results in the following dispersion equation relating wave vector  $K$  and frequency  $\omega$

$$K^4 a \nu + K^2 i \omega (\nu + a) - \omega^2 - \beta \gamma = 0 \quad (3-6-43)$$

For the waves to attenuate at infinity, it is necessary that the wave vector should be complex, i.e.  $K = K_1 + iK_2$ , and the imaginary part of the

wave vector should be negative ( $K_2 < 0$ ). The frequency is assumed real. Then from (3-6-43) it is easy to get

$$1. \left. \begin{aligned} r = \omega^2(v - a)^2 - 4av\beta\gamma \geq 0 \\ K_1^2 = K_2^2 \end{aligned} \right\} \quad (3-6-44)$$

$$\left. \begin{aligned} K_2 = - \left\{ A \pm (4av)^{-1} r^{\frac{1}{2}} \right\}^{\frac{1}{2}}, \quad \beta\gamma > 0 \\ K_2 = - \left\{ \pm A + (4av)^{-1} r^{1/2} \right\}^{1/2}, \quad \beta\gamma < 0, \quad |\beta\gamma| \geq \omega^2 \end{aligned} \right\} \quad (3-6-45)$$

$$A = \frac{\omega(v + a)}{4av} \quad (3-6-46)$$

$$2. \quad r < 0; \quad K_1 K_2 = -A \quad (3-6-47)$$

$$K_2 = - \left\{ \pm B + (A^2 + B^2)^{1/2} \right\}^{\frac{1}{2}} \quad (3-6-48)$$

$$B = \mp (4av)^{-1} (-r)^{\frac{1}{2}} \quad (3-6-49)$$

From expressions (3-6-44)-(3-6-48) it is evident that essentially different types of temperature waves may propagate in fluids depending on the absolute value and the sign of  $\beta\gamma\omega^{-2}$ . For all frequencies  $\omega$ , at  $\beta\gamma < 0$  and for frequencies  $\omega \geq \frac{2(av\beta\gamma)^{1/2}}{|v - a|}$  when  $\beta\gamma > 0$ , as for isothermal fluids without consideration of gravity forces, only the strongly attenuated waves propagate [(3-6-45)-(3-6-47)]. The penetration (attenuation) depth  $L = |K_2^{-1}|$  is of the same order as the wave length  $l = |2\pi K_1^{-1}|$ . This case is not of much interest.

Propagation of waves with frequencies  $\omega < \frac{2(av\beta\gamma)^{1/2}}{|v - a|}$  in the medium with  $\beta\gamma > 0$  is of great interest. In a fluid whose density decreases as temperature grows,  $\beta < 0$ . It means that the temperature gradient in the chosen coordinate system must be negative, and vice versa. For gases the thermal expansion coefficient is less than zero, i.e. the gravitational field and temperature gradient should be parallel. In this case, as is seen from (3-6-47) and (3-6-48), two waves propagate with given frequency  $\omega$ . One corresponding to the plus sign at  $B$  in expression (3-6-48) is strongly attenuated. The other, whose wave vector components

$$\left. \begin{aligned} K_1 &= A [B (\sqrt{1 + A^2 B^{-2}} - 1)]^{-\frac{1}{2}} \\ K_2 &= - \sqrt{B} (\sqrt{1 + A^2 B^{-2}} - 1)^{\frac{1}{2}} \end{aligned} \right\} \quad (3-6-50)$$

may be weakly attenuating under certain conditions.

We shall define these conditions. First it is necessary that the absolute value of the attenuation length  $L = |K_2^{-1}|$  be large. It is possible when  $A^2 B^{-2} \ll 1$ . Second, the penetration depth in the scale of wave lengths should be large. To understand the conditions under which this requirement is fulfilled, it is necessary to analyse the expression  $Ll^{-1}$

$$\frac{L}{l} = \frac{K_1}{2\pi K_2} = \frac{A}{2\pi B (\sqrt{1 + A^2 B^{-2}} - 1)} \quad (3-6-51)$$

With allowance made for the constraint  $A^2 B^{-2} \leq 1$ , we simplify the expression (3-6-51)

$$\frac{L}{l} \approx \frac{[4a\nu\beta\gamma - \omega^2(\nu - a)^2]^{1/2}}{\pi\omega(\nu + a)} \quad (3-6-52)$$

It may easily be seen that the ratio  $L/l$  is directly proportional to the thermal expansion coefficient and the temperature gradient, and inversely proportional to frequency. The greater the ratio, the more will be the coefficient of thermal expansion and the temperature gradient. The lesser the frequency the closer will be the values of kinematic viscosity coefficients and thermal diffusivity. The absolute value of the temperature gradient, as mentioned earlier, has a maximum; therefore, a finite upper limit of frequencies exists for waves propagating with small attenuation,  $\omega < \sqrt{\beta\gamma}$ . As shown by calculations, low frequency oscillations are weakly attenuating. We shall mention some numerical estimates. The passing of high frequencies is best achieved in fluids with  $c\nu \approx a$ . Air is one such fluid

$$\nu = 13 \times 10^{-6} \text{ m}^2/\text{s}; \quad a = 19 \times 10^{-6} \text{ m}^2/\text{s}$$

For a temperature gradient of  $10^2$  deg/m at frequency of  $10^{-3}$  Hz, the penetration depth is 10 m or  $10^3$  wave lengths.

Thus, in nonisothermal fluids with temperature gradients parallel to the gravitational field (or antiparallel, if  $\beta > 0$ ) the weakly attenuating temperature waves may propagate in the direction normal to  $\vec{g}$  and  $\nabla T$ .

Amplitudes and phases of these waves depend on conditions at the boundary surface. In order to satisfy boundary conditions (3-6-40), the solution will be sought as the sum of weakly and strongly attenuating waves with frequency  $\omega$  and wave vectors  $K = K_1 + iK_2$ ;  $\hat{K} = \hat{K}_1 + i\hat{K}_2$

$$\left. \begin{aligned} T &= \text{Re} \{ \theta \exp i(\omega\tau - Kx) + \hat{\theta} \exp i(\omega\tau - \hat{K}x) \} \\ v_y &= \text{Re} \{ v_y \exp i(\omega\tau - Kx) + \hat{v}_y \exp i(\omega\tau - \hat{K}x) \} \end{aligned} \right\} \quad (3-6-53)$$

where

$$\theta = \theta_1 + i\theta_2; \quad \hat{\theta} = \hat{\theta}_1 + i\hat{\theta}_2;$$

$$v_y = v_{y1} + iv_{y2}; \quad \hat{v}_y = \hat{v}_{y1} + i\hat{v}_{y2}$$

Omitting the details of time consuming calculations, we shall write out actual expressions for temperature and transverse component of velocity

$$T = \theta_1 \cos^{-1} \xi \exp K_2 x \cos (\omega \tau - K_1 x + \xi) \\ + \hat{\theta}_1 \cos^{-1} \hat{\xi} \exp \hat{K}_2 x \cos (\omega \tau - \hat{K}_1 x + \hat{\xi}) \quad (3-6-54)$$

$$v_y = (m\theta_1 - n\theta_2) \cos^{-1} \chi \exp K_2 x \cos (\omega \tau - K_1 x + \chi) \\ + (m\hat{\theta}_1 + n\hat{\theta}_2) \cos^{-1} \hat{\chi} \exp \hat{K}_2 x \cos (\omega \tau - \hat{K}_1 x + \hat{\chi}) \quad (3-6-55)$$

where

$$\theta_1 = \frac{m\theta_0 + v_{y0} \cos \varepsilon}{2m}; \quad \theta_2 = \frac{v_{y0} \sin \varepsilon - n\theta_0}{2m} \quad (3-6-56)$$

$$\hat{\theta}_1 = \frac{m\theta_0 - v_{y0} \cos \varepsilon}{2m}; \quad \hat{\theta}_2 = -\theta_2 \quad (3-6-57)$$

$$\xi = \tan^{-1} \theta_2 \theta_1^{-1}; \quad \hat{\xi} = \tan^{-1} \hat{\theta}_2 \hat{\theta}_1^{-1} \quad (3-6-58)$$

$$\chi = \tan^{-1} \frac{m\theta_2 + n\theta_1}{m\theta_1 - n\theta_2}; \quad \hat{\chi} = \tan^{-1} \frac{m\hat{\theta}_2 - n\hat{\theta}_1}{m\hat{\theta}_1 + n\hat{\theta}_2} \quad (3-6-59)$$

$$m = -\frac{a\sqrt{-r}}{2avr}; \quad n = \omega \frac{a - v}{2\gamma v} \quad (3-6-60)$$

Let us analyse the above expressions. First it is necessary to determine, whether one weakly attenuating wave can propagate and if so, then for which boundary conditions? It turns out that propagation is possible. Let the amplitude of transverse oscillations of the wall ( $x = 0$ ) be  $v_{y0} = m\theta_0 \cos^{-1} \varepsilon$ , and the phase shift between oscillations of temperatures and transverse velocity at the wall be  $\varepsilon = \tan^{-1} nm^{-1}$ . Then as follows from (3-6-56) and (3-6-57),  $\hat{\theta}_1 = \hat{\theta}_2 = 0$  and only weakly attenuating thermal waves propagate

$$\left. \begin{aligned} T &= \theta_0 \exp K_2 x \cos (\omega \tau - K_1 x) \\ v_y &= v_{y0} \exp K_2 x \cos (\omega \tau - K_1 x + \tan^{-1} nm^{-1}) \end{aligned} \right\} \quad (3-6-61)$$

The maximum amplitude of thermal and velocity oscillations coincides with the amplitude of vibrations at the boundary surface, and the phase shift between thermal and velocity waves at any point of the wave coincides with the phase shift at the wall.

Such boundary conditions may be selected when a weakly attenuating wave is suppressed and the whole energy at the boundary is spent only in generating a strongly attenuating wave. This takes place when

$$v_{y0} = n\theta_0 \sin^{-1} \varepsilon; \quad \varepsilon = -\tan^{-1} nm^{-1}.$$

Two more questions may be of great interest. Which waves are generated only due to temperature oscillations at the wall and only due to oscillations of the boundary surface? It may be easily demonstrated that in both cases both velocity and temperature oscillations are generated, i.e. we may speak of excitation of transverse oscillations by thermal ones and of excitation of temperature oscillations by transverse oscillations of the boundary surface in a nonisothermal fluid present in the gravitational field.

Let the amplitude of temperature oscillations at the wall  $\theta_0$  be zero, then from equations (3-6-54)-(3-6-60) we get

$$T = \frac{v_{y0}}{2m} [\exp K_2 x \cos(\omega\tau - K_1 x) - \exp \hat{K}_2 x \cos(\omega\tau - \hat{K}_1 x)] \quad (3-6-62)$$

$$v_y = v_{y0} (2 \cos \chi)^{-1} [\exp K_2 x \cos(\omega\tau - K_1 x + \chi) - \exp \hat{K}_2 x \cos(\omega\tau - \hat{K}_1 x - \chi)] \quad (3-6-63)$$

where  $\chi = \tan^{-1} nm^{-1}$ .

Now let the wall be at rest ( $v_{y0} = 0$ ). Then expressions (3-6-62) and (3-6-63) remain valid if the wave amplitude is substituted by  $\theta_0(2 \cos \chi)^{-1}$  and  $\theta_0(m^2 + n^2)(2m)^{-1}$ , respectively. In both cases both types of waves (weakly and strongly attenuating) are generated.

Up to now we have spoken of oscillations of temperature and transverse velocity. Substituting expressions for  $T$  and  $v_y$  into the equation of state (3-6-37), continuity equation (3-6-38) and the equation of motion (3-6-39), we may easily find strongly and weakly attenuating waves of density, longitudinal velocity and pressure. All the temperature, velocity, sound and density waves propagating in nonisothermal fluids in a gravity field only due to thermal compressibility will be called thermoconvective waves. The above predicted and analysed weakly attenuating thermal waves are certainly of the greatest interest among thermoconvective waves.

# CONJUGATE HEAT-TRANSFER PROBLEMS

## 4

### 4-1. PHYSICAL BASIS OF CONJUGATE HEAT-TRANSFER PROBLEMS

Convective heat-transfer processes are of great importance in many fields of modern engineering. These processes play an important role in thermal engineering, atomic power engineering and other branches of modern engineering. Heat-transfer problems are increasingly important in design of modern high-speed aircrafts due to considerable aerodynamic heating of the structure.

In analytical studies of heat transfer in solids in a fluid or gas flow at the solid-fluid interface, the so-called third kind boundary conditions are generally considered. These are proportionality conditions for the heat flux at the wall  $q(x, \tau)$ , and the difference between wall temperature and a characteristic temperature of the main flow ( $T_w - T_\infty$ ). In addition, a proportionality factor  $\alpha(x, \tau)$ , known as the heat-transfer coefficient is introduced.

$$q(x, \tau) = \alpha(x, \tau) (T_w - T_\infty) = \text{Nu}(x, \tau) \frac{\lambda}{l} (T_w - T_\infty) \quad (4-1-1)$$

where  $l$  is the characteristic dimension and  $\lambda$  is the thermal conductivity of the fluid.

This relation is usually called Newton's law.

The heat-transfer coefficient is found both analytically (from the solution of boundary layer equations) and experimentally.

During theoretical calculations it is generally assumed that the conditions at the wall are preassigned and constant (this makes it possible to consider the problem as a similarity one which facilitates the solution). It may be noted that the wall temperature, for example, may be assumed constant (independent of space coordinates) only in the exceptional case when the thermal conductivity of the solid body is infinitely large. But, often in practice, the surface temperature of the body in a flow cannot be assumed constant. This primarily refers to high-rate heat-transfer processes (for example, when a body is in a flow having a temperature considerably different from that of the body).

The problem of heat transfer between the solid and fluid under constant wall conditions has been studied adequately both theoretically and experi-



mentally, whereas for variable surface temperature it has not been studied in detail. An attempt in this direction was first made in [4-1]. The author studied heat transfer in a longitudinal plate streamlined by a laminar supersonic compressible gas flow. In this case the wall temperature was given by a variable as a power series. The parabolic relation between the wall temperature and the longitudinal coordinate was considered

$$T_w = T_e + T_e(0.25 - 0.83x + 0.33x^2) \quad (4-1-2)$$

At this temperature of the surface in a flow, local values of the heat flux, heat-transfer coefficient and Nusselt number were obtained. It was discovered that there were points at which the heat-transfer coefficient was negative or had an infinite discontinuity which, of course, is physically unacceptable. An explanation of such contradicting results is given in [4-2] where an incompressible fluid flow past a plate has been considered. A qualitative analysis of temperature distribution in a boundary layer is presented therein with the condition that surface temperature varies according to the preassigned law (Fig. 4-1). It may be concluded that at the leading

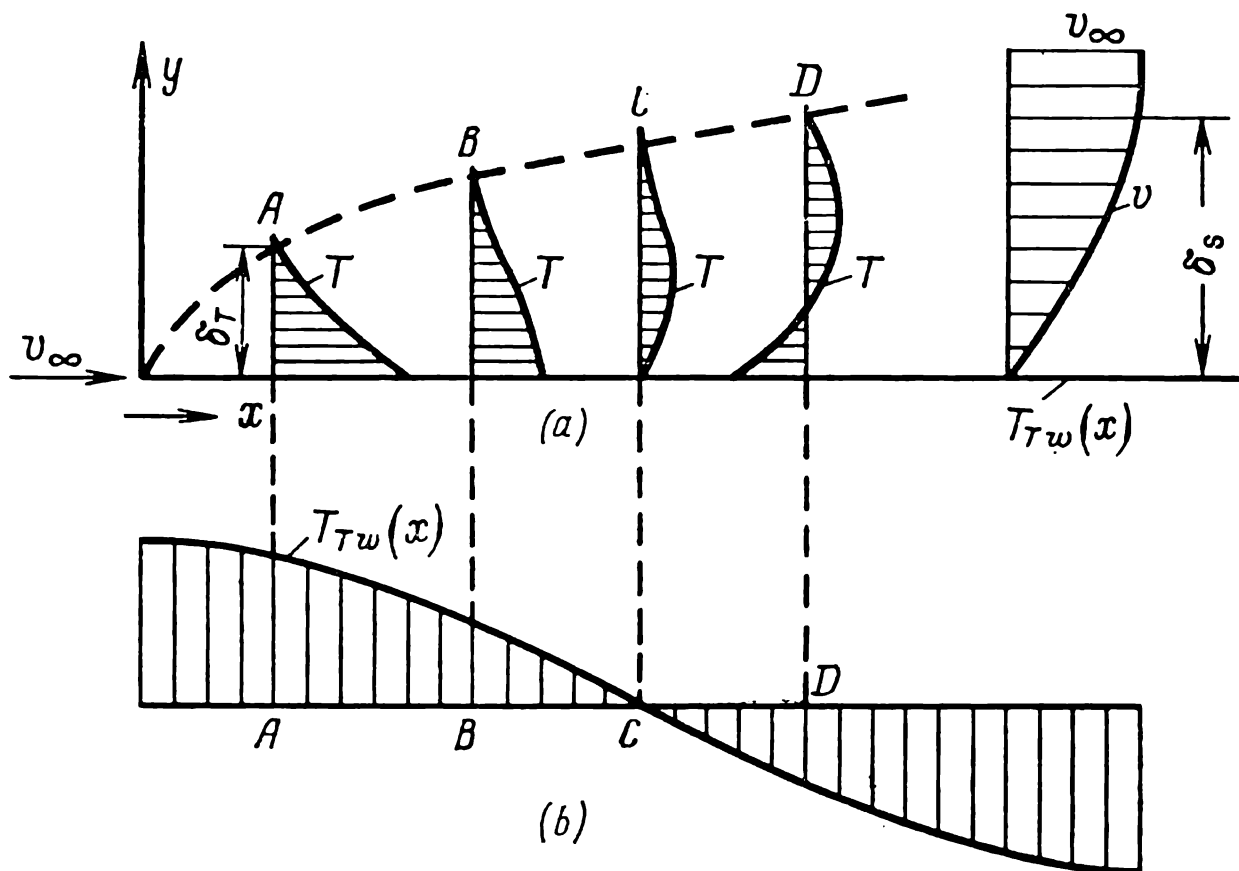


Fig. 4-1. Profile of temperature in a boundary layer (a) and distribution of temperature along the solid surface (b)

edge the temperature profile is almost of type A, which is similar to the ordinary profile for a constant wall temperature (Fig. 4-1). A decrease in the wall temperature down the flow ( $\partial T_w / \partial x < 0$ ) produces an effect, primarily, in that part of the boundary layer which is close to the surface

in a flow. The external layers are cooled much farther down the flow. This results in  $(\partial T/\partial y)_w > 0$  at point  $C$  where the wall temperature is the same as that of the external flow, i.e. its sign is opposite to that of the gradient at point  $A$ . The heat flux at the wall may be written in two forms in the Fourier form

$$q = -\lambda \left( \frac{\partial T}{\partial y} \right)_w \quad (4-1-3)$$

and in the Newton form

$$q = \alpha(T_w - T_\infty) = \text{Nu} \frac{\lambda}{l} (T_w - T_\infty) \quad (4-1-4)$$

At point  $C$ , when equation (4-1-3) is used, it appears that heat is transferred from the fluid to the wall, though the wall temperature is the same as the ambient temperature, and hence, from relation (4-1-4), the heat flux equals zero.

In the downflow region from point  $C$  the fluid temperature near the wall proves to be lower, and upon moving away from the surface, greater than at the wall. Then point  $B$  should exist between the points  $A$  and  $C$  for which  $(\partial T/\partial y)_w = 0$ , i.e. from equation (4-1-3) the heat flux equals zero, whereas from equation (4-1-4) it does not go to zero. Over the whole length  $BC$  heat is transferred from the fluid to the wall, although  $T_w > T_\infty$ . In a formal consideration, if equation (4-1-4) were used, we would have been forced to take  $\text{Nu} < 0$  in this interval and at  $C$   $\text{Nu} \rightarrow \infty$ . Based on the aforementioned, we may conclude that for a variable temperature of the solid surface in a flow, the Newton law of cooling does not hold and therefore the Nusselt numbers found from equation (4-1-4) are physically senseless. Reference [4-3] contains a review of works on heat transfer for variable wall conditions. The results presented therein are similar to those considered earlier. In particular, the Nusselt numbers for constant and variable conditions at the boundary differ widely.

In all the earlier mentioned works it is demonstrated that for prescribed variable conditions at the solid surface of the body (close to reality) the use of Newton's law and, consequently, of the heat-transfer coefficient cannot be allowed. However, a law describing the dependence of the wall temperature on time and coordinates cannot be prescribed *a priori*, but should be obtained by simultaneous solution of equations for heat propagation in a fluid and solid and equations of motion, the temperatures and heat fluxes at the solid-fluid interface being equal. This is the so-called conjugate heat-transfer problem [4-4], [4-5]. This formulation includes the mutual thermal effect of the solid and fluid, which was ignored earlier. As a result heat transfer proves to be independent of the properties of the body, of its thermophysical properties, sizes, distribution of sources in the body, etc., that, evidently, contradicts the physical meaning. This

is of particular importance to consider heat-transfer problems as conjugate ones in the case of unsteady-state heat transfer. Even in the extreme case when the thermal conductivity of the solid is very large ( $\lambda_s \rightarrow \infty$ ), the surface temperature cannot be assumed constant since, although it is independent of the coordinates of the surface points, it varies with time. However, unlike the steady-state heat transfer, even in this extreme case a law governing surface temperature variation with time cannot be prescribed, and hence almost all the problems of unsteady-state convective heat transfer should be formulated as conjugate ones.

At present a few papers using this formulation have been published.

Boundary conditions of the fourth kind for external heat-transfer problems have been known for a long time. Mention may be made of the studies on free convection by G. A. Ostroumov and his associates [4-6]; it should be emphasized that the solution methods considered by them are approximate. These are linearization by series expansion with respect to parameter  $Gr Pr$  which is large in some real cases; the resulting series may therefore be divergent.

In [4-4] the problem of unsteady-state heat transfer in an incompressible flow through a rectangular pipe with a semi-infinite solid inside has been considered. Despite simplifying assumptions (constant flow velocity, averaging of the energy equation along the flow), the obtained solution gives a qualitative picture of the process. In [4-21] conjugate heat-transfer problems in circular and plane ducts are solved both analytically and numerically.

The external conjugate heat-transfer problem was originally stated in [4-4], which demonstrated usefulness of such a formulation. A. A. Pomerantsev [4-7] considered a gas flow over a solid wall; the wall temperature was prescribed by the power law. The problem was solved with simplifying assumptions, such as averaged temperature along the transverse coordinate and neglected thermal conductivity along the longitudinal coordinate. T. L. Perelman [4-5] considered a steady-state problem of an incompressible flow over a plate with heat sources. The problem was solved with a constraint for the Prandtl numbers ( $Pr \leq 1$ ). In [4-8] a steady-state problem of a gas flow over a sheet (which is a very limiting condition) was solved.

A. Zargary [4-11] derived a generalized Laplace transformation theorem to solve the conjugate problems of convective heat transfer in pipes. Numerical methods for solving conjugate problems have been developed in [4-10], [4-13]. Papers [4-13] and [4-14] contain solutions of conjugate problems on free convection. Recently a number of papers dedicated to conjugate problems [4-15], [4-16] have appeared.

Solution of conjugate heat-transfer problems involves principle mathematical difficulties. The main difficulty lies in solving a system of partial differential equations which have different forms in different intervals. Steady-state problems involve differential equations of different types: for fluids partial differential equations are of a parabolic type while for

solids they are of elliptical type. The most reasonable approach to the solution of conjugate problems is to introduce at the conjugation boundary an unknown function equal to the temperature or heat flux at that boundary, which allows the system of partial differential equations to be reduced to two "independent" boundary-value problems. The unknown function is subsequently determined from the remaining conjugation conditions.

For external conjugate problems both steady-state and unsteady-state asymptotic solutions were obtained for a supersonic gas flow over a plate. It should be pointed out that for the considered external problems no exact solutions can be obtained since the boundary layer equations do not hold at the leading edge, therefore all the solutions based on the boundary layer theory are asymptotic.

Despite the large number of analytical solutions of conjugate problems, very few relevant experimental works are available. In [4-14] the effect of a plate temperature field on the temperature field and velocity profile in a boundary layer has been convincingly demonstrated (see Figs. 4-2, 4-3). Here, the following notations are used:  $X$  and  $Y$  are dimensionless coordinates ( $X = x/l$ ,  $Y = y/l$ ),  $\theta$  is the dimensionless temperature

$$\theta = \frac{g\beta l^3(T - T_\infty)}{\nu^2} \quad (4-1-5)$$

where  $\beta$  is the volume expansion coefficient of the gas.

Figures 4-2 and 4-3 show that the finite difference and integral solutions coincide and are close to the experimental data.

In Fig. 4-4 the temperature distribution along the plate is shown for different ratios  $\lambda_s/\lambda_f$ . It is seen from this figure that at  $\lambda_s/\lambda_f = 2500$  the

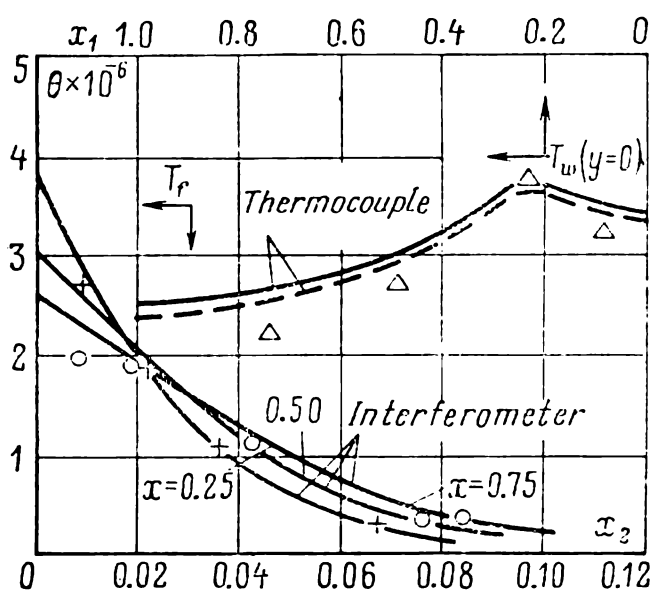


Fig. 4-2. Temperature field in a plate (ceramics) and in the boundary layer of fluid heated from below

$T_f$  — fluid temperature;  $T_w$  — plate temperature; — finite difference solution; - - - integral solution

temperature variation is about  $5^\circ\text{C}$  per  $65^\circ\text{C}$ . For large  $\lambda_s/\lambda_f$  the plate thermal conductivity plays a leading role, and thermal interaction mainly occurs through the plate. At  $\lambda_s/\lambda_f = 1$  (thermal conductivity of the plate

is very small) thermal interaction mainly occurs through the boundary layer, and a greater part of the heat from the sources remains in their vicinity.

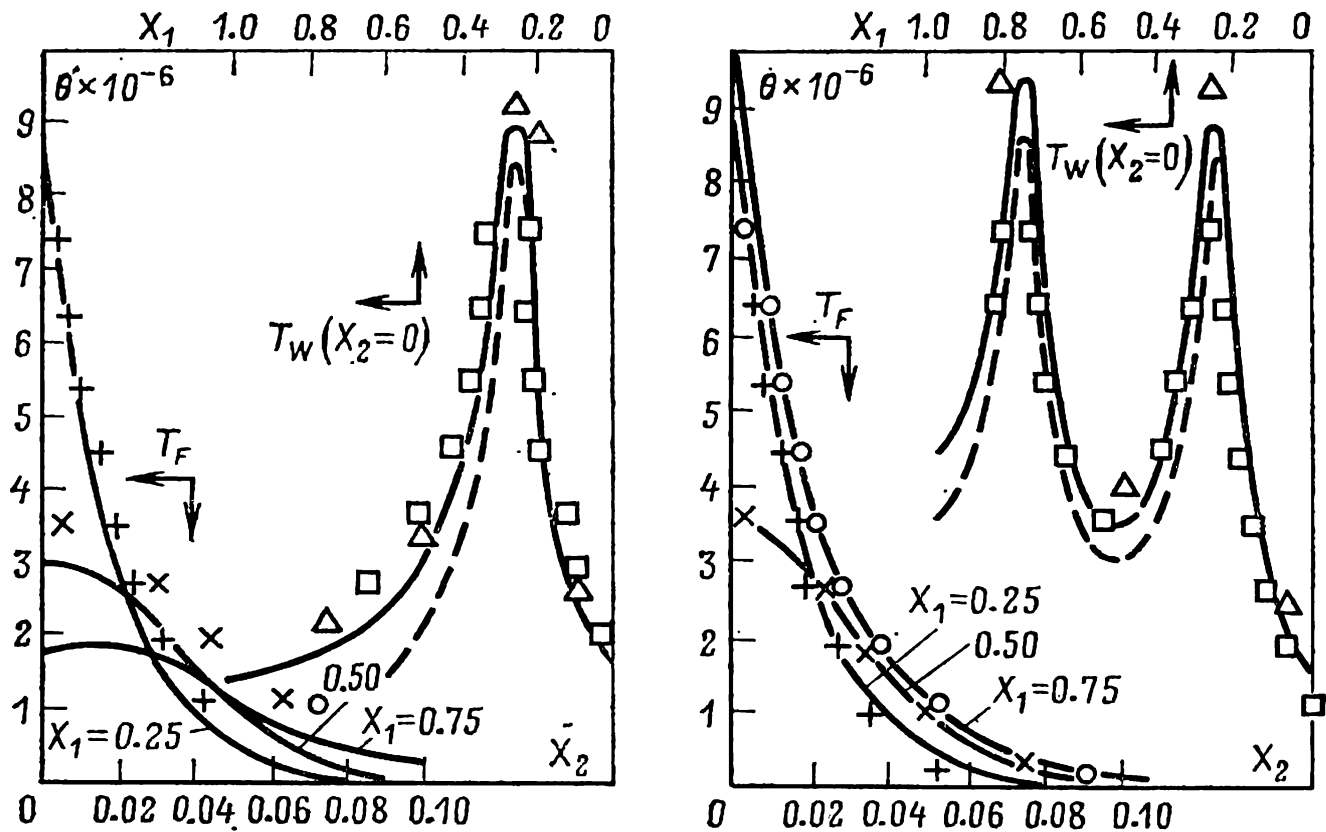


Fig. 4-3. Temperature field in a glass plate (heated from below) and in the fluid boundary layer. Notations as in Fig. 4-2

Figure 4-5 is a plot of the local Nusselt number  $Nu_x$  versus the coordinate  $X$  for ceramic and glass plates. As can be seen from Fig. 4-5,  $Nu_x$  greatly depends on the plate thermal conductivity. At certain values of  $X$

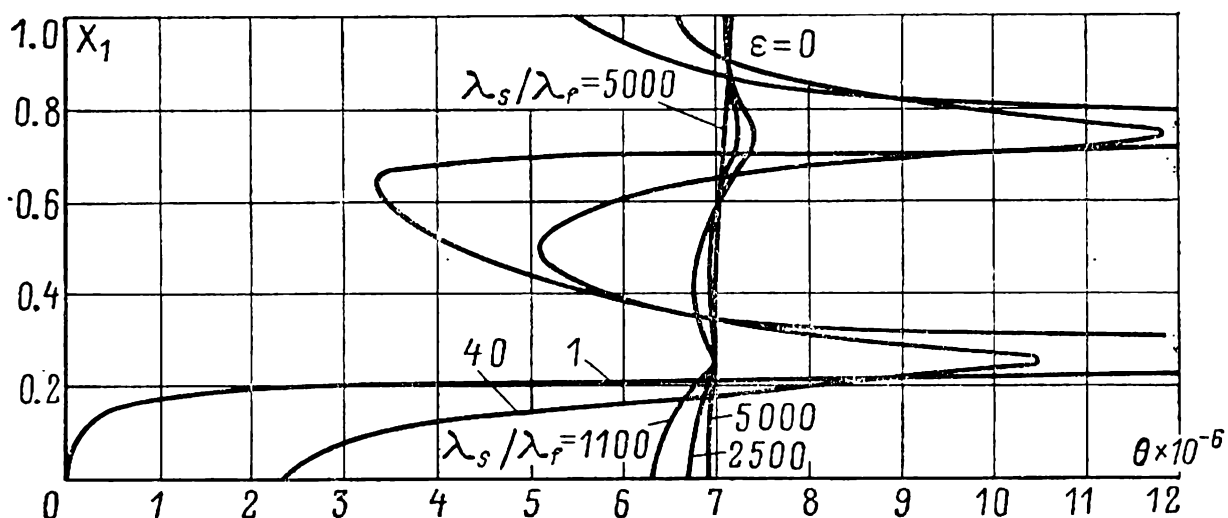


Fig. 4-4. Effect of plate's thermal conductivity on the distribution of vertical plate surface temperature in the absence of radiation

Nusselt number is negative which experimentally verifies the earlier given analysis (see Fig. 4-1). Thus; we have demonstrated both theoretically

and experimentally that a conjugative statement of convective heat-transfer problems is correct. It is important to determine the conditions for which convective heat-transfer problems may be solved by conventional methods

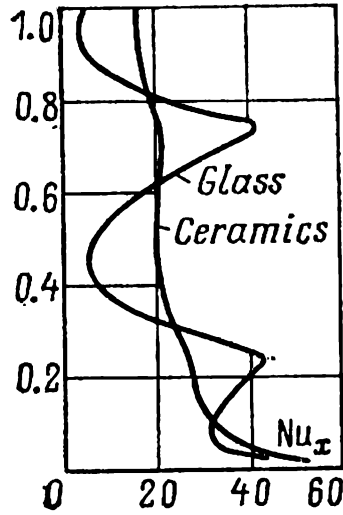


Fig. 4-5 Local Nusselt number (ceramic and glass plates with switched on lower heater) versus  $x$ -coordinate

without accounting for thermal conductivity of bodies in a fluid flow. For this aim the Brun conjugation number (Br) may be recommended.

#### 4-2. CONJUGATION NUMBER

In the solution of convective heat-transfer problems the boundary conditions of the fourth kind are substituted for boundary conditions of the third kind

$$-\lambda_f \left( \frac{\partial T_f}{\partial y} \right)_w = -\lambda_s \left( \frac{\partial T_s}{\partial y} \right)_w + Q_w \quad (4-2-1)$$

$$(T_f)_w = (T_s)_w \quad (4-2-2)$$

where  $Q_w$  is the heat source at the surface of the body; subscripts  $f$  and  $s$  refer to fluid and solid, respectively, and subscript  $w$ , to the surface of the body.

Without heat sources ( $Q_w = 0$ ) we get from boundary condition (4-2-1)

$$\frac{(\Delta T_s)_b}{(\Delta T_f)_{\delta'_T}} = \frac{\lambda_f}{\lambda_s} \frac{b}{\delta'_T} \quad (4-2-3)$$

where  $b$  is the wall thickness and  $\delta'_T$  is the conventional thickness of a thermal boundary layer (see Fig. 3-10).

Consequently, the ratio of the temperature drop across the wall  $(\Delta T_s)_b$  to the temperature drop in the fluid boundary layer  $(\Delta T_f)_{\delta'_T}$  depends not only on the ratio of thermal conductivities of the fluid and wall  $\lambda_f/\lambda_s$  but also on the ratio of the wall thickness to a conventional thickness of the

thermal boundary layer  $b/\delta'_T$ . Here, it should be emphasized that substitution of the surface heat flux at the solid surface, i.e.

$$-\lambda_s \left( \frac{\partial T_s}{\partial y} \right)_w \approx \frac{\lambda_s (\Delta T_s)_b}{b} \quad (4-2-4)$$

holds only for a linear temperature distribution across the wall. This will occur at small thicknesses or large thermal conductivities in the absence of heat sources in the body.

If the temperature distribution across the wall is nonlinear, then according to the similarity theory rules (the theory of generalized variables) we may write

$$\left( \frac{\partial T_s}{\partial y} \right)_w = \varepsilon \frac{(\Delta T_s)_b}{b} \quad (4-2-5)$$

where  $\varepsilon$  is the correction factor for the distortion of temperature profile across the wall, which is a single-valued function of  $(\Delta T_s)_b/b$ .

Thus

$$(\Delta \theta)_b = \frac{(\Delta T_s)_b}{(\Delta T_f)_{\delta'_T}} = \varepsilon \frac{\lambda_f}{\lambda_s} \frac{b}{x} \left( \frac{x}{\delta'_T} \right) \quad (4-2-6)$$

where  $(\Delta \theta)_b$  is the relative temperature drop across the wall. Assumption of a constant temperature at the plate surface corresponds to infinitely large thermal conductivity of the wall ( $\lambda_s \gg \lambda_f$ ) which results in an infinitesimal temperature drop  $(\Delta T_s)_b \rightarrow 0$ .

If the relative temperature drop  $(\Delta \theta)_b < 0.01$ , then the temperature drop in the wall may be neglected with an accuracy of 1% and the wall temperature may be assumed constant  $(T_s)_w = \text{const.}$

One can see from formula (4-2-6) that the relative temperature drop across the wall depends not only on the ratio of thermal conductivities and the wall thickness, but also on the thickness  $\delta'_T$  of conventional boundary layer, which in its turn depends on the fluid velocity, viscosity, and varies in the flow direction (along the  $x$  axis).

According to formula (3-1-86)  $x/\delta'_T$  is the local Nusselt number ( $\text{Nu}_x = x/\delta'_T$ ). The local Nusselt number  $\text{Nu}_x$  is defined by an approximate relation of the form

$$\text{Nu}_x = A_x \text{Pr}^m \text{Re}_x^n$$

where  $A_x$ ,  $m$  and  $n$  are constants.

Then

$$(\Delta \theta)_b = A_x \varepsilon \frac{b}{x} \frac{\lambda_f}{\lambda_s} \text{Pr}^m \text{Re}_x^n \quad (4-2-7)$$

The relative temperature drop  $(\Delta\theta)_b$  is thus a single-valued function of dimensionless quantity  $[(\lambda_f/\lambda_s) \text{Pr}^m \text{Re}_x^n]$  called the Brun number \*

$$(\Delta\theta)_B = f(\text{Br}_x) \quad (4-2-8)$$

where  $\text{Br}_x$  is the local Brun number

$$\text{Br}_x = \frac{\lambda_f b}{\lambda_s x} \text{Pr}^m \text{Re}_x^n \quad (4-2-9)$$

The local Brun number is a conjugation criterion.

In the case of a small  $\text{Br}_x$  ( $\text{Br}_x < \text{Br}_{x \min}$ ) convective heat-transfer problems may be solved by the conventional procedure without taking into account conjugation with the temperature field within the wall.  $\text{Br}_{x \min}$  may be found from exact analytical solutions or experimentally. According to formula (4-2-7), it depends on the flow pattern (laminar or turbulent). In a turbulent flow  $\text{Br}_x$  is larger than in a laminar flow, the ratio  $\lambda_f b / \lambda_s x$  being equal. Therefore  $\text{Br}_{x \min}$  should be determined depending on the flow pattern.

The mean Brun number is

$$\text{Br} = \frac{2\lambda_f b}{\lambda_s l} \text{Pr}^m \text{Re}_l^n \quad (4-2-10)$$

$\text{Br}_{\min}$  may be estimated approximately. Let us assume that  $(\Delta\theta)_b \leq 0.05$  (accuracy is 5%), then with a linear temperature distribution across the wall ( $\varepsilon = 1$ ) and a laminar flow over a plate ( $A = 0.662$ )  $\text{Br}_{\min} < 0.08$ , or approximately  $\text{Br}_{\min} < 0.1$ . In [4-15] it was found from an analytical solution that in a laminar air flow over a flat plate the convective heat-transfer problems should be solved as conjugate ones if the dimensionless parameter  $\kappa$  is more than unity ( $\kappa \geq 1$ ). Parameter  $\kappa$  and the Brun number  $\text{Br}_x$  are related by

$$\text{Br}_x = 2\text{Pr}^{1/3} \xi^{-1/2} \kappa = 2\text{Pr}^{1/3} \left( \frac{b}{x} \right)^{1/2} \kappa \quad (4-2-11)$$

For air ( $\text{Pr} = 0.7$ ) at  $b/x = 0.01$ ; for  $\kappa \geq 1$ ,  $\text{Br}_x$  will be more than 0.18 ( $\text{Br}_x \geq 0.18$ ).

Evaluation of  $\text{Br}$  for a laminar air and water flow over flat plates of various materials is of interest. Let  $b/x = 0.05$ , then for a steel plate [ $\lambda_s = 46 \text{ W/(m} \cdot ^\circ\text{C)}$ ] in a water flow [ $\lambda_f = 0.68 \text{ W/(m} \cdot ^\circ\text{C)}$ ] at  $\text{Re} = 10^6$ ,  $\text{Br} = 1.8$ . For the same plate in an air flow [ $\lambda_f = 0.028 \text{ W/(m} \cdot ^\circ\text{C)}$ ]  $\text{Br} = 0.074$ , i.e. much less than 0.1.

For a glass plate [ $\lambda_s = 0.96 \text{ W/(m} \cdot ^\circ\text{C)}$ ] in a water flow at  $\text{Re} = 10^6$ ,  $\text{Br} = 87$  and at  $\text{Re} = 10^4$ ,  $\text{Br} = 8.7$ . If a glass plate is in an air flow

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\* E. Brun — a famous French scientist.



at  $Re = 10^6$ , then  $Br = 4.8$ , and at  $Re = 10^4$   $Br = 0.48$ . Thus, the problem of a glass plate in an air or water flow should be solved as a conjugate one.

#### 4-3. APPROXIMATE SOLUTIONS OF A PROBLEM FOR A PLATE IN A LAMINAR FLOW

As an approximate solution of a conjugate problem for a laminar flow over an infinitely long plate, we may use the solution of problems dealing with a flow over a porous plate with retraction of the evaporation surface. In this case the distance  $\xi$  of the evaporation surface from the plate surface is taken as the plate thickness ( $\xi = b = \text{const}$ ). The evaporation surface temperature  $T_e$  will be equal to the temperature of one of the plate surfaces  $T_b = (T_e = T_b = \text{const})$ . Moreover, evaporation is absent ( $j_1 = 0$ ,  $B_2 = 0$ ,  $B = B_1$ ). The other notations are the same as before. The problem is thus asymmetric, i.e. one of the plate surfaces ( $y = 0$ ) is streamlined by a laminar flow of heated fluid, the opposite surface ( $y = -b$ ) is maintained at a constant temperature  $T_b$  (boundary conditions of the first kind). Solution of the problem is given by (3-2-46)-(3-2-48).

The coefficient  $B$  depends on  $Pr$  alone ( $B = 0.53 \sqrt{Pr}$ ) and the coefficient  $K$  is inversely proportional to  $Br_x$

$$\frac{1}{K} = 0.81 Pr^{\frac{1}{6}} Br_x \quad (4-3-1)$$

For air ( $Pr = 0.7$ )  $Br_x = 1.32 (1/K)$ .

Formulas (3-2-59)-(3-2-60) may then be used. Here  $N^*$  is the ratio of the local Nusselt number  $Nu_x$  of the conjugate problem to the local Nusselt number  $Nu_{x0}$  of the ordinary problem, i.e. when  $Br_x = 0$  (the temperature across the plate is uniform and constant  $T_e = T_w = \text{const}$ ).

Hence at small  $Br_x$  ( $Br_x < 1$ ) we get

$$N^* - 1 = 0.81 Pr^{1/3} \beta Br_x \quad (4-3-2)$$

where  $\beta$  is taken from Table 3-5. For air ( $\beta = 0.78$ ,  $B = 0.44$ ) we have

$$N^* - 1 = 0.63 Br_x \quad (4-3-3)$$

A comparison of the above expression with a similar relation, obtained from an exact solution, is of interest. If use is made of the plots shown in [4-15] and of expression (4-3-1), we get

$$N^* - 1 = 0.48 Br_x \quad (4-3-4)$$

which is similar to (4-2-14). If we bear in mind that in [4-15] a more general problem, when  $T_e$  varies along the plate length, was solved:  $T(-b, x) = T_e(x)$ .

At large  $Br_x$  ( $Br_x > 1$ ) formula (3-2-60) may be used;  $(N^* - 1)$  will then be a linear function of  $Br_x^{-1}$ .

The stated problem may be solved using integral equation (3-1-75).

The temperature and velocity profiles in the boundary layer are assumed in the form of a semi-cubic parabola

$$\vartheta = \vartheta_w + \frac{3}{2} \frac{\vartheta_\infty - \vartheta_w}{\delta_T} y - \frac{1}{2} \frac{\vartheta_\infty - \vartheta_w}{\delta_T^3} y^3 \quad (4-3-5)$$

$$\frac{v_x}{v_\infty} = \frac{3}{2} \frac{y}{\delta} - \frac{1}{2} \left( \frac{y}{\delta} \right)^3 \quad (4-3-6)$$

where  $\vartheta = T_f - T_b$ ;  $\vartheta_w = T_w - T_b$ ;  $T_w$  is the temperature of the plate surface exposed to the fluid flow [ $T_w = T(0x)$ ];  $T_f$  is the fluid temperature.

Temperature distribution across the plate is assumed to be linear

$$\vartheta' = \vartheta_w \left( 1 + \frac{y}{b} \right) \quad (4-3-7)$$

where  $\vartheta' = T_s - T_b$ ;  $T_s$  is the plate temperature.

Substitution of expressions (4-3-5) and (4-3-6) into the integral boundary layer equation

$$\frac{d}{dx} \int_0^{\delta_T} (\vartheta_\infty - \vartheta) v_x(y) dy = a \left( \frac{d\vartheta}{dy} \right)_{y=0} \quad (4-3-8)$$

yields the differential equation

$$\frac{3}{20} \vartheta_\infty v_\infty \frac{d}{dx} \left\{ \frac{1}{(1+z)'} \frac{\delta_T^2}{\delta} - \frac{1}{14} \frac{\delta_T^4}{\delta^3} \right\} = \frac{3}{2} a \frac{\vartheta_w}{\delta_T(1+z)} \quad (4-3-9)$$

where

$$z = 3\lambda_f b / 2\lambda_s \delta_T \quad (4-3-10)$$

In the derivation of equation (4-3-9) use is made of the fourth kind boundary condition

$$\frac{3}{2} \lambda_f \frac{\vartheta_\infty - \vartheta_w}{\delta_T} = \lambda_s \frac{\vartheta_w}{b} \quad (4-3-11)$$

from which we get the expression for unknown temperature  $\vartheta_w$

$$\vartheta_w = \frac{\vartheta_\infty z}{1+z} \quad (4-3-12)$$

If  $z = 0$  is assumed (the plate temperature field effect on the boundary layer temperature field is neglected), we get formula (3-1-77). Thus, the boundary layer equation (4-3-9) differs from an ordinary boundary layer

equation by the additional quantity  $z$ , which is directly proportional to  $Br_x$  [see formula (4-3-9)].

If the second term in parentheses of expression (4-3-9) is neglected in comparison with the first, we get

$$\frac{d}{dx} \left[ \frac{1}{z^2(1+z)\delta} \right] = \frac{10az}{\gamma^3(1+z)v_\infty} \quad (4-3-13)$$

where

$$\gamma = 3\lambda_f b / 2\lambda_s$$

Two asymptotic solutions may be found easily

1. when  $z \rightarrow 0$  and  $x \rightarrow \infty$

$$\frac{\delta}{\delta_T} = \sqrt[3]{\frac{14}{13} Pr} \quad (4-3-14)$$

which is the classical solution [see formula (3-1-78)];

2. when  $z \rightarrow \infty$  and hence  $x \rightarrow 0$

$$\frac{\delta}{\delta_T} = \sqrt[3]{\frac{28}{13} Pr} \quad (4-3-15)$$

We get more accurate solutions at  $z \rightarrow 0$  and  $z \rightarrow \infty$  assuming

$$\frac{\delta}{\delta_T} = Pr^{1/3} [A + \varphi(x)] \quad (4-3-16)$$

where  $\varphi(x)$  is a small function, so that  $\varphi^2(x) = 0$  and  $A$  is a constant.

Then

$$z = \frac{A\gamma Pr^{1/3}}{\delta} \left[ 1 + \frac{1}{A} \varphi(x) \right] \quad (4-3-17)$$

Substitution of (4-3-17) into (4-3-13) gives the differential equation of the first order with respect to  $z$ .

Asymptotic solutions of the above equation may be used for the determination of the local Nusselt numbers

$$Nu_{x0} = \frac{3}{2} \frac{x}{\delta} \sqrt[3]{\frac{14}{13} Pr} \quad (4-3-18)$$

where  $Nu_{x0}$  is the Nusselt number for the classical problem;

$$Nu_x = \frac{3}{2} \frac{x}{\delta} Pr^{1/3} A \left[ 1 + \frac{1}{A} \varphi(x) \right] \quad (4-3-19)$$

where  $Nu_x$  is the Nusselt number for the conjugate problem.

Then

$$N^* = \frac{\text{Nu}_x}{\text{Nu}_{x0}} = 1 + \left(\frac{13}{14}\right)^{1/3} \left\{ A \left[ 1 + \frac{1}{A} \varphi(x) \right] - \left(\frac{14}{13}\right)^{1/3} \right\} \quad (4-3-20)$$

For small  $z$  and, consequently, for small  $\text{Br}_x$ , we get

$$N^* - 1 = 0.33 \text{Br}_x; \quad \text{Br}_x \ll 1 \quad (4-3-21)$$

For large  $\text{Br}_x$

$$N^* - 1 = 0.26 - 0.07 \text{Br}_x^{-1}, \quad \text{Br}_x \ll 1 \quad (4-3-22)$$

i.e. the relations between  $N^*$  and  $\text{Br}_x$  are similar qualitatively, the only difference being in the values of the coefficient. The difference is explained by the variation ranges of  $\text{Br}_x$ . For practical calculations, when  $\text{Br}_x$  varies from 0 to a certain finite number  $N$  ( $0 < \text{Br}_x < N$ ), we may use the formula

$$N^* - 1 = C \text{Br}_x^m \quad (4-3-23)$$

where  $C$  and  $m$  are constants found from the exact solution of the problem or experimentally.

#### 4.4. EXACT SOLUTIONS OF HEAT-TRANSFER PROBLEMS

##### FOR A PLATE (WITH A HEAT SOURCE) IN COMPRESSIBLE GAS FLOW

Let us first consider a flow past a plate. For this we introduce the following notations:

$u$  denotes the longitudinal velocity  $v_x$  ( $v_x \equiv u$ ),  $v$  denotes the transverse velocity  $v_y$  ( $v_y \equiv v$ ) and  $\mu$  stands for dynamic viscosity. Then, according to the scheme shown in Fig. 4-6 the system of transfer equations takes the form \*

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) \quad (4-4-1)$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \quad (4-4-2)$$

$$\rho u \frac{\partial \theta}{\partial x} + \rho v \frac{\partial \theta}{\partial y} = \frac{1}{\text{Pr}} \frac{\partial}{\partial y} \left( \mu \frac{\partial \theta}{\partial y} \right) + \frac{\mu}{c_p} \left( \frac{\partial u}{\partial y} \right)^2 \quad (4-4-3)$$

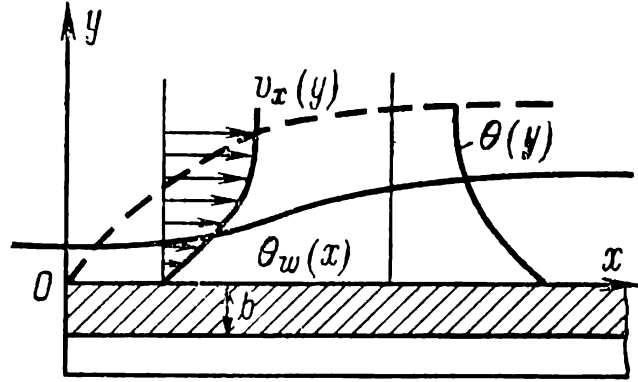
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\*  $\mu$  is used instead of  $\eta$  because in external heat-transfer problems  $\eta$  usually denotes a generalized coordinate.

Let us assume that  $Pr = \text{const}$  and  $c_p = \text{const}$ , and take the following boundary conditions

$$\left. \begin{array}{l} \text{at } y = 0 \text{ and } u = 0 \quad \theta = \theta_w(x) \\ \text{at } y = \infty \text{ and } u = u_\infty \quad \theta = T_\infty \end{array} \right\} \quad (4-4-4)$$

Fig. 4-6. Profiles of velocity  $v_x(y)$  and temperature  $\theta(y)$  in a boundary layer flow past a plate



The relation between the viscosity and temperature is as follows [4-1]

$$\frac{\mu}{\mu_\infty} = \sqrt{\frac{\bar{\theta}_w}{T_\infty}} \frac{T_\infty + B}{\bar{\theta}_w + B} \left( \frac{\theta}{T_\infty} \right) \quad (4-4-5)$$

where  $B$  is a constant and  $\bar{\theta}_w$  is the mean surface temperature. We shall quote the results reported in [4-15]. The plate temperature is found from the solution of the differential equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = -\frac{Q(x, y)}{\lambda_s} \quad (4-4-6)$$

where  $Q(x, y)$  is the heat source (continuous function of  $x, y$ ) provided

$$\left( \frac{\partial T}{\partial x} \right)_{x=0} = \left( \frac{\partial T}{\partial x} \right)_{x=L} = 0 \quad (4-4-7)$$

For a symmetrical flow over a plate of thickness  $2b$

$$\left( \frac{\partial T}{\partial y} \right)_{y=-b} = 0 \quad (4-4-8)$$

At the interface the ordinary boundary conditions hold

$$\left[ -\lambda_f(\theta) \frac{\partial \theta}{\partial y} \right]_{y=0} = -\lambda_s \left( \frac{\partial T}{\partial y} \right)_{y=0} \quad (4-4-9)$$

$$\theta(x, 0) \equiv \theta_w(x) = \theta_e + \vartheta(x) = (T)_{y=0} \quad (4-4-10)$$

where  $\theta_e$  is the temperature of the thermally insulated surface.

The problem may consist either in the determination of the surface temperature and the heat flux or in the determination of the kind of the

heat source  $\theta(x, y)$  necessary for the surface temperature or heat flux through the surface to attain the prescribed value.

For a thin plate, equation (4-4-6) with the use of (4-4-8) through (4-4-10) may be written as

$$\frac{d^2\theta_w(x)}{dx^2} - \frac{1}{\lambda_s b} q(x) = -\frac{W(x)}{\lambda_s} \quad (4-4-11)$$

where  $q(x)$  is the heat flux through the plate-flow surface and  $W(x)$  is the averaged heat source

$$W(x) = \sum_{n=0}^{\infty} W_n x^n$$

Condition (4-4-7) becomes

$$\left( \frac{d\theta_w(x)}{dx} \right)_{x=0} = \left( \frac{d\theta_w(x)}{dx} \right)_{x=L} = 0 \quad (4-4-12)$$

The energy equation may be written as

$$\frac{\partial^2 \theta^*}{\partial \eta^2} + \text{Pr} f \frac{\partial \theta^*}{\partial \eta} - 2\text{Pr} f' x^* \frac{\partial \theta^*}{\partial x^*} = -\frac{\text{Pr}}{4} (\gamma - 1) M_{\infty}^2 (f'')^2 \quad (4-4-13)$$

where  $x^* = x/L$  and  $\theta^* = \theta/T_{\infty}$  are dimensionless variables, and  $\eta$  is defined by

$$f(\eta) = \frac{\psi^*}{\sqrt{x^*}} \quad (4-4-14)$$

where  $\psi^*$  is the dimensionless stream function

$$\psi^* = \frac{\psi}{\sqrt{v_{\infty} v_{\infty} L C}} \quad (4-4-15)$$

Function  $f(\eta)$  is the solution of the Blasius equation

$$\left. \begin{aligned} f''' + ff'' &= 0 \\ f(0) &= 0; \quad f'(0) = 0; \quad f'(\infty) = 2 \end{aligned} \right\} \quad (4-4-16)$$

The boundary conditions for equation (4-4-13) are written as

$$\theta^*(x^*, 0) = \theta_w^*(x^*) = \theta_e^* + \vartheta^*(x^*); \quad \theta^*(x^*, \infty) = 1 \quad (4-4-17)$$

The solution of the energy equation has the form

$$\begin{aligned} \theta^*(x^*, \eta) &= N(\eta) + \sum_{n=0}^{\infty} [A_n Y_{\alpha n}(\eta) + P_n Y_{\alpha n + \beta} x^{*\beta} \\ &+ Q_n Y_{\alpha n + \gamma} x^{*\gamma}] x^{*\alpha n} \end{aligned} \quad (4-4-18)$$

Function  $Y_p(\eta)$  may be found from the equation

$$Y_p'' + \text{Pr} f Y_p' - 2\text{Pr} f' p Y_p = 0 \quad (4-4-19)$$

with boundary conditions

$$Y_p(0) = 1; \quad Y_p(\infty) = 0 \quad (4-4-20)$$

$$N(\eta) = 1 + \frac{\gamma - 1}{2} M_\infty^2 r(\eta) \quad (4-4-21)$$

$$r(\eta) = \frac{1}{2} \text{Pr} \int_{\eta}^{\infty} [f''(\xi)]^{\text{Pr}} \int_0^{\xi} [f''(\zeta)]^{2-\text{Pr}} d\zeta d\xi \quad (4-4-22)$$

here  $p = \alpha n$ ;  $\alpha n + \beta$ ;  $\alpha n + \gamma$  ( $n = 0, 1, 2, \dots$ );  $\alpha, \beta, \gamma > 0$  (positive numbers). Coefficients  $A_n, P_n$  and  $Q_n$  are found from appropriate formulas [4-25].

The heat-transfer coefficient  $\alpha$  is found from the equality

$$q(x^*) = \alpha T_\infty [\theta_w^*(x^*) - \theta_e^*] = - \frac{\lambda_\infty T_\infty}{2} C_w(x^*) \sqrt{\frac{u_\infty}{\nu_\infty L x^* C}} \times \sum_{n=0}^{\infty} [A_n Y'_{3/2n}(0) + P_n x^{*1/2} Y'_{3/2n+1/2}(0) + Q_n x^* Y'_{3/2n+1}(0)] x^{*3/2n} \quad (4-4-23)$$

Taking into account that  $A_0 = \theta^*(0, 0) - \theta_e^*$ ,  $P_0 = Q_0 = 0$ , we get

$$\frac{\alpha}{\bar{\alpha}} = \frac{C_w(x^*) [\theta^*(0, 0) - \theta_e^*]}{C [\theta_w^*(x^*) - \theta_e^*]} - \frac{\lambda_\infty C_w(x^*)}{2 [\theta_w^*(x^*) - \theta_e^*]} \sqrt{\frac{u_\infty}{\nu_\infty L x^* C}} \sum_{n=1}^{\infty} [A_n Y'_{3/2n}(0) + P_n x^{*1/2} Y'_{3/2n+1/2}(0) + Q_n x^* Y'_{3/2n+1}(0)] x^{*3/2n}, \quad (4-4-24)$$

where  $\bar{\alpha}$  is the heat-transfer coefficient at  $\theta_w = \text{const}$

$$C_w(x^*) = \sqrt{\theta_w^*(x^*)} \frac{1 + \frac{B}{T_\infty}}{\theta_w^*(x^*) + \frac{B}{T_\infty}} \quad (4-4-25)$$

The eigenfunctions  $Y_p(\eta)$  are calculated from the formulas given in [4-16].

From the solution of the conjugate problem it follows that the temperature at the surface cannot be prescribed in the form of the Taylor series in terms of powers of  $x$ . Except in trivial cases, the surface temperature is not an analytical function of  $x$  but has a singular point at  $x = 0$ . Consequently, the surface temperature, if it is variable, cannot be specified.

## 4-5. ASYMMETRIC PROBLEM WITHOUT A HEAT SOURCE

*(a) Boundary Conditions of the First Kind*

In this section we shall restrict ourselves to a consideration of the conjugate steady-state problem for a plate (of length  $L$ , thickness  $b$  and thermal conductivity  $\lambda_f$ ) in a longitudinal gas flow of constant velocity  $u_\infty$  and temperature  $T_\infty$  at infinity; the temperature at the internal surface is assumed to be constant (a similar solution is obtained for the case when the heat flux is prescribed at the internal surface) [4-16].

Gas velocity is assumed to be temperature-dependent and to follow a linear law; the Prandtl number is considered to be constant.

1. Mathematically the problem reduces to the solution of the equation for gas in the dimensionless variables

$$\frac{\partial^2 \vartheta}{\partial \xi^2} + \text{Pr} \varphi \frac{\partial \vartheta}{\partial \xi} - 2 \text{Pr} \varphi'(\xi) \frac{\partial \vartheta}{\partial \xi} = - \frac{\text{Pr}(k-1)}{4} M_\infty^2 [\varphi''(\xi)]^2 \quad (4-5-1)$$

with boundary conditions

$$(\vartheta)_{\xi=0} = \vartheta_e + \chi(\xi); \quad (\vartheta)_{\xi \rightarrow \infty} = 1 \quad (4-5-2)$$

and the equation for the solid

$$\frac{\partial^2 \theta}{\partial \xi^2} + \frac{\partial^2 \theta}{\partial y^2} = 0 \quad (4-5-3)$$

with boundary conditions

$$(\theta)_{\xi \rightarrow 0} = \vartheta_{00}; \quad \left( \frac{\partial \theta}{\partial \xi} \right)_{\xi \rightarrow \infty} = 0 \quad (4-5-4)$$

where  $\vartheta_{00}$  is the stagnation temperature and

$$(\theta)_{y=0} = \vartheta_e + \chi(\xi); \quad (\theta)_{y=-1} = \vartheta_b \quad (4-5-5)$$

where  $\vartheta_e$  is the recovery temperature.

At the fluid-solid interface the conjugation conditions are given as

$$\vartheta_{\xi=0} = \theta_{y=0}; \quad \kappa \xi^{-0.5} \left( \frac{\partial \vartheta}{\partial \xi} \right)_{\xi=0} = \left( \frac{\partial \theta}{\partial y} \right)_{y=0} \quad (4-5-6)$$

where  $\vartheta$  is the dimensionless fluid temperature and  $\theta$  is the dimensionless temperature of the solid wall.

The notations used here are as follows:  $\chi = [\chi_1/\chi_1(\infty)]$  is the dimensionless temperature at the solid-gas interface;  $\chi_1$  is the temperature at the solid-fluid (gas) interface;  $y = x_2/b$  are the dimensionless spatial coordi-



nates;  $\xi = x_1/b$  is the ordinate

$$\zeta = 0.5 \xi^{-0.5} \int_0^{y_1} \rho dy; \quad y_1 = y \sqrt{\text{Re}_\infty}$$

Quantity  $\kappa$  is

$$\kappa = 0.5 \frac{\lambda_f}{\lambda_s} \sqrt{\text{Re}_\infty} = 0.5 \frac{\lambda_f}{\lambda_s} \sqrt{u_\infty b / \nu_\infty} \quad (4-5-7)$$

where subscript  $\infty$  refers to the fluid flow far from the wall. Later the unknown function  $\chi(\xi)$  is determined from the second conjugation condition for flows (4-5-6).

As is known, in the case of a linear dependence of viscosity on temperature, the hydrodynamic problem is autonomous, i.e. functions  $\varphi(\zeta)$ ,  $\varphi'(\zeta)$ ,  $\varphi''(\zeta)$  are known; this is an ordinary Blasius profile.

The solution of equations (4-5-1) and (4-5-2) for a gas is to be sought in the form

$$\vartheta(\zeta, \xi) = 1 + \frac{K-1}{8} M_\infty^2 \psi(\zeta) + \bar{\vartheta}(\zeta, \xi) \quad (4-5-8)$$

where

$$\psi(\zeta) = 2 \text{Pr} \int_\zeta^\infty [\varphi''(\zeta_1)]^{\text{Pr}} d\zeta_1 \int_0^{\zeta_1} [\varphi''(\zeta_2)]^{2-\text{Pr}} d\zeta_2$$

and  $\bar{\vartheta}$  satisfies the boundary-value problem

$$\frac{\partial^2 \bar{\vartheta}}{\partial \zeta^2} + \text{Pr} \varphi \frac{\partial \bar{\vartheta}}{\partial \zeta} - 2 \text{Pr} \varphi' \xi \frac{\partial \bar{\vartheta}}{\partial \xi} = 0 \quad (4-5-9)$$

$$(\bar{\vartheta})_{\zeta=0} = \chi(\xi); \quad (\bar{\vartheta})_{\zeta \rightarrow \infty} = 0 \quad (4-5-10)$$

Using the generalized Fourier sine transformation

$$u_s(\alpha) = \lim_{\mu \rightarrow 0} \int_0^\infty u(\xi) \sin \alpha \xi \exp(-\mu \xi) d\xi \quad (4-5-11)$$

we may rewrite (4-5-9) and (4-5-10) in the form of transforms

$$\frac{\partial^2 \bar{\vartheta}}{\partial \zeta^2} + \text{Pr} \varphi \frac{\partial \bar{\vartheta}_s}{\partial \zeta} + 2 \text{Pr} \varphi'(\zeta) \left[ \bar{\vartheta}_s + \alpha \frac{\partial \bar{\vartheta}_s}{\partial \alpha} \right] = 0 \quad (4-5-12)$$

$$(\bar{\vartheta}_s)_{\zeta=0} = \chi_s(\alpha); \quad (\bar{\vartheta})_{\zeta \rightarrow \infty} = 0 \quad (4-5-13)$$

The solution of equations (4-5-12) and (4-5-13) is to be sought in the form of a series

$$\vartheta_s(\zeta, \alpha) = \sum_i C_i z_{vi} \alpha^{vi} \quad (4-5-14)$$

The solution of (4-5-14) and the relation

$$\alpha^{\nu_i} \rightarrow \frac{2}{\pi} \frac{\Gamma(\nu_i + 1) \sin \left[ \frac{1}{2} (\nu_i + 1) \pi \right]}{\zeta^{1+\nu_i}} \quad (4-5-15)$$

will be written as

$$\begin{aligned} \theta(\xi, \zeta) &= \frac{2}{\pi} \sum_i \xi^{(\nu_i+1)} C_i \Gamma(\nu_i + 1) \sin \left[ \frac{1}{2} (\nu_i + 1) \pi \right] Z_{\nu_i}(\zeta) \\ &= \frac{2}{\pi} \sum_i \xi^{-(\nu_i+1)} D_i Z_{\nu_i}(\zeta) \end{aligned} \quad (4-5-16)$$

where  $Z(\zeta)$  satisfies the boundary-value problem

$$Z''(\zeta) + \text{Pr } \varphi Z'(\zeta) + 2 \text{Pr } \varphi'(1 + \nu) Z(\zeta) = 0 \quad (4-5-17)$$

$$Z|_{\zeta=0} = 1 \quad Z|_{\zeta \rightarrow \infty} = 0 \quad (4-5-18)$$

$D_i, \nu_i$  are constants to be determined later on.

Here the transform of the unknown function is

$$D_i = C_i \Gamma(\nu_i + 1) \sin \left[ \frac{1}{2} (\nu_i + 1) \pi \right] \quad (4-5-19)$$

The solution in transforms (4-5-3) and (4-5-5), with consideration of (4-5-11), may be written as

$$\begin{aligned} \theta_s &= \frac{\vartheta_{00}}{\alpha} + \frac{\sinh(\alpha)(1+y)}{\sinh \alpha} \chi_s(\alpha) \\ &\quad - \frac{(\vartheta_{00} - \vartheta_e) \sinh(\alpha)(1+y) + (\vartheta_b - \vartheta_{00}) \sinh \alpha y}{\alpha \sinh \alpha} \end{aligned} \quad (4-5-20)$$

Differentiation of equations (4-5-17) and (4-5-20) and substitution of obtained results into the second conjugation condition (4-5-6) for flows, which has been written for transforms, yield an equation from which constants  $\nu_i$  and  $D_i$  are determined

$$\begin{aligned} \propto \sum_i D_i \frac{Z'_{\nu_i}(0) \alpha^{\nu_i} \cosh \alpha}{\Gamma\left(\nu_i + \frac{3}{2}\right) \sin \left[ \frac{1}{2} \left(\nu_i + \frac{3}{2}\right) \pi \right]} &= (\vartheta_e - \vartheta_{00}) \cosh \alpha \\ + (\vartheta_{00} - \vartheta_b) + \alpha \sum_i D_i \left\{ \Gamma(\nu_i + 1) \sin \left[ \frac{1}{2} (\nu_i + \pi) \right] \right\}^{-1} \alpha^{\nu_i} \cosh \alpha & \end{aligned} \quad (4-5-21)$$

By expanding the functions appearing in (4-5-21) into a power series and equating the coefficients at equal powers of  $\alpha$ , we get  $v_i = 0.5$  ( $i = 2$ ). Thereafter, with allowance made for (4-5-15), we finally obtain an expression for temperature distribution in the boundary layer

$$\begin{aligned} \vartheta(\zeta, \xi) = & 1 + [(K-1)/8] M_\infty^2 \psi(\zeta) + (\vartheta_b - \vartheta_e) \int_{\zeta}^{\infty} [\varphi''(\zeta_1)]^{\text{Pr}} d\zeta_1 \left( \int_0^{\infty} [\varphi''(\zeta_1)]^{\text{Pr}} d\zeta_1 \right)^{-1} \\ & + \kappa Z'_{-1}(0) (\vartheta_b - \vartheta_e) \{ [\varphi''(\zeta)/\varphi''(0)]^{\text{Pr}} \xi^{-0.5} + 0.25 Z_{1.5}(\zeta) \xi^{-2.5} \\ & + 0.25 \kappa Z'_{1.5}(0) Z_2(\zeta) \xi^{-3} + 0.25 \kappa^2 Z'_2(0) Z'_{1.5}(0) Z_{2.5}(\zeta) \xi^{-3.5} \\ & + (7.5\sqrt{2}/16\sqrt{\pi}) \kappa^3 Z'_{2.5}(0) Z'_2(0) Z'_{1.5}(0) Z_3(\zeta) \xi^{-4} + \dots \} \end{aligned} \quad (4-5-22)$$

At  $\zeta = 0$  the temperature of a surface in a gas flow is obtained from equation (4-5-22)

$$\begin{aligned} \vartheta_w = (\vartheta)_{\zeta=0} = (\vartheta)_{y=0} = & \vartheta_b + \kappa Z'_{-1}(0) (\vartheta_b - \vartheta_e) \{ \xi^{-0.5} + 0.25 \xi^{-2.5} \\ & + 0.25 \kappa Z'_{1.5}(0) \xi^{-3} + 0.25 \kappa^2 Z'_2(0) Z'_{1.5}(0) \xi^{-3.5} + \dots \} \end{aligned} \quad (4-5-23)$$

It may be mentioned that functions  $Z_{v_i}$  in equations (4-5-22) and (4-5-23) for the obtained values of  $v_i$  are determined as the solution of the boundary-value problem

$$Z''_{v_i} + \text{Pr } \varphi Z'_{v_i} (1 + v_i) Z_{v_i} = 0 \quad (4-5-24)$$

$$(Z_{v_i})_{\zeta=0} = 1; \quad (Z_{v_i})_{\zeta \rightarrow \infty} = 0 \quad (4-5-25)$$

the exact solutions of which were obtained in two cases, when  $v_0 = -1$  and  $v_i = -0.5$

$$Z_{-1}(\zeta) = \int_{\zeta}^{\infty} [\varphi''(\zeta_1)]^{\text{Pr}} d\zeta_1 \left( \int_0^{\infty} [\varphi''(\zeta_1)]^{\text{Pr}} d\zeta_1 \right)^{-1} \quad (4-5-26)$$

$$Z_{-0.5}(\zeta) = \left[ \frac{\varphi''(\zeta)}{\varphi''(0)} \right]^{\text{Pr}} \quad (4-5-27)$$

For other values of  $v_i$  the functions  $Z_{v_i}$  may be obtained only numerically, to accomplish this it is necessary to find first an asymptotic solution of equations (4-5-24) and (4-5-25). The Blasius functions  $\varphi(\zeta)$  and  $\varphi'(\zeta)$  entering into equations (4-5-24) and (4-5-25) are not expressed analytically, but for large  $\zeta$  ( $\zeta \geq 3.5$ ). They may be represented as

$$\varphi(\zeta) \approx 2(\zeta - 0.86); \quad \varphi'(\zeta) = 2; \quad \zeta \geq 3.5 \quad (4-5-28)$$

Substitution of (4-5-28) into (4-5-24) and (4-5-25) yields an equation for large  $\zeta$

$$Z''_{v_i}(\zeta) + 2 \text{Pr}(\zeta - 0.86) Z'_{v_i}(\zeta) + 4 \text{Pr}(1 + v_i) Z(\zeta) = 0 \quad (4-5-29)$$

$$(Z_{v_i})_{\zeta \rightarrow \infty} = 0$$

the solution of which is

$$Z_{v_i}|_{\zeta > 3.5} = C \frac{d^{2v+1}}{d\zeta^{2v+1}} \{\exp[-0.5(\zeta - 0.86)^2]\} \quad (4-5-30)$$

Numerical calculation is performed with regard for (4-5-30). In so doing the constant  $C$  is determined at the end of the calculation at  $\zeta = 0$ . The results obtained by the finite-difference method are given in [4-9]. Note that functions  $Z_{v_i}$  at  $v_i \geq 0$  are not monotonous.

It should be emphasized that solutions (4-5-22) and (4-5-23) contain a new criterion  $\kappa$  [see (4-5-7)], which comprises characteristics of both the main flow and the body. When  $\kappa \rightarrow 0$  (this corresponds to an infinite heat conductivity of a solid or to an infinitesimal thickness of a plate  $b \rightarrow 0$ ), we obtain the known gas temperature profile at constant temperature of the solid body in a flow

$$\begin{aligned} \vartheta^*(\zeta, \xi) &= \lim_{\kappa \rightarrow 0} \vartheta(\zeta, \xi) = 1 + [(K - 1)/8] M_\infty^2 \psi(\zeta) \\ &+ (\vartheta_b - \vartheta_e) \int_{\xi}^{\infty} [\varphi''(\zeta_1)]^{\text{Pr}} d\zeta_1 \left( \int_0^{\infty} [\varphi''(\zeta_1)]^{\text{Pr}} d\zeta_1 \right)^{-1} \end{aligned} \quad (4-5-31)$$

Generally speaking, the criterion  $\kappa$  may take any value from 0 to  $\infty$  (for a majority of real situations  $\kappa \approx 1$ ); note that the extreme case  $\kappa \rightarrow \infty$  corresponds to thermal insulation of a body in a flow, and  $\kappa \rightarrow 0$  is characteristic of the case when it is possible to use Newton's law with an error factor that may be determined.

In all other cases ( $0 < \kappa < \infty$ ) the heat-transfer coefficient may be only formally introduced. Hence the Nusselt number with regard for equation (4-5-22) is determined as follows [4-16]

$$\begin{aligned} \text{Nu}(\xi) &= 0.5 \sqrt{\text{Re}_\xi} \{0.664 \sqrt[3]{\text{Pr}} - \kappa Z'_{-1}(0) [0.25 Z'_{1.5}(0) \xi^{-2.5} \\ &+ 0.25 \kappa Z'_2(0) Z'_{1.5}(0) \xi^{-3} + 0.25 \kappa^2 Z'_{2.5}(0) Z'_2(0) Z'_{1.5}(0) \xi^{-3.5} + \dots]\} \\ &\times \{1 + \kappa Z'_{-1}(0) [\xi^{-0.5} + 0.25 \xi^{-2.5} + 0.25 \kappa Z'_{1.5}(0) \xi^{-3} + \dots]^{-1}\} \end{aligned} \quad (4-5-32)$$

It is easy to see that at  $\kappa \rightarrow 0$  the well-known expression for the Nusselt number is obtained from (4-5-32)

$$\text{Nu}^* = \lim_{\kappa \rightarrow 0} \text{Nu} = 0.332 \sqrt[3]{\text{Re}_\xi} \sqrt[3]{\text{Pr}} \quad (4-5-33)$$

where  $\text{Re}_\xi = v_\infty x / \nu_\infty$ .

2. Let us now consider two examples of calculation from formulas derived in [4-9].

(a) Poor conductor ( $\lambda_f = 2.52 \text{ W/m} \cdot \text{g} \cdot ^\circ\text{C}$ ). The plate thickness  $b$  is 0.01 m. The characteristics of the main flow are:  $M = 3$ ;  $T_b = 280^\circ\text{K}$ ;  $T_\infty = 223^\circ\text{K}$ ;

$$\nu_\infty = 9.54 \times 10^{-6} \text{ m}^2/\text{s}; \quad \lambda_\infty = 2.04 \times 10^{-2} \text{ W/m} \cdot ^\circ\text{C};$$

$$U_\infty^{(1)} = 900 \text{ m/s}; \quad u_\infty^{(2)} = 400 \text{ m/s}.$$

In this case  $\kappa^{(1)} = 3.95$ ;  $\kappa^{(2)} = 2.64$ . From equations (4-5-23) and (4-5-32) the temperature of the surface and the Nusselt numbers were calculated. The results are presented in Figs. 4-7 and 4-8.

(b) Good conductor ( $\lambda_f = 15.6 \text{ W/m} \cdot ^\circ\text{C}$ ). In this case  $\kappa^{(1)} = 0.639$ . The results of calculations similar to the case (a) are presented in Figs. 4-7 and 4-8.

From the graphs one may see that the values of  $\vartheta_w$  and  $Nu$  differ considerably from those usually used (at  $\kappa \rightarrow 0$ ). Thus, in case (a) at certain points the values of  $Nu$  differ several times from the conventional value of  $Nu^*$ ; the same refers to the temperature of the surface in a flow (since when  $\kappa \rightarrow 0$ ,  $\vartheta_w^* = \vartheta_b$ ). At the same time for a good conductor, case (b), the values of  $\vartheta_w$  and  $Nu$  do not differ very much (by not more than 25%) from the respective values of  $\vartheta_w^*$  and  $Nu^*$  (when  $\kappa \rightarrow 0$ ).

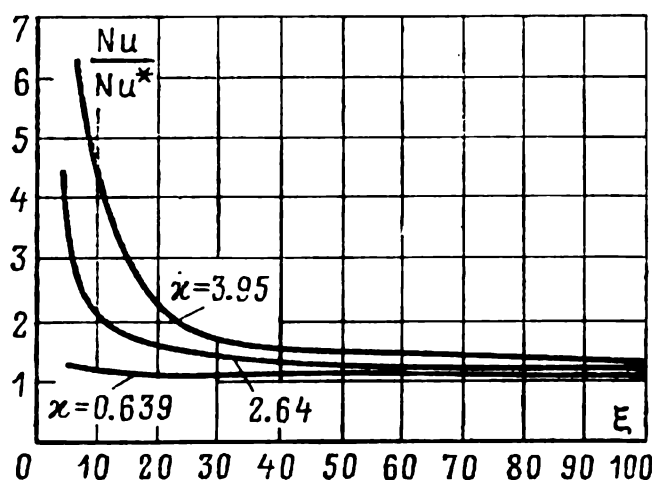


Fig. 4-7. Plot of  $Nu(\xi)/Nu^*$  versus  $\xi$  and  $\kappa$

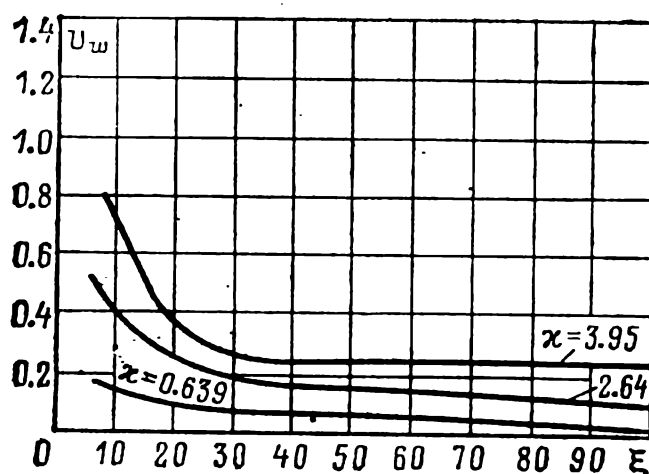


Fig. 4-8. Plot of  $\vartheta_w$  (relative difference of temperatures) versus  $\kappa$  and  $\xi$

### (b) Boundary Conditions of the Second Kind

The solution presented above was obtained for a given temperature of the inner surface of a solid in a flow. Here we shall give the solution for a given heat flux at the inner surface

$$\left( \frac{\partial \theta}{\partial y} \right)_{y=-1} = \frac{b}{\lambda_s T_\infty} q_1 \quad (4-5-34)$$

The solution of the boundary-value problem (4-5-1)-(4-5-2) may be transformed thus

$$\vartheta_s(\alpha, \zeta) = \frac{1}{\alpha} \left[ 1 - \frac{K-1}{8} M_\infty^2 v(\zeta) \right] + \sum_i C_i Z_{v_i}(\zeta) \alpha^{v_i} \quad (4-5-35)$$

The solution of the boundary-value problem (4-5-3)-(4-5-4) for a solid (the condition at the inner surface) is substituted by (4-5-34) and may be reduced to

$$\theta_s = \frac{\vartheta_{00}}{\alpha} + \left[ \chi_s(\alpha) + \frac{\vartheta_e - \vartheta_{00}}{\alpha} \right] \frac{\cosh \alpha(1+y)}{\cosh \alpha} - \frac{\bar{q} \sinh \alpha y}{\alpha^2 \cosh \alpha} \quad (4-5-36)$$

Differentiation of expressions (4-5-35) and (4-5-36) and their substitution into conjugation condition (4-5-6) in the transformed form give

$$\kappa \sum_i C_i A_{ii} \alpha^{v_i+1.5} \cosh \alpha = (\vartheta_e \vartheta_{00}) \alpha \sinh \alpha - \bar{q} + \sum_i C_i \alpha^{v_i+2} \sinh \alpha \quad (4-5-37)$$

From the above equation, by equating the coefficients at equal powers of  $\alpha$ , we find

$$v_i = \frac{3}{2}(i-1); \quad C_i = -\bar{q} \left[ \kappa^{i+1} \prod_{k=0}^i A_k \right]^{-1} \quad (4-5-38)$$

where

$$A_i = Z'_{v_i}(0) \Gamma(v_i+1) \sin \left[ \frac{(v_i+1)\pi}{2} \right] \left\{ \Gamma \left( v_i + \frac{3}{2} \right) \sin \left[ \frac{\left( v_i + \frac{3}{2} \right) \pi}{2} \right] \right\}^{-1} \quad (4-5-39)$$

Using these values of  $v_i$  and  $C_i$  and transforming (4-5-35) with due regard for (4-5-14) and the inversion formula

$$\frac{1}{\alpha^{\mu+1}} \leftrightarrow \frac{\xi^\mu}{\Gamma(\mu+1) \sin \left[ (\mu+1) \frac{\pi}{2} \right]} \quad (4-5-40)$$

we get the temperature distribution in a boundary layer

$$\begin{aligned} \vartheta(\xi, \zeta) = & 1 + \frac{K-1}{8} M_\infty^2 \Psi(\zeta) - \frac{bq_1}{\lambda_s \kappa Z'_{-1.5}(0) T_\infty} \\ & \times \left\{ Z_{-1.5}(\zeta) \xi^{0.5} + \frac{Z'_0(\zeta) \xi^{-1}}{4\kappa Z'_0(0)} - \frac{Z_{1.5}(\zeta) \xi^{-2.5}}{2\kappa^2 Z'_0(0) Z'_{1.5}(0)} + \dots \right\} \end{aligned} \quad (4-5-41)$$

From equation (4-5-41) at  $\zeta = 0$  the temperature of the surface in a flow is obtained

$$\vartheta_w = \vartheta_{\zeta=0} = \vartheta_{y=0} = \vartheta_e - \frac{bq_1}{T_\infty \kappa \lambda_f Z'_{-1.5}(0)} \left[ \xi^{0.5} + \frac{\xi^{-1}}{4\kappa Z'_0(0)} - \frac{\xi^{-2.5}}{2\kappa^2 Z'_0(0) Z'_{1.5}(0)} + \frac{35\xi^{-4}}{8\kappa^3 Z'_0(0) Z'_{1.5}(0) Z'_3(0)} + \dots \right] \quad (4-5-42)$$

Let us now consider the extreme cases of formulas (4-5-41) and (4-5-42).

(a)  $\kappa \rightarrow \infty$ , a thermally insulated surface. Expressions (4-5-41) and (4-5-42) give the known relations

$$\vartheta_w^* = \lim_{\substack{\kappa \rightarrow \infty \\ \lambda_s \rightarrow 0}} \vartheta_w = \vartheta_e \quad (4-5-43)$$

$$\vartheta^*(\xi, \zeta) = \lim_{\substack{\kappa \rightarrow \infty \\ \lambda_s \rightarrow 0}} \vartheta = 1 + \frac{K-1}{8} M_\infty^2 v(\zeta) \quad (4-5-44)$$

(b)  $b/\lambda_s \rightarrow 0$ , a thin body. Expressions (4-5-41) and (4-5-42) have the form

$$\vartheta_w^* = \lim_{b/\lambda_s \rightarrow 0} \vartheta_w = \vartheta_e - \frac{2q_1 \sqrt{x}}{\lambda_\infty Z'_{-1.5}(0) T_\infty} \quad (4-5-45)$$

$$\vartheta^* = \lim_{b/\lambda_s \rightarrow 0} \vartheta = 1 + \frac{K-1}{8} M_\infty^2 v(\zeta) - \frac{2q_1 Z_{-1.5}(\xi) \sqrt{x}}{\lambda_\infty Z'_{-1.5}(0) T_\infty} \quad (4-5-46)$$

It should be pointed out that (4-5-45) is of the same form as the formula presented in [4-3] for a constant heat release in a thin wall.

This is one more confirmation of the universality of the solutions (4-5-41) and (4-5-42) which are the generalized forms of available results for some particular cases (good or poor heat conductors, etc.) and may be used in all practical cases.

By using equations (4-5-41) and (4-5-42) and employing formally Newton's law, we may find an expression for the local Nusselt number

$$\begin{aligned} \text{Nu}(x) = & - \frac{\sqrt{\text{Re}_x}}{2} \left\{ Z'_{-1.5}(0) \xi^{0.5} + \frac{\xi^{-1}}{4\kappa} - \frac{\xi^{-2.5}}{2\kappa^2 Z'_0(0)} \right. \\ & + \frac{35\xi^{-4}}{8\kappa^3 Z'_0(0) Z'_{1.5}(0)} + \dots \left. \right\} \left\{ \xi^{0.5} + \frac{\xi^{-1}}{4\kappa Z'_0(0)} - \frac{\xi^{-2.5}}{2\kappa^2 Z'_0(0) Z'_{1.5}(0)} \right. \\ & + \left. \frac{35\xi^{-4}}{8\kappa^3 Z'_0(0) Z'_{1.5}(0) Z'_3(0)} + \dots \right\}^{-1} \quad (4-5-47) \end{aligned}$$

As regards external problems, in case (a),  $x \rightarrow \infty$ , equations (4-5-43)-(4-5-44) imply that the Nusselt number equals zero; in case (b),  $b/\lambda_s \rightarrow 0$ , from equations (4-5-36)-(4-5-37) we get the expression for the Nusselt number

$$\text{Nu}_x^* = \lim_{b/\lambda_s \rightarrow 0} \text{Nu} = 0.5 Z'_{-1.5}(0) \sqrt{\text{Re}_x} \quad (4-5-48)$$

Comparison of the above expression with the formula obtained in [4-3] for a similar problem

$$\text{Nu}_x = 0.455 \text{Pr}^{1/3} \sqrt{\text{Re}_x} \quad (4-5-49)$$

shows their agreement (at  $\text{Pr} = 0.72$ ,  $Z'_{-1.5}(0) = 0.82$ ).

To conclude the section it should be mentioned that the temperature distribution in a solid for the considered problems has been dealt with in greater detail in [4-9].

#### 4-6. INTERNAL CONJUGATE PROBLEMS

The general method developed for internal heat-transfer conjugate problems is based on reducing the problem to the solution of a singular integral equation for an unknown temperature of the surface in a flow [4-9]. The method permits exact solutions to be obtained both for the steady-state and unsteady-state heat transfer with laminar and turbulent flows.

Here we shall restrict ourselves (due to the lack of space) to a consideration of a steady-state heat-transfer problem with laminar forced convection in circular and plane tubes for a developed Poiseuille velocity distribution with allowance made for mechanical energy dissipation.

1. Mathematically the problem reduces to the solution of equations for a fluid in dimensionless variables

$$\text{Pe}(1 - \eta^2) \frac{\partial \theta_m}{\partial \xi} = \frac{1}{\eta^{m-1}} \frac{\partial}{\partial \eta} \left( \eta^{m-1} \frac{\partial \theta_m}{\partial \eta} \right) + H\eta^2; \quad \begin{matrix} 0 \leq \eta \leq 1 \\ 0 \leq \xi \leq \infty \end{matrix} \quad (4-6-1)$$

with boundary conditions

$$\theta_m|_{\xi=0} = \theta_{m0} \quad \theta_m|_{\xi \rightarrow \infty} = A_m(1 - \eta^4) + 1 \quad (4-6-2)$$

$$\left. \frac{\partial \theta_m}{\partial \eta} \right|_{\eta=0} = 0 \quad \theta_m|_{\eta=1} = \chi_m(\xi) \quad (4-6-3)$$



and to equations for a solid body

$$\frac{\partial^2 \theta_{fm}}{\partial \xi^2} + \frac{1}{\eta^{m-1}} \frac{\partial}{\partial \eta} \left( \eta^{(m-1)} \frac{\partial \theta_{fm}}{\partial \eta} \right) = 0; \quad \begin{matrix} 1 \leq \eta \leq \delta \\ 0 \leq \xi < \infty \end{matrix} \quad (4-6-4)$$

with the boundary conditions

$$\theta_{fm}|_{\xi=0} = \theta_{fm0} \quad \theta_{fm}|_{\xi \rightarrow \infty} = F_m(\eta) \quad (4-6-5)$$

$$\theta_{fm}|_{\eta=1} = \chi_m(\xi) \quad \theta_{fm}|_{\eta=\delta} = \psi_m(\xi) \quad (4-6-6)$$

At the solid-fluid interface the conjugation conditions are

$$\theta_m|_{\eta=1} = \theta_{fm}|_{\eta=1} = \chi_m(\xi) \quad (4-6-7)$$

$$\beta \frac{\partial \theta_m}{\partial \eta} \Big|_{\eta=1} = \frac{\partial \theta_{fm}}{\partial \eta} \Big|_{\eta=1} \quad (4-6-8)$$

Note that the conditions at  $\xi \rightarrow \infty$  in equations (4-6-2) and (4-6-3) have been obtained from the solution of the stated problem at  $\xi \rightarrow \infty$ .

2. Solutions (4-6-1) through (4-6-3) are readily obtainable by applying the superposition principle in the form

$$\begin{aligned} \theta_m = & A_m(1 - \eta^4) - \sum_{n=1}^{\infty} A_{nm} R_{nm}(\eta) \int_0^{\xi} \chi(\xi_1) \\ & \times \exp[-b_{nm}(\xi - \xi_1)] d\xi_1 + \sum_{n=1}^{\infty} B_{nm} R_{nm}(\eta) \exp[-b_{nm} \xi] \end{aligned} \quad (4-6-9)$$

where

$$\begin{aligned} A_{nm} = & - \frac{\int_0^1 B_{nm}(\eta_1) \eta_1^{m-1} (1 - \eta_1^2) d\eta_1}{\int_0^1 R_{nm}^2(\eta_1) \eta_1^{m-1} (1 - \eta_1^2) d\eta_1} = \frac{2}{P_{nm} \left( \frac{\partial R_{nm}}{\partial P_{nm}} \right)_{\eta=1}} \\ B_{nm} = & \frac{2 \int_0^1 R_{nm}(\eta_1) \eta_1^{m-1} (1 - \eta_1^2) [\theta_0 - A_m(1 - \eta_1^4)] d\eta_1}{[(\partial R_{nm} / \partial P_{nm})(\partial R_{nm} / \partial \eta)]_{\eta=1}} \end{aligned}$$

$$R_{nm}(\eta) = \exp[-0.5 P_{nm} \eta^2] {}_1F_1[\alpha_m, \beta_m, P_{nm} \eta^2]$$

$\alpha_1 = \frac{1 - P_{n1}}{4}$ ;  $\alpha_2 = \frac{2 - P_{n2}}{4}$ ;  $\beta_1 = 0.5$ ;  $\beta_2 = 1$ ;  $P_{nm}^2 = b_{nm}$ ;  $P_{nm}$  are the roots of the characteristic equation

$${}_1F_1(\alpha_m, \beta_m, P_m) = 0 \quad (4-6-10)$$

which are available, for example, in [4-20].

3. The solution of boundary-value problem for a solid body is found from (4-6-4)-(4-6-6) by the generalized Fourier sine transformation in the form

$$\begin{aligned} \theta_{f1} = \theta_{f10} + \lim_{\mu \rightarrow 0} \frac{1}{\pi} \left\langle \int_0^\infty \left[ \chi(\alpha) - \frac{\theta_{f10}}{\alpha} \right] \sinh \alpha(b - \eta) \right. \\ \left. + \left[ \psi_1(\alpha) - \frac{\theta_{f10}}{\alpha} \right] \sinh \alpha(\eta - 1) \right\} \frac{\exp[-\mu\alpha]}{\sinh \alpha(b - 1)} \sin \alpha\xi d\alpha \rangle \end{aligned} \quad (4-6-11)$$

$$\begin{aligned} \theta_{f2} = 1 - 0.25 \beta H \log_e \eta + \lim_{\mu \rightarrow 0} \frac{1}{\pi} \left\langle \int_0^\infty \left[ \chi(\alpha) - \frac{1}{\alpha} \right] \right. \\ \times \frac{K_0(\alpha\delta) I_0(\alpha\eta) - I_0(\alpha\delta) K_0(\alpha\eta)}{I_0(\alpha) K_0(\alpha\delta) - I_0(\alpha\delta) K_0(\alpha)} - \int_1^\delta G(\eta, \eta_1, \alpha) \\ \left. \times [0.25 \beta H \log_e \eta_1 - \psi_2(\infty)] d\eta_1 \right\} \exp[-\mu\alpha] \sin \alpha\xi d\alpha \rangle \end{aligned} \quad (4-6-12)$$

$$G(\eta, \eta_1, \alpha) = \begin{cases} \frac{\alpha[K(\alpha\eta_1)I_0(\alpha\delta) - K_0(\alpha\delta)I_0(\alpha\eta_1)]}{I_0(\alpha\delta)} I_0(\alpha\eta); & 1 \leq \eta \leq \eta_1 \\ \frac{\alpha[K_0(\alpha\eta)I_0(\alpha\delta) - K_0(\alpha\delta)I_0(\alpha\eta)]}{I_0(\alpha\delta)} I_0(\alpha\eta_1); & \eta_1 \leq \eta \leq \delta \end{cases}$$

Having substituted (4-6-9), (4-6-11) and (4-6-12) into the conjugation condition (4-6-8) written for the transforms, we obtain the equation for the transform of unknown function  $\chi_m(\alpha)$

$$\varphi_m(t)a_m(t) - b_m(t) \int_0^\infty \frac{\varphi_m(\tau)}{\tau - t} d\tau = f_m(t)$$

where

$$\varphi_m(t) = \chi_m(\alpha); \quad t = \alpha^2$$

$$\left. \begin{aligned} a_1(t) &= \sum_{n=1}^{\infty} \frac{A_{n1} R'_{n1}(1) b_{n1}^2}{b_{n1}^2 + t} - \sqrt{t} \beta^{-1} \coth \sqrt{t} (\delta - 1) \\ b_1(t) &= 0.5 \beta \sum_{n=1}^{\infty} \frac{\sqrt{t} A_{n1} R'_{n1}(1) b_{n1}}{b_{n1}^2 + t} \\ f_1(t) &= -\frac{\beta H}{3 \sqrt{t}} + \beta \sum_{n=1}^{\infty} B_{n1} R'_{n1}(1) \frac{\sqrt{t}}{b_{n1}^2 + t} \\ &\quad - \theta_{f10} \coth \sqrt{t} (\delta - 1) + \frac{\theta_{f10} - \sqrt{t} \psi_1(t)}{\sinh \sqrt{t} (\delta - 1)} \\ a_2(t) &= \sum_{n=1}^{\infty} \frac{A_{n2} R'_{n2}(1) b_{n2}^2}{b_{n2}^2 + t} - \frac{\sqrt{t} [K_0(\delta \sqrt{t}) I_1(\sqrt{t}) + I_0(\delta \sqrt{t}) K_1(\sqrt{t})]}{\beta [I_0(\sqrt{t}) K_0(\sqrt{t} \delta) - I_0(\sqrt{t} \delta) K_0(\sqrt{t})]} \\ b_2(t) &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{A_{n2} R'_{n2}(1) b_{n2} \sqrt{t}}{b_{n2}^2 + t} \\ f_2(t) &= \sum_{n=1}^{\infty} \frac{B_{n2} R'_{n2}(1) \sqrt{t}}{b_{n2}^2 + t} + \frac{1}{\beta} \sqrt{t} \int_1^{\delta} G'(1, \rho_1, \sqrt{t}) \\ &\quad \times \left[ \frac{H \beta \log_e \rho_1}{4} - \psi_2(\infty) \right] d\rho_1 \\ &\quad + \frac{K_0(\delta \sqrt{t}) I_1(\sqrt{t}) + I_0(\delta \sqrt{t}) K_1(\sqrt{t})}{I_0(\sqrt{t}) K_0(\delta \sqrt{t}) - I_0(\delta \sqrt{t}) K_0(\sqrt{t})} \end{aligned} \right\} \quad (4-6-13)$$

4. Equation (4-6-13) is a singular integral equation containing the Cauchy kernel (4-6-14). To solve it, we shall take advantage of the concept of analytical extension to the complex region and reduce equation (4-6-13) to the Riemann boundary-value problem with discontinuity coefficients. Let us introduce the piecewise analytical function

$$\hat{\Phi}(z) = \frac{1}{2\pi i} \int_L \frac{\varphi(\tau) d\tau}{\tau - z} \quad (4-6-14)$$

where the contour  $L$  is a positive part of the real axis.

Using Sokhotsky-Plemel's formulae, we arrive at

$$\varphi(t) = \Phi^+(t) - \Phi^-(t) \quad (4-6-15)$$

$$\frac{1}{\pi i} \int_L \frac{\varphi(\tau)}{\tau - t} d\tau = \Phi^+(t) + \Phi^-(t) \quad (4-6-16)$$

Substitution of (4-6-15) and (4-6-16) into (4-6-13) yields

$$\Phi^+(t) = G(t) \Phi^-(t) + g(t) \quad (4-6-17)$$

where

$$G(t) = \frac{a(t) + b(t)\pi i}{a(t) - b(t)\pi i}; \quad g(t) = \frac{f(t)}{a(t) - b(t)\pi i}$$

The inhomogeneous boundary-value Riemann problem (4-6-15) may be solved in the general form, if the index of the problem is not negative. It is easy to demonstrate that in the present case the index equals zero [4-9] and solution (4-6-13) is of the form

$$\begin{aligned} \chi(\alpha) = & 0.5g(\alpha^2)[1 + G^{-1}(\alpha^2)] + \sqrt{G(\alpha^2)} \times \exp[\Gamma(\alpha^2)][1 - G^{-1}(\alpha^2)] \\ & \times \frac{1}{\pi i} \int_0^\infty \frac{q(y^2)y dy}{\sqrt{G(y^2)}(y^2 - \alpha^2) \exp[\Gamma(y^2)]} \end{aligned} \quad (4-6-18)$$

where

$$\Gamma(t) = \frac{1}{2\pi i} \int_L \frac{\log_e G(\tau)}{\tau - t} d\tau$$

The unknown temperatures  $\theta_m$  and  $\theta_{fm}$  are found from equations (4-6-9), (4-6-11) and (4-6-12) by taking into account (4-6-18) and the relationship

$$\chi_m(\xi) = (1/\pi \lim_{\mu \rightarrow 0}) \int_0^\infty \chi_m(\alpha) \exp[-\mu\alpha] \sin \alpha\xi d\alpha$$

#### 4-7. UNSTEADY-STATE HEAT TRANSFER WITH LAMINAR FLOW OF INCOMPRESSIBLE FLUID IN PLANE AND CIRCULAR TUBES

In this section exact analytical solutions of unsteady-state conjugate problems of laminar forced convection heat transfer in plane and circular tubes with Poiseuille velocity distribution are presented. The system

of equations (4-5-3)-(4-5-4) in this case becomes (in dimensionless variables)

$$\frac{\partial \theta_j}{\partial \text{Fo}} + \text{Pe}(1 - \eta^2) \frac{\partial \theta_j}{\partial \xi} = \frac{1}{\eta^{j-1}} \frac{\partial}{\partial \eta} \left( \eta^{j-1} \frac{\partial \theta_j}{\partial \eta} \right) + H\eta^2 \quad (4-7-1)$$

$$\theta_j|_{\text{Fo}=0} = 0, \quad 0 < \eta < 1, \quad 0 < \xi < \infty \quad (4-7-2)$$

$$\theta_j|_{\xi=0} = 1; \quad \theta_j|_{\xi \rightarrow \infty} = \theta_{j\infty}(\eta, \text{Fo}) \quad (4-7-3)$$

$$\left. \frac{\partial \theta_j}{\partial \eta} \right|_{\eta=0} = 0; \quad \theta_j|_{\eta=0} = \frac{\chi_j(\xi, \text{Fo}) - T_0}{T_1 - T_0} \quad (4-7-4)$$

$$\frac{\partial \theta_{sj}}{\partial \text{Fo}} = K_a^{-1} \left[ \frac{1}{\eta^{j-1}} \frac{\partial}{\partial \eta} \left( \eta^{j-1} \frac{\partial \theta_{sj}}{\partial \eta} \right) \right] + \frac{\partial^2 \theta_{sj}}{\partial \xi^2} \quad (4-7-5)$$

$$\theta_{sj}|_{\text{Fo}=0} = 0; \quad 1 < \eta < \delta; \quad 0 < \xi < \infty \quad (4-7-6)$$

$$\theta_{sj}|_{\xi=0} = 1; \quad \theta_{sj}|_{\xi \rightarrow \infty} = \theta_{s\infty}(\eta, \text{Fo}) \quad (4-7-7)$$

$$\theta_{sj}|_{\eta=1} = \frac{\chi_j(\xi, \text{Fo}) - T_{s0}}{T_{s1} - T_{s0}}; \quad \theta_{sj}|_{\eta=\delta} = \frac{\psi_j(\xi, \text{Fo}) - T_{s0}}{T_{s1} - T_{s0}} \quad (4-7-8)$$

$$[(T_1 - T_0)\theta + T_0]|_{\eta=1} = [(T_{s1} - T_{s0})\theta_s + T_{s0}]|_{\eta=1} \quad (4-7-9)$$

$$K_\lambda \left. \frac{\partial \theta_j}{\partial \eta} \right|_{\eta=1} = \left. \frac{\partial \theta_{sj}}{\partial \eta} \right|_{\eta=1} \quad (4-7-10)$$

where  $\psi_j(\xi, \text{Fo})$  is a known function given in the general form;  $\chi_j(\xi, \text{Fo})$  is an unknown function which will be determined later on, and  $\theta_{\infty j}$  and  $\theta_{s\infty j}$  are found as the solution of the system of equations (4-7-1) through (4-7-10) at  $\xi \rightarrow \infty$  and are of the form

$$\begin{aligned} \theta_{\infty 1}(\eta, \text{Fo}) &= H(\text{Fo} + \text{Fo}^2) + (T_1 - T_0)^{-1} \frac{\partial}{\partial \eta} \int_0^{\text{Fo}} [\chi_1(\infty, \text{Fo}_1) - T_0] \times \\ &\times \mathfrak{I}_1 [0.5\eta, \text{Fo} - \text{Fo}_1] d\text{Fo}_1 \\ \theta_{s\infty 1}(\eta, \text{Fo}) &= \frac{[T_{s1} - T_{s0}]^{-1}}{K_a(\delta - 1)} \int_0^{\text{Fo}} \left\{ [\psi_1(\infty, \text{Fo}) - T_{s0}] \frac{\partial}{\partial \eta} \times \right. \\ &\times \mathfrak{I}_3 \left[ \frac{\eta - 1}{2(\delta - 1)}, \frac{\text{Fo} - \text{Fo}_1}{K_a(\delta - 1)^2} \right] \left. \right\} d\text{Fo}_1 \end{aligned} \quad (4-7-11)$$

$$\begin{aligned}
\theta_{\infty 2}(\eta, \text{Fo}) &= \frac{H}{16} (1 - \eta^4) + 2 \sum_{n=1}^{\infty} \frac{\mu_n J_0(\mu_n \eta)}{J_1(\mu_n)} (T_1 - T_0)^{-1} \times \\
&\times \int_0^{\text{Fo}} [\chi_2(\infty, \text{Fo}_1) - T_0] \exp[-\mu_n^2(\text{Fo} - \text{Fo}_1)] d\text{Fo}_1 - \frac{H}{8} \sum_{n=1}^{\infty} \frac{J_0(\mu_n \eta)}{J_1^2(\mu_n)} \times \\
&\times \exp(-\mu_n^2 \text{Fo}) \int_0^1 (1 - \eta_1^4) \eta_1 J_0(\mu_n \eta_1) d\eta_1 \\
\theta_{\infty 2} &= \pi \sum_{n=1}^{\infty} \frac{J_0^2(\alpha_n) \alpha_n^2 u_0(\alpha_n \eta)}{(T_{s1} - T_{s0})[J_0^2(\alpha_n) - J_0^2(\delta \alpha_n)]} \int_0^{\text{Fo}} [\chi_2(\infty, \text{Fo}) - T_{s0}] \exp \times \\
&\times [-\alpha_n^2(\text{Fo} - \text{Fo}_1)] d\text{Fo}_1 - \pi (T_{s1} - T_{s0})^{-1} \sum_{n=1}^{\infty} \frac{J_0(\delta \alpha_n) J_0(\alpha_n) u_0(\eta \alpha_n) \alpha_n^2}{J_0^2(\alpha_n) - J_0^2(\delta \alpha_n)} \times \\
&\times \int_0^{\text{Fo}} [\psi_2(\infty, \text{Fo} - \text{Fo}_1) - T_{s0}] \exp [-\alpha_n^2 \text{Fo}_1] d\text{Fo}_1 \quad (4-7-12)
\end{aligned}$$

where  $\alpha_n, \mu_n$  are respectively the roots of the equations

$$J_0(\alpha) Y_0(\alpha \delta) - J_0(\alpha \delta) Y_0(\alpha) = 0; \quad J_0(\mu) = 0, \quad (4-7-13)$$

$$u_0(\alpha \eta) = J_0(\alpha \eta) Y_0(\alpha \delta) - J_0(\alpha \delta) Y_0(\alpha \eta) \quad (4-7-14)$$

The boundary-value problem (4-7-1) through (4-7-10) is solved by the Laplace transformation with respect to time; the transformed problem in this case looks like the corresponding steady-state problem (it is solved in the same way as described above). Omitting the intermediate calculations [4-9], we shall give temperature fields in a fluid and solid plane and circular tubes, respectively

$$\begin{aligned}
\theta_j(\xi, \eta, \text{Fo}) &= \theta_{\infty j}(\eta, \text{Fo}) - 1 + 2 \sum_{n=1}^{\infty} \beta_{nj} \exp\left(-\frac{\beta_{nj}^2 \xi}{\text{Pe}}\right) \int_0^1 \left\{ 2\Phi_{1j}(\eta, \eta_1, \text{Fo}) \right. \\
&- \theta_{\infty j}(\eta_1, 0) \Phi_{1j}(\eta, \eta_1, \text{Fo}) - \int_0^{\text{Fo}} \Phi_{1j}(\eta, \eta_1, \text{Fo}_1) \theta_{\infty j}(\eta, \text{Fo} - \text{Fo}_1) d\text{Fo}_1 \Big\} d\eta_1 \\
&+ \frac{2}{\text{Pe}(T_1 - T_0)} \sum_{n=1}^{\infty} \beta_{nj}^3 \int_0^1 d\eta_1 \left\{ \int_0^{\xi} \exp[-\beta_{nj}^2(\text{Fo} - \text{Fo}_1)] \int_0^{\text{Fo}} \varphi_{1j}(\eta, \eta_1, \text{Fo}_1) \right. \\
&\times [\chi_j(\xi_1, \text{Fo} - \text{Fo}_1) - \chi_j(\infty, \text{Fo} - \text{Fo}_1)] d\text{Fo}_1 \Big\} d\xi_1 \quad (4-7-15)
\end{aligned}$$

$$\begin{aligned}
 \theta_{s1}(\xi, \eta, Fo) = & \frac{(T_{s1} - T_{sa})^{-1}}{K_a(\delta - 1)} \int_0^{Fo} \left\{ [\psi_1(\infty, Fo_1) - T_{s0}] \frac{\partial}{\partial \eta} v_3 \right. \\
 & \times \left[ \frac{\eta - 1}{2(\delta - 1)}, \frac{Fo - Fo_1}{K_a(\delta - 1)^2} \right] - [\chi_1(\infty, Fo_1) \\
 & \left. - T_{s0}] \frac{\partial}{\partial \eta} v_3 \left[ \frac{\delta - \eta}{2(\delta - 1)}, \frac{Fo - Fo_1}{K_a(\delta - 1)^2} \right] \right\} \times dFo_1 \\
 & + \frac{(T_{s1} - T_{s0})^{-1}}{K_a(\delta - 1)} \left\{ \frac{2}{\pi} \int_0^\infty \sin \alpha \xi d\alpha \int_0^\infty \frac{\partial}{\partial \eta} v_3 \left[ \frac{\delta - \eta}{2(\delta - 1)}, \frac{Fo_1}{K_a(\delta - 1)^2} \right] \right. \\
 & \times \exp \left( -\frac{\alpha^2 Fo_1}{K_a} \right) \psi_1 \left( \alpha, Fo - Fo_1 \right) - \frac{\partial}{\partial \eta} v_3 \left[ \frac{\eta - 1}{2(\delta - 1)}, \frac{Fo_1}{K_a(\delta - 1)^2} \right] \\
 & \times \exp \left( -\frac{\alpha^2 Fo_1}{K_a} \right) \chi_1(\alpha, Fo - Fo_1) dFo_1 \left. \right\} + 2 \frac{(T_{s1} - T_{s0})^{-1}}{K_a \sqrt{\pi}(\delta - 1)} \\
 & \times \int_0^{Fo} \operatorname{erf} \left( \frac{\xi \sqrt{K_a}}{2 \sqrt{Fo_1}} \right) \left\{ \frac{\partial}{\partial \eta} v_3 \left[ \frac{\eta - 1}{2(\delta - 1)}, \frac{Fo_1}{K_a(\delta - 1)^2} \right] \chi_1(\infty, Fo - Fo_1) \right. \\
 & \left. - \frac{\partial}{\partial \eta} v_3 \left[ \frac{\delta - \eta}{2(\delta - 1)}, \frac{Fo_1}{K_a(\delta - 1)^2} \right] \psi_1(\infty, Fo - Fo_1) \right\} dFo_1 \\
 & + \frac{\xi K_a}{4 \sqrt{\pi}(\delta - 1)} \times \int_1^\delta d\eta_1 \int_0^{Fo} \left\{ \hat{v}_3 \left[ \frac{\eta - \eta_1}{2(\delta - 1)}, \frac{Fo_1}{K_a(\delta - 1)^2} \right] \right. \\
 & \left. - \hat{v}_3 \left[ \frac{\eta + \eta_1 - 2}{2(\delta - 1)}, \frac{Fo_1}{K_a(\delta - 1)^2} \right] \right\} \\
 & \times \frac{\exp \left( \frac{\xi^2 K_a}{4 Fo_1} \right)}{Fo_1^{1.5}} [\theta_{s\infty 1}(\eta, Fo - Fo_1) - 1] dFo_1 + \theta_{s\infty 1}(\eta, Fo) \quad (4-7-16)
 \end{aligned}$$

$$\begin{aligned}
 \theta_{s2}(\xi, \eta, Fo) = & \theta_{s\infty 2}(\eta, Fo) + \frac{2}{\pi} \int_0^\infty \int_0^{Fo} \frac{I_0(\alpha \delta) K_0(\alpha \eta) - I_0(\alpha \eta) K_0(\alpha \delta)}{I_0(\alpha \delta) K_0(\alpha) - I_0(\alpha) K_0(\alpha \delta)} \\
 & \times \sin \alpha \xi \chi'_{s1}(\alpha, Fo_1) d\alpha dFo_1
 \end{aligned}$$

$$\begin{aligned}
& + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\alpha_n^2 J_0(\alpha_n) I_0(\delta \alpha_n) [J_0(\alpha_n \eta) Y_0(\alpha_n) - J_0(\alpha_n) Y_0(\alpha_n \eta)]}{(T_{s1} - T_{s0}) [J_0^2(\alpha_n) - J_0^2(\delta \alpha_n)]} \int_0^{\infty} \sin \alpha \xi d\alpha \\
& \times \int_0^{\text{Fo}} \exp [-K_a \text{Fo}_1 (\alpha^2 + \alpha_n^2)] \frac{\chi_2'(\alpha, \text{Fo} - \text{Fo}_1)}{\alpha^2 + \alpha_n^2} d\text{Fo}_1 - \frac{2(T_{s1} - T_{s0})^{-1}}{\pi} \\
& \times \int_0^{\infty} \sin \alpha \xi d\alpha \int_0^{\text{Fo}} F_2(1, \delta, \eta, \alpha) \chi_2(\infty, \text{Fo} - \text{Fo}_1) d\text{Fo}_1 - \frac{2(T_{s1} - T_{s0})^{-1}}{\pi} \\
& \times \sum_{n=1}^{\infty} \frac{\alpha_n^2 J_0(\alpha_n) J_0(\delta \alpha_n) [J_0(\alpha_n \eta) Y_0(\alpha) - J_0(\alpha_n) Y_0(\alpha_n \eta)]}{J_0^2(\alpha_n) - J_0^2(\delta \alpha_n)} \int_0^{\infty} \sin \alpha \xi d\alpha \\
& \times \int_0^{\infty} \frac{\exp [-K_a \text{Fo}_1 (\alpha^2 + \alpha_n^2)]}{\alpha + \alpha_n^2} \left[ \psi_{s2}(\alpha, \text{Fo} - \text{Fo}_1) - \frac{\psi_2(\infty, \text{Fo} - \text{Fo}_1)}{\alpha} \right] d\text{Fo}_1 \\
& + \frac{2}{\pi} \int_0^{\infty} \alpha \sin \alpha \xi \frac{I_0(\alpha \eta) K_0(\alpha) - K_0(\alpha \eta) I_0(\alpha)}{K_0(\alpha) I_0(\alpha \delta) - I_0(\alpha) K_0(\alpha \delta)} d\alpha \int_1^{\delta} \int_0^{\text{Fo}} [I_0(\alpha \eta_1) K_0(\alpha \delta) \\
& - I_0(\alpha \delta) K_0(\alpha \eta_1)] [\theta'_{s\infty 2}(\eta_1, \text{Fo} - \text{Fo}_1) - \alpha] d\text{Fo}_1 d\eta_1 + \frac{2}{\pi} \int_0^{\infty} \sin \alpha \xi d\alpha \\
& \times \sum_{n=1}^{\infty} \frac{\alpha_n^2 J_0(\alpha_n) J_0(\delta \alpha_n)}{J_0^2(\alpha_n) - J_0^2(\delta \alpha_n)} [J_0(\alpha_n \eta) Y_0(\alpha_n) - Y_0(\alpha_n \eta) J_0(\alpha_n)] \\
& \times \int_1^{\delta} \int_0^{\text{Fo}} [J_0(\alpha_n \eta_1) Y_0(\alpha_n \delta) - Y_0(\alpha_n \eta_1) J_0(\alpha_n \delta)] \\
& \times \frac{\exp [-K_a \text{Fo}_1 (\alpha^2 + \alpha_n^2)] \alpha}{\alpha^2 + \alpha_n^2} [\theta'_{s\infty 2}(\eta_1, \text{Fo} - \text{Fo}_1) - 1] d\eta d\text{Fo}_1 \quad (4-7-17)
\end{aligned}$$

where

$$\begin{aligned}
\Phi_{1j}(\eta, \eta_1, \text{Fo}) &= \frac{R_{nj}(\eta, 0) R_{nj}(\eta_1, 0) \eta_1^j (1 - \eta_1^2)}{\left( \frac{\partial R_{nj}}{\partial \alpha_{nj}} \frac{\partial R_{nj}}{\partial \eta} \right)_{\eta=1}} \\
&+ \sum_{i=1}^{\infty} \frac{R_{nj}(\eta, P_i) R_{nj}(\eta_1, P_i) \eta_1^j (1 - \eta_1^2) \exp(P_{ij} \text{Fo})}{\left\{ \frac{\partial}{\partial P_j} \left[ \frac{\partial R_{nj}}{\partial \beta_{nj}} \frac{\partial R_{nj}}{\partial \eta} \right] \right\}_{P=P_i}}
\end{aligned}$$



$$R_{nj} = \exp\left(-\frac{\beta\eta^2}{2}\right) {}_1F_1\left[\frac{P}{4\beta} - \frac{\beta}{4} + \frac{j+1}{4}, \frac{j+1}{2}, \beta\eta^2\right] \quad (4-7-18)$$

$P_{ij}$  and  $\dot{\beta}_{nj}$  are found from the system of transcendental equations

$${}_1F_1\left[\frac{P}{4\beta} - \frac{\beta}{4} + \frac{j+1}{4}, \frac{j+1}{2}, \beta\right] = 0$$

$$\left[\frac{\partial R}{\partial \beta} \frac{\partial R}{\partial \eta}\right]_{\eta=1} \quad (4-7-19)$$

$\alpha_n$  are the roots of the equation

$$J_0(\alpha\delta)Y_0(\alpha) - J_0(\alpha)Y_0(\alpha\delta) = 0 \quad (4-7-20)$$

$$F_2(1, \delta, \eta, \mu) = \frac{I_0(\mu\eta)K_0(\mu) - I_0(\mu)K_0(\mu\eta)}{I_0(\mu\delta)K_0(\mu) - I_0(\mu)K_0(\mu\delta)} \quad (4-7-21)$$

$$\mu = (\rho + \alpha^2 K_a)^{0.5} K_a^{-0.5} \quad (4-7-22)$$

The unknown function  $\chi(\xi, Fo)$  will be found by applying to equations (4-7-15)-(4-7-17) the double integral transformation, i.e. the Saplace transformation with respect to time and the generalized Fourier sine transformation with respect to  $\xi$ . Then for the transformation of the unknown function

$$\chi_{sj}(\alpha, P) = \lim_{\sigma \rightarrow 0} \int_0^\infty \int_0^\infty \chi(\xi, Fo) \exp(-P Fo) \exp(-\sigma\xi) \sin \alpha\xi d\xi dFo$$

we get the singular integral equation with the Cauchy kernel

$$\varphi_j(t, P) a_j(t, P) - b_j(t, P) \int_0^\infty \frac{\varphi_j(\tau, P)}{\tau - t} d\tau = f_j(t, P) \quad (4-7-23)$$

where

$$t = \alpha^2; \quad \chi_{sj}(\alpha, P) = \varphi_j(t, P)$$

$$a_1(t, P) = \frac{r}{(T_1 - T_0)} \sum_{n=1}^\infty A_n R'_{n1}(1) \frac{\bar{b}_{n1}^2}{\alpha^2 + \bar{b}_{n1}^2} - \frac{K_\lambda \sqrt{t + PK_a} \coth \sqrt{t + PK_a} (\delta - 1)}{T_{s1} - T_{s0}}$$

$$b_1(t, P) = \frac{r \sqrt{t}}{2(T_1 - T_0)} \sum_{n=1}^\infty \frac{\bar{b}_{n1}}{[t - (\bar{b}_{n1})^2]}$$

$$\begin{aligned}
f_1(t, P) &= r \sqrt{t} \sum_{n=1}^{\infty} \frac{B_{n1} R_{n1}(1)}{\bar{b}_{n1}^2 + t} - \frac{r}{(T_1 - T_0) \text{Pe}} \sum_{n=1}^{\infty} \frac{A_{n1} R_{n1}(1) \bar{b}_{n1}^2(\infty, P)}{t + \bar{b}_{n1}^2} \\
&\quad - K_{\lambda}^{-1} P \frac{\psi(t, P) - \psi(\infty, P)}{T_{s1} - T_{s0}} \frac{1}{\sqrt{t}} \sqrt{PK_a + t} \cosh \sqrt{PK_a + t} (\delta - 1) \\
&\quad - K_{\lambda}^{-1} \frac{\chi(\infty, P \sqrt{PK_a + t})}{T_{s1} - T_{s0}} \coth \sqrt{PK_a + t} (\delta - 1) \\
&\quad - K_{\lambda}^{-1} \sqrt{t} \int_1^{\delta} G'_1(\sqrt{t}, \eta, \eta_1, P) \left[ -\frac{1}{P} + \theta_{\infty 1}(\eta_1 P, \sqrt{t}) \right] d\eta_1 \\
G_1(\eta, \eta_1, \alpha, P) &= \begin{cases} \frac{\sinh [\sqrt{PK_a + \alpha^2} (\delta - \eta_1)] \sinh [\sqrt{PK_a + \alpha^2} (\eta - 1)]}{\sqrt{PK_a + \alpha^2} \sinh [\sqrt{PK_a + \alpha^2} (\delta - 1)]} & 1 < \eta < \eta_1 < \delta \\ \frac{\sinh [\sqrt{PK_a + \alpha^2} (\delta - \eta)] \sinh [\sqrt{PK_a + \alpha^2} (\eta_1 - 1)]}{\sqrt{PK_a + \alpha^2} \sinh [\sqrt{PK_a + \alpha^2} (\delta - 1)]} & 1 < \eta_1 < \eta < \delta \end{cases} \\
A_{nj} &= \frac{2}{\beta_{nj} P} \left[ \frac{1}{\left( \frac{\partial R_{n1}}{\partial \beta_{n1}} \right)_{\eta=1}} - P \frac{\int_0^1 R_{nj}(\eta_1) d\eta_1}{\left( \frac{\partial R_{nj}}{\partial \beta_{nj}} \right)_{\eta=1} \left( \frac{\partial R_{nj}}{\partial \eta} \right)_{\eta=0}} \right] \\
B_{nj} &= \frac{2\beta_{nj} \int_0^1 \left[ \frac{2}{P} - \theta_{\infty j}(\eta_1, P) \right] R_{nj}(\eta_1, P) \eta_1^j (1 - \eta_1^2) d\eta_1}{\left( \frac{\partial R_{nj}}{\partial \beta_{nj}} \frac{\partial R_{nj}}{\partial \eta} \right)_{\eta=1}} \\
a_2(t, P) &= \frac{r}{(T_1 - T_0)} \sum_{n=1}^{\infty} \frac{\bar{b}_{n2}}{\bar{b}_{n2}^2 + t} - K_{\lambda}^{-1} \frac{F'_1(1, \delta, 1, \mu)}{T_{s1} - T_{s0}} \\
b_2(t, P) &= \frac{r \sqrt{t}}{2(T_1 - T_0)} \sum_{n=1}^{\infty} \frac{\bar{b}_{n2}^2}{\bar{b}_{n2}^2 + t} \\
f_2(t, P) &= r \sqrt{t} \sum_{n=1}^{\infty} B_{n2} R'_{n2}(1) \frac{1}{t + \bar{b}_{n2}^2} - \frac{r}{(T_1 - T_0)} \sum_{n=1}^{\infty} A_{n2} R'_{n2}(1) \\
&\quad \times \frac{\bar{b}_{n2}^2}{t + \bar{b}_{n2}^2} \chi(\infty, P) - \frac{\psi_s(\alpha, P) - \frac{T_{s0}}{\sqrt{t}}}{T_1 - T_0} K_{\lambda}^{-1} F'_2(1, \delta, \eta, \mu)
\end{aligned}$$

$$\begin{aligned}
 & - \frac{K_\lambda^{-1} \chi(\infty, P)}{\alpha(T_{s1} - T_{s0})} F_1'(1, \delta, 1, \mu) - \frac{r}{\alpha} \frac{\partial \theta_\infty(1, P)}{\partial \eta} - K_\lambda^{-1} \sqrt{t} \int_0^\delta G_2'(\sqrt{t}, \eta, \eta_1, P) \\
 & \times \left[ \theta_{s\infty 2}(\eta, \eta_1, \sqrt{t}) - \frac{1}{P} \right] d\eta_1 - \frac{K_\lambda^{-1}}{\alpha} \frac{\partial \theta_{s\infty}(1, P)}{\partial \eta} \\
 G_2(\eta, \eta_1, P) = & \begin{cases} \frac{[I_0(\mu\eta)K_0(\mu) - K_0(\mu\eta)I_0(\mu)][I_0(\mu\eta_1)K_0(\mu\delta) - I_0(\mu\delta)K_0(\mu\eta_1)]}{K_0(\mu)I_0(\mu\delta) - I_0(\mu)K_0(\mu\delta)} & 1 < \eta < \eta_1 < \delta \\ \frac{[I_0(\mu\eta_1)K_0(\mu) - K_0(\mu\eta_1)I_0(\mu)][I_0(\mu\eta)K_0(\mu\delta) - I_0(\mu\delta)K_0(\mu\eta)]}{K_0(\mu)I_0(\mu\delta) - I_0(\mu)K_0(\mu\delta)} & 1 < \eta_1 < \eta < \delta \end{cases} \\
 F_1(1, \delta, \eta, \mu) = & \frac{I_0(\mu\delta)K_0(\mu\eta) - I_0(\mu\eta)K_0(\mu\delta)}{I_0(\mu\delta)K_0(\mu) - I_0(\mu)K_0(\mu\delta)} \\
 F_2(1, \delta, \eta, \mu) = & \frac{I_0(\mu\delta)K_0(\mu) - I_0(\mu)K_0(\mu\eta)}{I_0(\mu\delta)K_0(\mu) - I_0(\mu)K_0(\mu\delta)}
 \end{aligned}$$

The integral equation (4-7-23) is solved in a manner similar to the one described in Section 2-2. Therefore, omitting the intermediate calculations, we can write immediately the final transform of the unknown function  $\chi_j(\xi, \text{Fo})$ . As before, it is easy to demonstrate that the index in the Riemann boundary-value problem, corresponding to integral equation (4-7-23), equals zero

$$\begin{aligned}
 \chi_{sj}(\alpha, P) = & 0,5 g_j(\alpha^2, P) [1 + G_j^{-1}(\alpha^2, P)] + \sqrt{G_j(\alpha^2, P)} \exp[\Gamma_j(\alpha^2, P)] \\
 & \times [1 - G_j^{-1}(\alpha^2, P)] \frac{1}{\pi i} \int_0^\infty \frac{g_i(y^2, P) y dy}{\sqrt{G_j(y^2, P)} \exp[\Gamma_j(y^2, P)] (y^2 - \alpha^2)} \quad (4-7-24)
 \end{aligned}$$

where

$$\begin{aligned}
 G_j(t, P) = & \frac{a_j(t, P) + b_j(t, P)\pi i}{a_j(t, P) - b_j(t, P)\pi i}; \quad g_j(t, P) = \frac{f_j(t, P)}{a_j(t, P) - b_j(t, P)\pi i}; \\
 \Gamma_j(t, P) = & \frac{1}{2\pi i} \int_0^\infty \frac{\log_e G_j(P, \tau)}{\tau - t} d\tau
 \end{aligned}$$

The unknown function is found from the inversion formula

$$\begin{aligned}
 \chi_j(\xi, \text{Fo}) = & \frac{1}{\pi^2 i} \lim_{\sigma \rightarrow 0} \int_0^\infty \sin \alpha \xi d\alpha \int_{c-i\infty}^{c+i\infty} \exp(-\sigma\alpha) \times \\
 & \times \exp(P \text{ Fo}) \chi_s(\alpha, P) dP \quad (4-7-25)
 \end{aligned}$$

The temperature distributions in a fluid and solid are finally found by substituting (4-7-25) into (4-7-15)-(4-7-17). The problem is thus solved completely.

The present problem has been solved by a different method in [4-11]. The fluid velocity  $v_f$  in a tube is assumed constant and uniform in the cross section. But the fluid temperature at the fluid-wall interface [ $\theta(R, x, \tau) = \Phi(x, \tau)$ ] is assumed different from the tube wall temperature [ $T(R, x, \tau) = \varphi(x, \tau)$ ], where  $R$  is the internal tube radius. Thus at the fluid-wall interface a sudden change in temperature occurs, whereas heat fluxes are the same. These assumptions, although somewhat unusual, allow the complete solution of the problem to be found.

The fluid temperature is denoted through  $\theta(x, r, \tau)$ , and that of the wall through  $T(x, r, \tau)$ , where  $x$  is the direction along the tube (the same as the fluid flow direction).

The operators  $\mathcal{L}_1$  and  $\mathcal{L}_2$  defined by

$$\mathcal{L}_1(\cdot) = \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} - \frac{v_f}{a_1} \frac{\partial}{\partial x} - \frac{1}{a_1} \frac{\partial}{\partial \tau} \quad (4-7-26)$$

$$\mathcal{L}_2(\cdot) = \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} - \frac{1}{a_2} \frac{\partial}{\partial \tau} \quad (4-7-27)$$

are introduced, where  $a_1$  and  $a_2$  are the thermal diffusivities of the fluid and the wall, respectively.

The system of differential equations is of the form

$$\mathcal{L}_1(\theta) = 0; \quad 0 < r < R; \quad x \geq 0; \quad \tau \geq 0 \quad (4-7-28)$$

$$\mathcal{L}_2(T) = 0; \quad R < r < R_e; \quad x \geq 0; \quad \tau \geq 0, \quad (4-7-29)$$

where  $R_e$  is the external tube radius;  $(R_e - R)$  is the tube wall thickness.

The initial conditions

$$\left. \begin{array}{l} \theta = 0; \quad 0 < r < R \\ T = 0; \quad R < r < R_e \end{array} \right\} x \geq 0; \quad \tau \geq 0 \quad (4-7-30)$$

The boundary conditions

$$\left. \begin{array}{l} \frac{\partial \theta}{\partial r} = 0 \text{ at } r = 0; \quad \frac{\partial T}{\partial r} = 0 \text{ at } r = R_e \\ \theta = 1 \quad 0 < r < R \\ T = 1 \quad R < r < R_e \end{array} \right\} \bar{x} = 0 \quad \tau > 0 \quad \left. \vphantom{\begin{array}{l} \frac{\partial \theta}{\partial r} = 0 \text{ at } r = 0; \quad \frac{\partial T}{\partial r} = 0 \text{ at } r = R_e \\ \theta = 1 \quad 0 < r < R \\ T = 1 \quad R < r < R_e \end{array}} \right\} x \geq 0; \quad \tau \geq 0 \quad (4-7-31)$$

The conjugation conditions

$$\lambda_1 \left( \frac{\partial \theta}{\partial r} \right)_{r=R} = \alpha(\psi - \Phi); \quad \lambda_2 \left( \frac{\partial T}{\partial r} \right)_{r=R} = \alpha(\psi - \Phi); \quad x > 0 \quad \tau \geq 0 \quad (4-7-32)$$

where  $\Phi$  and  $\psi$  are the prescribed temperatures of the fluid and the tube wall at the boundary of their contact

$$\Phi(x, \tau) = \theta(R, x, \tau); \quad \psi(x, \tau) = T(R, x, \tau) \quad (4-7-33)$$

In a particular case, when the heat-transfer coefficient  $\alpha \rightarrow \infty$ ,  $\psi \equiv \Phi$ , the conjugation is obtained in the form of the fourth kind boundary condition

$$\lambda_1 \left( \frac{\partial \theta}{\partial r} \right)_{r=R} = \lambda_2 \left( \frac{\partial T}{\partial r} \right)_{r=R}; \quad x \geq 0, \quad \tau \geq 0 \quad (4-7-34)$$

Functions  $\Phi$  and  $\psi$  are interrelated by

$$\psi = \Phi + \frac{\lambda_1}{\alpha} \left( \frac{\partial \theta}{\partial r} \right)_{r=R} \quad (4-7-35)$$

For the solution of the problem under consideration the Laplace integral transformation is used. With this aim the theorem of integral transformation of the generalized function was used.

Let the function of the Laplace transform  $f(s)$  be of the form

$$f(s) = \frac{1}{s} \exp [-\sqrt{m(s)}] \quad (4-7-36)$$

where  $m(s)$  is the meromorphous function with its zeroes  $s = -a_k$  and poles  $s = -b_k$  satisfying the inequality

$$0 < a_1 < b_1 < a_2 < b_2 < \dots a_N < b_N \leq \infty \quad (4-7-37)$$

Then for  $\tau > 0$  the Laplace inverse transformation gives the original function  $f(s)$

$$F(t) = L^{-1}\{f(s)\} = \exp [-\mu(0)]$$

$$- \frac{1}{\pi} \sum_{k=1}^N \int_{a_k}^{b_k} \frac{1}{x} \exp (-x\tau) \sin [\mu(x)] dx \quad (4-7-38)$$

where

$$\mu(x) = \sqrt{|m(-x)|} \quad (4-7-39)$$

If we have a finite number  $(N - 1)$  of pairs  $a_k < b_k$  and an additional zero  $s = -a_N$  ( $a_N > b_{N-1}$ ), the  $N$ -th pole  $s = -b_N = -\infty$  should be added so as to make  $N$  pairs. Given an infinite number of pairs  $a_k < b_k$ ,  $N = \infty$  is assumed.

The resultant solution is

$$\theta(r, x, \tau) = \sum_{n=1}^{\infty} [Q_{1,n}(r)X_{1,n}(x, \tau) + Q_{2,n}(r) \int_0^{\infty} \int_0^{\tau} \Phi(\xi, \tau - \eta) \times K_{1,n}(x, \xi, \tau, \eta) d\eta d\xi] \quad (4-7-40)$$

$$T(r, x, \tau) = \sum_{n=1}^{\infty} [Q_{3,n}(r)X_{2,n}(x, \tau) + Q_{4,n}(r) \times \int_0^{\infty} \int_0^{\tau} \psi(\xi, \tau - \eta) K_{2,n}(x, \xi, \tau, \eta) d\eta d\xi] \quad (4-7-41)$$

where the coefficients  $Q, X, K$  entering into the solution (4-7-40)-(4-7-41) are equal to

$$x_{1,n}(x, \tau) = \exp\left(\frac{v_f x}{2a_1}\right) \left[ e^{-v_n x} \operatorname{erfc}\left(\frac{x - 2v_n a_1 \tau}{2\sqrt{a_1 \tau}}\right) + e^{v_n x} \operatorname{erfc}\left(\frac{x + 2v_n a_1 \tau}{2\sqrt{a_1 \tau}}\right) \right] \quad (4-7-42)$$

$$v_n^2 = \beta_n^2 + (v_f/2a_1)^2 \quad (4-7-43)$$

where  $\beta_n$  are the roots of the characteristic equation

$$J_0(\beta R) = 0 \quad (4-7-44)$$

$$X_{2,n}(x, \tau) = e^{-\gamma_n x} \operatorname{erfc}\left(\frac{x - 2a_2 \gamma_n \tau}{2\sqrt{a_2 \tau}}\right) + e^{\gamma_n x} \times \operatorname{erfc}\left(\frac{x + 2a_2 \gamma_n \tau}{2\sqrt{a_2 \tau}}\right) \quad (4-7-45)$$

where  $\gamma_n$  are the roots of the characteristic equation

$${}_0\mathcal{P}_1(R, R_e, \gamma) = 0 \quad (4-7-46)$$

Here  ${}_0\mathcal{P}_1$  is the function defined by

$${}_k\mathcal{P}_n(r, R_e; \gamma) = J_k(\gamma r)Y_n(\gamma R_e) - Y_R(\gamma r)J_n(\gamma R_e) \quad (4-7-47)$$

where  $J$  and  $Y$  are the Bessel functions of the first and second kind

$$Q_{1,n}(r) = \frac{1}{\beta_n R} \frac{J_0(\beta_n \tau)}{J_1(\mu_n R)} \quad (4-7-48)$$

$$Q_{2,n}(r) = \frac{1}{\beta_n R} \frac{J_0(\beta_n r)}{J_1(\beta_n R)} \quad (4-7-49)$$

$$Q_{3,n} = \frac{R_1}{\gamma_n} \frac{{}_1\mathcal{P}_1(R, R_e; \gamma_n) {}_0\mathcal{P}_1(r, R_e; \gamma_n)}{R_{e1}^2 \mathcal{P}_1^2(R, R_e; \gamma_n) - R_{e0}^2 \mathcal{P}_1^2(R, R_e; \gamma_n)} \quad (4-7-50)$$

$$Q_{4,n} = -\frac{1}{\gamma_n R_e} \left[ \frac{{}_0\mathcal{P}_1(r, R_e, \gamma_n)}{{}_0\rho_0(R, R_e; \gamma_n) + \frac{R}{R_e} {}_1\mathcal{P}_1(R, R_e; \gamma_n)} \right] \quad (4-7-51)$$

$$K_{1,n}(x, \xi, \tau, \eta) = \sqrt{\frac{a_n}{\pi \eta}} \beta_n^2 \exp \left[ -\frac{v_f(x - \xi)}{2a_1} - \eta \left( \frac{v_f^2}{4a_1} + a_1 \beta_n^2 \right) \right] \quad (4-7-52)$$

$$\times \left\{ \exp \left[ -\frac{(\xi - x)^2}{4a_1 \eta} \right] - \exp \left[ -\frac{(\xi + x)^2}{4a_1 \eta} \right] \right\}$$

$$K_{2,n}(x, \xi, \tau, \eta) = \gamma_n^2 \sqrt{\frac{a_2}{\pi \eta}} \exp(-a_2 \gamma_n^2 \eta) \left\{ \exp \left[ -\frac{(\xi - x)^2}{4a_2 \eta} \right] - \exp \left[ -\frac{(\xi + x)^2}{4a_2 \eta} \right] \right\} \quad (4-7-53)$$

In order to obtain a complete solution of the problem, determination of functions  $\Phi$  and  $\psi$  is necessary. They are related by

$$\psi = \Phi + \frac{\lambda_1}{\alpha} \left( \frac{\partial \theta}{\partial r} \right)_{r=R} \quad (4-7-54)$$

Derivatives  $\left( \frac{\partial \theta}{\partial r} \right)_R$  and  $\left( \frac{\partial T}{\partial r} \right)_R$  are equal to

$$\left( \frac{\partial \theta}{\partial \tau} \right)_{r=R} = -\frac{1}{R} \sum_{n=1}^{\infty} \left[ X_{1,n}(x, \tau) + \int_0^{\tau} \int_0^{\infty} \Phi(\xi, \tau - \eta) K_{1,n}(x, \xi, \tau, \eta) d\eta d\xi \right] \quad (4-7-55)$$

$$\left(\frac{\partial T}{\partial \tau}\right)_{r=R} = \sum_{n=1}^{\infty} \left[ Q'_{3,n}(R) X_{2,n}(x, \tau) + R'_{4,n} \int_0^{\infty} \int_0^{\tau} \psi(\xi, \tau - \eta) K_{2,n}(x, \xi, \tau, \eta') d\eta d\xi \right] \quad (4-7-56)$$

If the boundary conditions (4-7-31) and formula (4-7-54) are used, we get the integral equation

$$A(x, \tau) + \int_0^{\infty} \int_0^{\tau} B(x, \xi, \tau, \eta) \Phi(\xi, \tau - \eta) d\xi d\eta = \frac{1}{\alpha} \left[ C(x, \tau) + \int_0^{\tau} \int_0^{\infty} \int_0^{\tau-\eta} D(x, \tau, \eta, \varepsilon, \beta) \Phi(\varepsilon, \tau - \eta - \beta) d\beta d\varepsilon d\eta \right] \quad (4-7-57)$$

where

$$\begin{aligned} A(x, \tau) &= \sum_{n=1}^{\infty} \left[ \frac{\lambda_1}{R} X_{1,n}(x, \tau) + \lambda_2 R'_{3,n}(R) X_{2,n}(x, \tau) \right] \\ B(x, \xi, \tau, \eta) &= \sum_{n=1}^{\infty} \left[ \frac{\lambda_1}{R} K_{1,n}(x, \xi, \tau, \eta) + \lambda_2 R'_{4,n}(R) K_{2,n}(x, \xi, \tau, \eta) \right] \\ C(x, \tau) &= \lambda_1 \lambda_2 \sum_{n=1}^{\infty} R'_{4,n}(R) \int_0^{\infty} \int_0^{\tau} \sum_{m=1}^{\infty} X_{1,m}(\xi, \tau - \eta) K_{2,n}(x, \xi, \tau, \eta) d\eta d\xi \\ D(x, \tau, \eta, \varepsilon, \beta) &= \lambda_1 \lambda_2 \sum_{n=1}^{\infty} R_{n,4}(R) \sum_{m=1}^{\infty} \int_0^{\infty} K_{1,m}(x, \xi, \tau - \eta, \beta) \\ &\quad \times K_{2,n}(x, \xi, \tau, \eta) d\xi \end{aligned}$$

Thus, even with considerable simplifications the problem is solved almost in the same way as in the case of exact solution.

#### 4-8. CONJUGATE HEAT-TRANSFER PROBLEM WITH TURBULENT FLUID FLOW

In this section we shall demonstrate that conjugate solutions considered in the previous sections for laminar flows may be successfully applied to turbulent flows. It should be pointed out that to avoid repetitions (and due to lack of space) only the main points of the solution will be given.



### (a) Incompressible Fluids

In this case the transfer equations in a dimensionless form will be written as

$$\text{Pe} f_j(\eta) \frac{\partial \theta_j}{\partial \xi} = \frac{1}{\eta^j} \frac{\partial}{\partial \eta} \left[ \eta^j g_j(\eta) \frac{\partial \theta_j}{\partial \eta} \right]; \quad \begin{matrix} 0 < \eta < 1 \\ 0 < \xi < \infty \end{matrix} \quad (4-8-1)$$

$$\theta_j|_{\xi=0} = 1; \quad \left. \frac{\partial \theta_j}{\partial \xi} \right|_{\xi \rightarrow \infty} = 0 \quad (4-8-2)$$

$$\left. \frac{\partial \theta_j}{\partial \eta} \right|_{\eta=0} = 0; \quad \theta_j|_{\eta=1} = \frac{\chi_j(\xi)}{T_1} \quad (4-8-3)$$

$$\frac{\partial^2 \theta_{sj}}{\partial \xi^2} + \frac{\partial^2 \theta_{sj}}{\partial \eta^2} = 0; \quad 1 < \eta < \delta; \quad 0 < \xi < \infty \quad (4-8-4)$$

$$\theta_{sj}|_{\xi=0} = 1; \quad \left. \frac{\partial \theta_{sj}}{\partial \xi} \right|_{\xi \rightarrow \infty} = 0 \quad (4-8-5)$$

$$\theta_{sj}|_{\eta=1} = \frac{\chi_j(\xi)}{T_{s1}}; \quad \theta_{sj}|_{\eta=\delta} = \frac{\psi_j(\xi)}{T_{s1}} \quad (4-8-6)$$

$$T_1 \theta_j|_{\eta=1} = T_{s1} \theta_{sj}|_{\eta=1} = \chi_j(\xi) \quad (4-8-7)$$

$$r_1 K_\lambda \left. \frac{\partial \theta_j}{\partial \eta} \right|_{\eta=1} = \left. \frac{\partial \theta_{sj}}{\partial \eta} \right|_{\eta=1} \quad (4-8-8)$$

where  $\psi_j(\xi)$  is the prescribed function;  $\chi_j(\xi)$  is an unknown function to be determined later on. The values of velocity distribution  $f_j(\eta)$  and turbulent thermal conductivity  $g_j(\eta)$  may be found, for example, in [4-18].

The temperature field in the fluid for plane and circular tubes may be found in a way similar to that presented in section 4-7. The solution of the boundary-value problem (4-8-1)-(4-8-3) may then be written as

$$\begin{aligned} \theta_j(\eta, \xi) = & \theta_\infty + \sum_{n=1}^{\infty} B_{nj} R_{nj}(\eta) \exp[-\bar{b}_n \xi] \\ & - T_1^{-1} \sum_{n=1}^{\infty} A_{nj} R_{nj}(\eta) \int_0^\xi \exp[-\bar{b}_n(\xi - \xi_1)] [\chi_j(\xi_1) - T_1] d\xi_1 \end{aligned} \quad (4-8-9)$$

where  $R_{nj}$  satisfies the Sturm-Liouville boundary-value problem

$$\frac{1}{\eta^j} \frac{d}{d\eta} \left[ \eta^j g(\eta) \frac{dR_j}{d\eta} \right] - b f(\eta) R_j = 0,$$

$$R_j|_{\eta=0} = 0; \quad R_j|_{\eta=1} = 0$$

Since  $f_j(\eta)$  and  $g_j(\eta)$  may be found experimentally and their analytical expression is very complicated,  $R_{nj}(\eta)$  is determined numerically (see for example [4-18]). Bearing in mind that  $R_{nj}(\eta)$  is orthogonal in the interval (0.1), the expressions for constants  $A_{nj}$  and  $B_{nj}$  are obtained

$$A_{nj} = - \frac{\int_0^1 \eta_1^j f_j(\eta_1) R_{nj}(\eta_1) d\eta_1}{\int_0^1 \eta_1^j f_j(\eta_1) R_{nj}^2(\eta_1) d\eta_1} = \frac{g_j(1) \frac{\partial R_{nj}(1)}{\partial \eta}}{\int_0^1 \eta_1 f_j(\eta_1) R_{nj}^2(\eta_1) d\eta_1}$$

$$B_{nj} = \frac{\int_0^1 [2 - \theta_\infty(\eta_1)] \eta_1 f_j(\eta_1) R_{nj}(\eta_1) d\eta_1}{\int_0^1 \eta_1 f_j(\eta_1) R_{nj}^2(\eta_1) d\eta_1}$$

The temperature distribution in a solid may be found from the inversion formulas discussed earlier, with the assumption that  $A_j = 0$ .

### (b) Compressible Fluids

At small Mach numbers  $M$  for a turbulent gas flow through a circular tube the basic system of equations is written in the form [4-19]

$$D \text{Re}^{0.25} (1 - \eta^2)^{\frac{1}{7}} \frac{\partial \theta}{\partial \xi} = \frac{1}{\eta} \frac{\partial}{\partial \eta} \left[ \eta (1 - \eta^2)^{\frac{6}{7}} \frac{\partial \theta}{\partial \eta} \right]; \quad \begin{matrix} 0 < \eta < 1 \\ 0 < \xi < \infty \end{matrix} \quad (4-8-10)$$

$$\theta|_{\xi=0} = 1; \quad \left. \frac{\partial \theta}{\partial \xi} \right|_{\xi \rightarrow \infty} = 0 \quad (4-8-11)$$

$$\left. \frac{\partial \theta}{\partial \eta} \right|_{\eta=0} = 0; \quad \theta|_{\eta=1} = \frac{\chi(\xi)}{T_1} \quad (4-8-12)$$

$$\frac{\partial^2 \theta_s}{\partial \xi^2} + \frac{\partial^2 \theta_s}{\partial \eta^2} = 0; \quad \begin{matrix} 1 < \eta < \delta \\ 0 < \xi < \infty \end{matrix} \quad (4-8-13)$$

$$\theta_s|_{\xi=0} = 1; \quad \left. \frac{\partial \theta_s}{\partial \xi} \right|_{\xi \rightarrow \infty} = 0; \quad (4-8-14)$$

$$\theta_s|_{\eta=1} = \frac{\chi(\xi)}{T_{s1}}; \quad \theta_s|_{\eta=\delta} = \frac{\psi(\xi)}{T_{s1}} \quad (4-8-15)$$

$$T_1\theta|_{\eta=1} = T_{s1}\theta_s|_{\eta=1} = \chi(\xi) \quad (4-8-16)$$

$$r_1 K_\lambda \frac{\partial \theta}{\partial \eta} \bigg|_{\eta=1} = \frac{\partial \theta_s}{\partial \eta} \bigg|_{\eta=1} \quad (4-8-17)$$

It may be easily seen that if in the system of equations (4-8-1)-(4-8-8)  $Pe \rightarrow D(Re)^{0.25}$ ,  $f(\eta) \rightarrow (1 - \eta)^{1/7}$ , and  $g(\eta) \rightarrow (1 - \eta)^{6/7}$  are assumed, the system coincides with the system of equations (4-8-10)-(4-8-17). Therefore the solution of the latter coincides with (4-8-9), if the earlier mentioned substitution is made in it.

In sections 4-6 through 4-8 the following notations are used:  $T_j$  is the fluid temperature;  $T_{sj}$  is the temperature of the body;  $\theta_j$  and  $\theta_{sj}$  are the dimensionless temperatures of the fluid and solid, respectively

$$\theta_j = T_j(\chi(\infty)); \quad \theta_{sj} = T_{sj}/\chi(\infty) \quad (4-8-18)$$

$\chi(x, \tau)$  is the temperature at the conjugation boundary;

$\psi_j(x, \tau)$  is the temperature at the external boundary of the tube.

For the steady-state problem these temperatures are designated by  $\tilde{\chi}(x)$  and  $\tilde{\varphi}_j(x)$ .

In unsteady-state problems  $\theta_j(x)$  and  $\theta_{sj}(x)$  are defined by

$$\theta_j = \frac{T_j - T_0}{T_1 - T_0}; \quad \theta_{sj} = \frac{T_{sj} - T_{s0}}{T_{s1} - T_{s0}} \quad (4-8-19)$$

where  $T_0$  and  $T_{s0}$  are the initial temperatures of the fluid and solid, respectively;  $T_1$  and  $T_{s1}$  are the temperatures of the fluid and solid, respectively, at  $x = 0$ ;  $x, y$  are the longitudinal and transverse coordinates;  $\xi$  and  $\eta$  are the dimensionless longitudinal and transverse coordinates;  $u_1$  is the mean velocity;  $\lambda$  and  $\lambda_s$  are the thermal conductivity coefficients of the fluid and solid;  $\alpha$  is the Fourier transformation parameter;  $P$  is the Laplace transformation parameter in the integral relations;  $\sigma = Pr$

$$\delta = \frac{R_1}{R}; \quad K_\lambda = \frac{\lambda}{\lambda_s}; \quad K_a = a/a_s \quad (4-8-20)$$

$$r = \frac{T_1 - T_0}{T_{s1} - T_{s0}} \quad (4-8-21)$$

$$Pe = \frac{u_{0j}R}{a}; \quad H_j = \frac{4u_{0j}^2 Pr}{c\tilde{\chi}_j(\infty)} \quad (4-8-22)$$

$$u_{01} = \frac{3}{2} u_1; \quad A_1 = \frac{H_1}{16}; \quad F_1(\eta) = 1 + K_\lambda \frac{H}{3} (1 - \eta) \quad (4-8-23)$$

$$u_{02} = 2u_1; \quad A_2 = \frac{H_2}{12}; \quad F_2(\eta) = 1 - K_\lambda \frac{H}{4} \log_e \eta \quad (4-8-24)$$

$$v_8(u, z) = 1 + 2 \sum_{k=1}^{\infty} \exp(-\pi^2 k^2 z) \cos 2\pi k u \quad (4-8-25)$$

$$\hat{v}_3(u, z) = \frac{1}{\sqrt{\pi z}} \left\{ \sum_{k=0}^{\infty} \exp \left[ -\frac{(u+k)^2}{z} \right] - \sum_{k=-1}^{\infty} \exp \left[ -\frac{(u+k)^2}{z} \right] \right\} \quad (4-8-26)$$

$$v_1(u, z) = 2 \sum_{k=0}^{\infty} (-1)^k \exp \left[ -\pi^2 \left( k + \frac{1}{2} \right)^2 z \right] \sin (2k + 1)u \quad (4-8-27)$$

The subscripts are as follows:  $j = 0$  (or  $j = 1$  in Sec. 4-7) refers to a plane tube;  $j = 1$  (or  $j = 2$  in Sec. 4-7), to a circular tube; 0 refers to initial conditions; 1, to surface conditions;  $s$ , to the Fourier sine transform;  $c$ , to the Fourier cosine transform;  $s$ , to a solid;  $\infty$  refers to the main flow;  $w$  indicates conditions at the wall;  $b$  refers to conditions at the internal surface.

Conjugate convective heat-transfer problems are very important for the study of chemical engineering processes. The works by N. M. Zhavoronkov and his associates [4-22, 4-23] have demonstrated that external heat transfer in a number of chemical reactors is controlled not only by the dynamics of the fluid but also by simultaneous heat and mass transfer. Conjugate problems dealing with the injection of a substance into a boundary layer, and other problems, are considered in [4-24].

## 5-1. STRUCTURAL PROPERTIES

### (a) Porous Properties

A porous medium is understood to be a solid containing pores. Exact geometrical definition of the concept "pores" is rather difficult. Pores are usually considered to be voids distributed in a solid. Pores may be communicating or non-communicating. The communicating portion of a pore is sometimes called effective pore space. The term is commonly used in filtration theory. Many porous bodies have a large total porosity but a small effective one.

Porosity  $\Pi_v$  is usually defined as the ratio of the pore volume  $V_{por}$  to the body volume  $V_{bod}$  ( $\Pi_v = V_{por}/V_{bod}$ ). Together with volume porosity there exists a concept of surface porosity  $\Pi_s$ , which is defined as a ratio of effective pore surface area  $A_{por}$  to the total surface area  $A_{bod}$  ( $\Pi_s = A_{por}/A_{bod}$ ). As a first approximation,  $\Pi_v = \Pi_s = \Pi$  may be assumed.

A system of spherical particles of small radius and equal sizes affords a simple model of a porous body.

Spherical particles may be arranged in different ways (see Fig. 5-1). They may have a loose packing (cubic arrangement,  $\vartheta = 90^\circ$ ) or dense

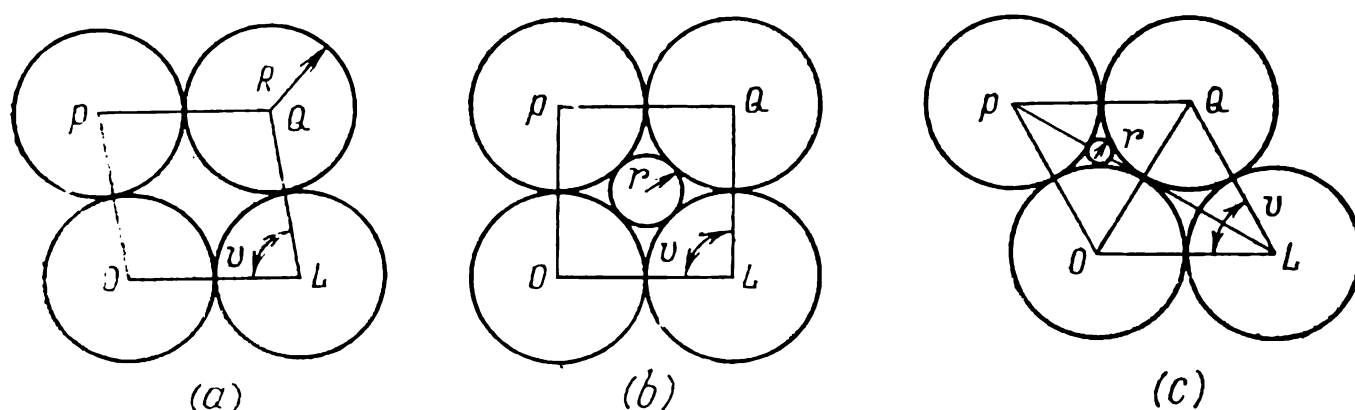


Fig. 5-1. Arrangement of spherical particles during cell formation

(a) general arrangement; (b) cubic arrangement ( $\vartheta = 90^\circ$ ); (c) hexagonal arrangement ( $\vartheta = 60^\circ$ )

packing (hexagonal arrangement,  $\vartheta = 60^\circ$ ). Porosity of such materials depends on the type of arrangement (the angle  $v$ ) rather than on the particle radius. With a cubic arrangement  $\Pi_v = 47.64\%$ , and with a hexagonal arrangement  $\Pi_v = 25.95\%$ . In the former case radii of the narrow-

est passages (the radius of a circle inscribed among four adjacent particles) equal  $0.41 R$  ( $R$  is the sphere radius). Pore radius in the widest passages is  $0.73 R$ .

With a hexagonal arrangement every particle is in contact with 12 neighbouring ones. Pores may be of two shapes, namely, tetrahedral and rhombohedral, the number of latter shapes being twice that of the former. Radius of the narrowest passages (Fig. 5-1b) is  $0.155 R$ . Radius of the widest passages is  $0.288R$  for tetrahedral shapes and  $0.414R$  for rhombohedral pores. Tetrahedral pores make up 7.37% of the entire volume and rhombohedral pores account for 18.58%.

Porosity of a system consisting of uniform spheres should be independent of the sphere radii. However, for natural loose materials such as sand, porosity increases as grain size decreases.

Such a porous system is a body penetrated with capillary tubes with a curved axis following the contour of spherical particles. The capillaries have a triangular cross-section, whose area ranges between a maximum and minimum.

There are some other geometrical properties of porous materials. Pore radius distribution curves  $F(r)$  are often used and granular materials may be characterized by particle size distribution curves (granulometric curves). Particle size distribution are sometimes assumed logarithmically normal and characterized by two parameters: mean dimension  $d$  (particle diameter) and dispersion  $D$ . In most of the reported works only one dimension is presented: either pore size  $d$  or grain size  $l$ .

In chemical engineering the quantity  $1/S_v$  is commonly used as a characteristic dimension. Here  $S_v$  is the total surface area of the particles per unit volume of the porous material (specific surface area of the porous material). This quantity may be readily found experimentally.

Specific surface area  $S_v$  (measured in inverse length units) may be found by the adsorption technique, statistical method or from permeability.

Permeability is the ability of porous materials to transmit liquid due to an adverse pressure gradient. Permeability is the conductivity of materials with respect to liquid.

Permeability or penetrability factor  $K_f$  is determined from the Darcy equation. If  $\Delta p$  is the pressure drop across the sample of thickness  $l$  and  $j_w$  is a volume flow rate of liquid, then

$$K_f = \frac{j_w \eta}{A(\Delta p/l)} \quad (5-1-1)$$

where  $\eta$  is the liquid viscosity and  $A$  is the cross-sectional area of the sample. The factor  $K_f$  is measured in squared length units. Permeability is a rough measure of root-mean-square pore diameter. Permeability is a macroscopic property of porous materials which depends on the geometry of porous structures. A comprehensive survey of the theories is presented in [5-5]. We shall consider some of them here.

In Koseni's theory a porous material is represented by a bundle of thin tubes having the same length. Permeability of such a system is

$$K_f = \frac{a\Pi^3}{S_v^2} \quad (5-1-2)$$

where  $a$  is a dimensionless constant dependent on the shape of the tube cross-section alone. For a circular cross-section  $a = 0.5$ , for a square shape  $a = 0.5619$ , for a cross-section such as an equilateral triangle  $a = 0.5974$ . The quantity  $a$  is called the Koseni constant. Koseni's theory was improved by introducing tortuosity factor  $\xi$ , defined as the ratio of the mean path length of a liquid particle in a material sample to the sample length. Koseni's formula then becomes

$$K_f = \frac{a\Pi^3}{\xi S_v^2} \quad (5-1-3)$$

It is difficult to check Koseni's formula in as much as it is hard to independently determine  $S_v$  and  $\xi$ .

An alternative approach to permeability is to introduce a minimum number of distribution parameters, such as particle size distribution, which is sufficient to find permeability.

The most natural assumption is that permeability is proportional to a squared mean characteristic dimension  $d_{sm}$  and depends on nonuniformity of particle composition and the packing:

$$K_f = d_{sm}^2 k_0(D, \Pi) \quad (5-1-4)$$

where  $D$  is the dispersion of grain distribution.

It is evident that at nonuniform particle distribution a dense parking is possible if fine particles fill the gaps between coarse grains so that  $k_0$  should decrease as  $D$  rises. Correlations of the type (5-1-4) may be composed for individual models of porous materials.

The rate of filtration through porous mediums (Darcy's law) is

$$\vec{v} = -\frac{K_f}{\eta} (\vec{\nabla} p + \rho \vec{\nabla} h) \quad (5-1-5)$$

where  $p$  is the external pressure and  $h$  is the pressure due to gravity.

If the liquid motion potential is defined by

$$\Phi = h + \int_{p_0}^p \frac{1}{\rho} dp \quad (5-1-6)$$

Darcy's law becomes

$$\vec{v} = -\frac{K_f \rho}{\eta} \vec{\nabla} \Phi \quad (5-1-7)$$

For particular liquids it is necessary to introduce into formula (5-1-7) a correction for the Klinkenberg effect [5-1]:

$$\bar{v} = \frac{\rho K_f}{\eta} \left( 1 + \frac{a}{p} \right) \bar{\nabla} \Phi = - \frac{K_a}{\eta} (\bar{\nabla} p + \rho \bar{\nabla} h) \quad (5-1-8)$$

where

$$K_a = K_f \left( 1 + \frac{a}{p} \right), \quad a \text{ is a constant.}$$

From the dimensional analysis and invariant properties it may be assumed that

$$\bar{\nabla} p = - \frac{\eta}{L^2} f \left( \frac{v \rho L}{\eta} \right) \bar{v} \quad (5-1-9)$$

Series expansion of  $f$  with respect to the argument powers and considering the first two terms only, we get

$$\bar{\nabla} p = - \frac{\eta}{k} \bar{v} - \frac{\beta \rho v}{\sqrt{k}} \bar{v} \quad (5-1-10)$$

which is the so-called binomial filtration law. The constant  $\beta$  is an additional characteristic of porous materials. M. D. Millionshchikov suggested  $\sqrt{k/\Pi}$  for a characteristic dimension and  $v_a (\text{Re} = \rho v_m \sqrt{k/\Pi^{3/2}} \eta)$  for a characteristic velocity which corroborates with the available data for the inertia region.

As  $\text{Re}$  increases the quadratic term in expression (5-1-10) becomes more influential: this is observed in all materials consisting of coarse particles. Analogous to pipe-flows, as  $\text{Re}$  increases the turbulent similarity flow arises, the wall roughness being substituted by tortuosity factor. Sometimes the term "turbulent filtration" is given to a flow for which the quadratic correction is essential. However, this is not true since the inertia component of the resistance at nonuniform motion becomes essential before transition to a turbulent flow.

At small filtration velocities deviations from Darcy's law are possible. An increase of filtration velocity, as the pressure gradient rises, has been reported in [5-3] through [5-5], the increase being more rapid than linear. This deviation is observed in colloidal capillary-porous materials.

In this case we may write

$$\bar{v} = - \frac{k}{\eta} \bar{\nabla} p \left( 1 - \frac{Q}{|\bar{\nabla} p|} \right), \quad |\bar{\nabla} p| \gg Q \quad (5-1-11)$$

where  $Q$  is a constant having the same dimension as the pressure gradient.



A similar situation is observed in rheological fluids filtrating through porous materials, such as quartz sand.

In [5-8] filtration of water and electrolyte solutions through argillaceous sandstone with a small amount of clay particles is reported. The results are presented in Fig. 5-2, which shows that the relation between

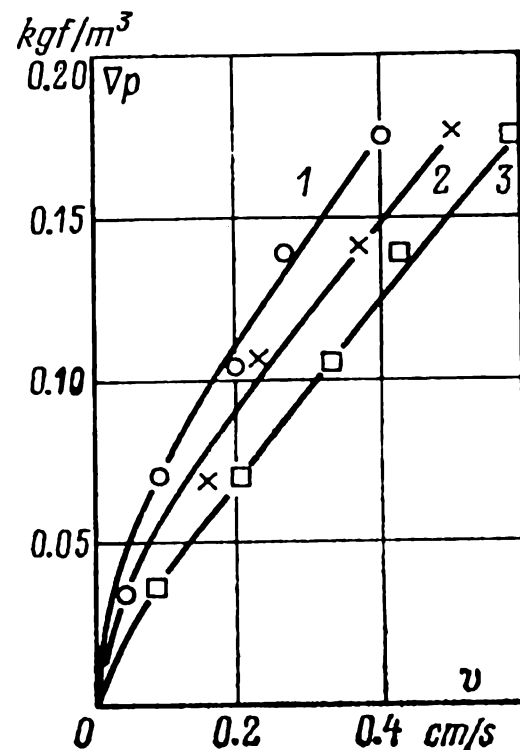


Fig. 5-2. Filtration of electrolytes through argillaceous sandstone  
1 - water; 2 - 10% solution of NaCl; 3 - 5% solution of NaCl

the pressure gradient and filtration velocity is nonlinear. One of the reasons for the deviation from Darcy's law is the stronger bond of water with the solid framework. In particular, osmotic water absorption and capillary liquid bonds are observed in colloidal capillary-porous materials. More thorough consideration of liquid bonds in porous materials is therefore necessary.

Some researchers determine the tortuosity factor  $\xi$  from experiments on vapour diffusion through a porous layer. The essence of the method is as follows. The diffusivity  $D$  of vapour moisture during evaporation from a porous material is determined. Once the diffusivity of vapour into air is known, it is possible to find the diffusion-resistance coefficient  $\mu_D$

$$\mu_D = D_{12}/D \quad (5-1-12)$$

The tortuosity factor will be found from the formula

$$\xi = \mu_D \Pi_s \approx \mu_D \Pi \quad (5-1-13)$$

It may be approximately assumed that  $\Pi_s = \Pi_v = \Pi$ .

It is possible to calculate the coefficient  $\mu_D$  for a system of regularly packed spherical particles [5-2]. A schematic diagram of the calculation pro-

cedure is shown in Fig. 5-3. The diffusion resistance coefficient will then be [5-2].

$$\mu_D = \frac{3A_0}{l} \frac{n_l}{n_p} \int_0^{r\sqrt{3}} \frac{dz}{A} \quad (5-1-14)$$

where  $A_0$  is the total exposed surface area;  $n_l$  is the number of particles within the layer;  $n_p$  is the number of individual passages through the bed;

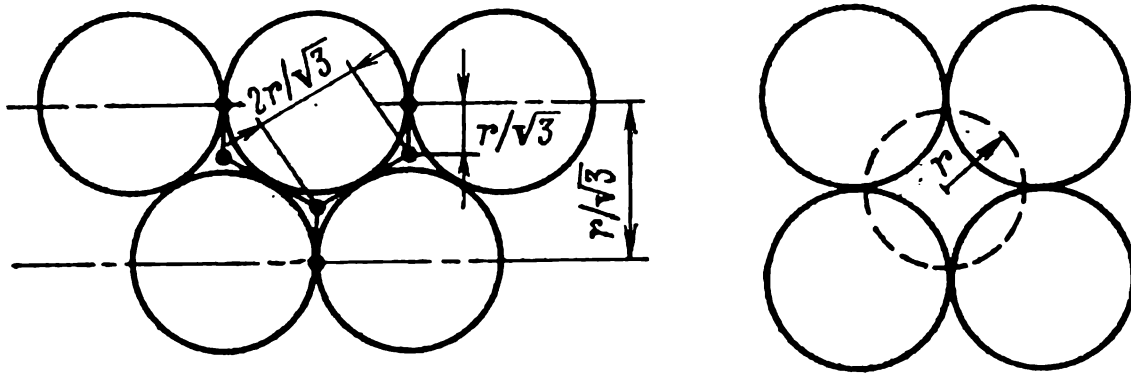


Fig. 5-3. Rhombic four-cornered arrangement of spherical particles

$A$  is the cross-sectional area of single, four-cornered passage that exists between touching spheres; and  $z$  is the spatial coordinate.

For a regular geometry

$$A = \pi \left[ \left( \frac{4}{\pi} - 1 \right) r^2 + z^2 \right] \quad (5-1-15)$$

Then integration of equation (5-1-14) yields

$$\mu_D = \frac{3A_0}{l} \frac{n_l}{n_p} \left\{ 2/\pi r [2\sqrt{3}/\pi - 1]^{1/2} \tan^{-1} 2 / \left( \frac{4}{\pi} - 1 \right) \right\}^{1/2} \quad (5-1-16)$$

Since  $l = \sqrt{3}rn_l$ ,  $A_0 = 4r^2n_p$  then  $\mu_D = 3.52$ . The porosity of the system under consideration is  $\Pi = 0.3955$ , then  $\xi = 0.3955 \times 3.52 = 1.39$ . Experimental and predicted tortuosity factors  $\xi$  are provided in Table 5-1. In experiments on steam diffusion through a bed (30 mm thick) of spherical particles with diameter of 2 mm at 8°C was studied. Predicted and experimental values in Table 5-1 are satisfactorily corroborated.

Krischer [5-76] experimentally found that coefficient  $\mu_D$  for building materials was directly proportional to  $\sim 3\Pi^{-3/2}$ ; tortuosity  $\xi$  is then proportional to

$$\xi \sim \Pi^{-1/2} \quad (5-1-17)$$

Table 5-1

**Comparison of Theoretical with Experimental  
Diffusion-Resistance Parameters for 2 mm Spheres**

Arrangement	Porosity $\Pi$		Diffusion-resistance coefficient $\mu_D$		Tortuosity $\xi$	
	Calc.	Expt.	Calc.	Expt.	Calc.	Expt.
Cubic	0.4764	0.485	2.65	2.95	1.26	1.43
Rhombic, 4-cornered	0.3955	0.4285	3.52	3.72	1.39	1.59
Rhombic, 3-cornered	0.3955	0.409	4.34	4.66	1.72	1.90
Octahedric	0.2595	0.330	5.00	5.05	1.53	1.67
Tetrahedric	0.2595	0.302	6.43	6.00	1.67	1.81
Random	—	0.37	—	4.25	—	1.57

Experimental data on tortuosity  $\xi$  and coefficient  $\mu_D$  for a number of materials are compiled in Table 5-2. It is evident that tortuosity changes

Table 5-2

**Diffusion Parameters for Moisture Movement  
Through Non-Swelling Porous Materials**

Material	Bulk density $\rho_b$ , kg/m <sup>3</sup>	Porosity $\Pi$ , m <sup>3</sup> /m <sup>3</sup>	Tortuosity $\xi$ , m/m	Diffusion-resistance coefficient $\mu_D$
<i>Beds of spheres</i>				
Glass spheres, 1.9 mm	—	0.365	1.13	3.1
Glass spheres, 0.5 mm	—	0.37	1.4	3.8
Sea-sand, 0.2 mm	—	0.36	1.7	4.7
<i>Building materials</i>				
Fibreboard	300-380	0.81-0.76	2.0-2.6 3.0-3.9	2.5-3.2 4.0-5.1
Foam concrete	650-840	0.71-0.62	4.1-5.3 4.3-5.5	5.8-7.5 6.9-8.9
Slag concrete	1140	0.50	4.3-5.0	8.5-10
Lime plaster, 1: 3 sand	1800	0.28	2.4-2.7	8.6-9.8
Cement plaster, 1: 3 gravel	2140	0.21	10.5-17.5	5.0-83.5
<i>Foodstuffs</i>				
Dried vegetables	135	0.907	1.6	1.7
Powdered egg	305	0.79	1.9	2.4
Coffee grounds	400	0.725	1.16	1.6
Chocolate pudding powder	725	0.5	3.4	6.8
Spray-dried milk	750	0.482	1.6	3.3
<i>Other materials</i>				
Full-chrome leather	—	0.280-0.405	1.4-2.6	3.2-7.5
Semi-chrome leather	—	—	1.8-7.7	7.3-33

over a wide range

$$1.13 < \xi < 17.5 \quad (5-1-18)$$

This method for determining the tortuosity factor from the diffusion-resistance coefficient may be criticized in that evaporation in moist materials may be accompanied not only with moisture diffusion but also with capillary and film flows (see Sec. 5-7). Therefore, in [5-10] calculations were made of the effusion-resistance coefficient  $\mu_E$  (resistance of porous material to vapour effusion within the body).

The correlation between theoretical and experimental data presented in Table 5-3 is satisfactory.

Table 5-3

Comparison of Theoretical with Experimental Effusion Parameters for 2-mm Spheres

Arrangement	Porosity $\Pi$ , $\text{m}^3/\text{m}^3$		Effusion-resistance coefficient $\mu_E$		Tortuosity $\xi$ , m/m	
	calc.	expt.	calc.	expt.	calc.	expt.
Cubic	0.4764	0.480	3.76	4.40	1.79	2.11
Rhombic, 4-cornered	0.3955	0.405	4.32	4.29	1.71	1.74
Rhombic, 3-cornered	0.3955	0.404	10.92	10.64	4.32	4.30
Octahedric	0.2595	0.321	9.13	8.54	2.37	2.74
Tetrahedric	0.2595	0.294	9.39	9.38	2.44	2.76
Random	—	0.38	—	5.52	—	2.1

Porous structures are usually characterized by uniformity, anisotropy and heterogeneity. The structural properties are macroscopic. Averaging is performed with respect to the volume element with radius  $R$  which should be sufficiently large to satisfy Darcy's law.

Heterogeneity, uniformity and anisotropy are determined from distribution of permeability probability of elementary volumes of porous materials.

The penetration factor or permeability is described by the differential surface distribution curve  $f(k)$  similar to the differential pore radius distribution curve  $f_s(r)$ .

A uniform material is a material for which  $f(k)$  is either a delta-function ( $\delta$ ) or a linear combination of delta-functions

$$f(k) = \sum_{i=1}^N \delta A_i \quad (5-1-19)$$

where  $A_i$  is a constant consistent with the constraint

$$\sum_{i=1}^N A_i = 1 \quad (5-1-20)$$

Here  $N$  is a finite value.

Examples of uniform and nonuniform materials are presented in Figs. 5-4 and 5-5, respectively.

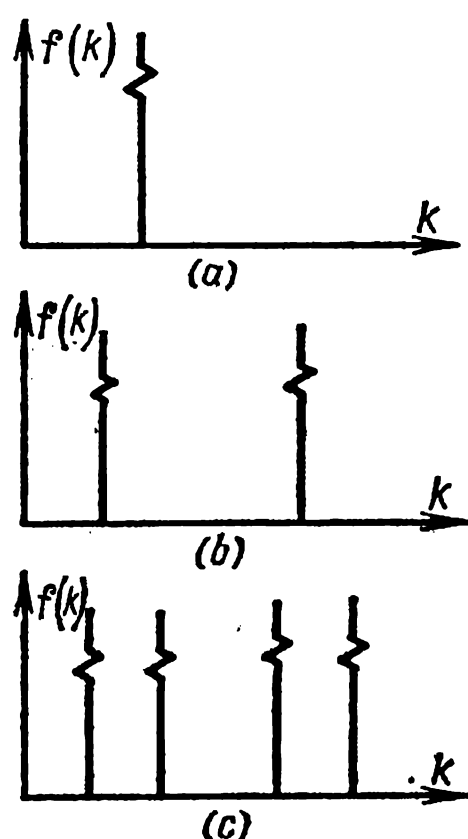


Fig. 5-4. Permeability of uniform materials: homogeneous (a) and heterogeneous (b, c)

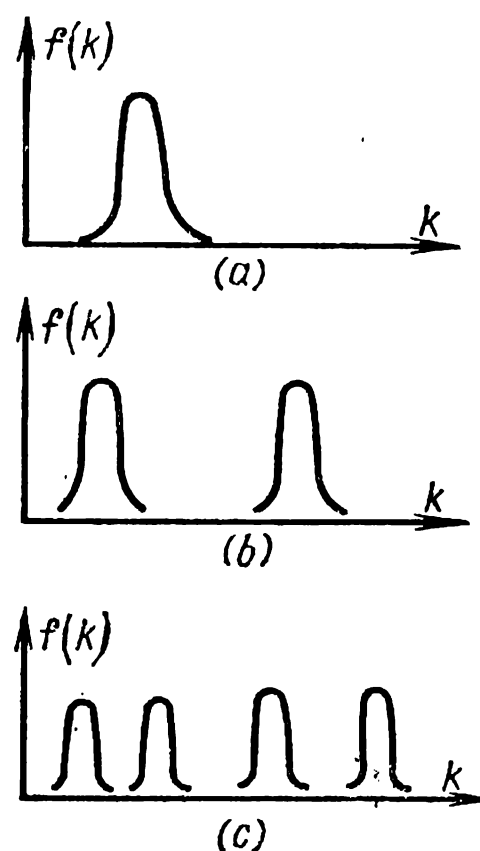


Fig. 5-5. Permeability of nonuniform materials: homogeneous (a) and heterogeneous (b, c)

A nonuniform material may thus be defined as a material for which  $f(k)$  cannot be expressed by a finite number of weighting delta-functions. A heterogeneous material is described by at least two permeability distribution curves (see Fig. 5-4 b, c and Fig. 5-5b, c). Here, it is implied that the material is homoscedastic. If  $f(k)$  depends on the location in space or orientation in a porous material, then we deal with second-kind non-ideality. If permeability in an elementary volume depends on the direction, the material is called anisotropic. In a general case the function  $f(k)$  depends on position and fixing; it may be characterized by five independent variables: rectangular coordinates  $x_i$  ( $i = 1, 2, 3$ ) for position and two angular

coordinates  $\theta$  and  $\psi$  for orientation:

$$p(k_1 \leq k \leq k_2) = \int_{k_1}^{k_2} f(x_i, \theta, \psi) dx_i d\theta d\psi \quad (5-1-21)$$

If the permeability distribution function  $f(k)$  is independent of  $\theta$  and  $\psi$ , the porous material is isotropic. If  $f(k)$  is expressed by a finite linear combination of  $\delta$ -functions, the material is uniform; if  $f(k)$  is monomodel, the material is homogeneous; the second kind non-ideality is characterized by variable  $f(k)$

$$\rho(\text{var}_1 \leq \text{var } k \leq \text{var}_2) = \int_{\text{var}_1}^{\text{var}_2} f(x_i, \theta, \psi) dx_i d\theta d\psi \quad (5-1-22)$$

A porous body may be considered capillary-porous and pores to be capillaries if  $\psi_v \gg \psi_g$ . In this case the effect of gravity on a capillary liquid may be neglected. If a capillary potential is comparable with a gravitational one, a body will be referred to as porous. Pores may be classified as cavities (macropores), macrocapillaries and microcapillaries.

A system of cylindrical communicating capillaries offers the simplest model of a capillary porous body. A porous body may often be reduced to an equivalent capillary of conical or slotted configuration.

The pore-radius distribution curve is the main structural characteristic of capillary-porous bodies. Integral pore distribution curves characterize variation of relative pore volume  $V^*$  (the ratio of pore volume to body volume) with respect to the capillary radius  $r$ . The curve  $V^* = f(r)$  begins at a certain value of  $r_{1 \min}$  (a minimum radius of a capillary pore)<sup>1</sup> and crosses the axis of volume  $V^*$  at  $r = r_{\max}$ . Thus over the range  $r_{\min} \leq r \leq r_{\max}$  relative volume  $V^*$  continuously increases with  $r$ . If on a certain section  $r_1 < r < r_2$  there are no pores with  $r_{\min}$ , the curve  $V^* = f(r)$  becomes a straight line parallel to the axis  $r$ . The relation  $dV^*/dr = f_v(r)$  is called a differential pore distribution curve or a differential equation for a volume pore characteristic. This curve may be obtained from the integral curve  $V^* = f(r)$  by graphical differentiation techniques.

In a section without pores ( $r_1 < r < r_2$ ) the differential pore-distribution curve  $f_v(r)$  discontinues and coincides with the axis  $r$  as  $dV^*/dr = 0$  in this section. The area below the curve  $f_v(r)$  in any section  $r$  is numerically equal to the volume of pores whose radii vary within the section.

<sup>1</sup>  $r_{\min}$  is the experimentally determined size of the smallest pores; the pore size varies between 0 and  $r_{\max}$ .

The total pore volume in a unit volume of the body, equal to the body porosity, is defined by the relation

$$\Pi_V = V_{\max}^* = \int_{r_{\min}}^{r_{\max}} \frac{dV^*}{dr} dr = \int_{r_{\min}}^{r_{\max}} f_V(r) dr \quad (5-1-23)$$

The maximum liquid content is

$$\omega_{\max} = V_{\max}^* \rho_l = \Pi_V \rho_l \quad (5-1-24)$$

The specific liquid content or relative liquid concentration (the ratio of liquid mass to a unit mass of a dry body) is

$$u = \frac{1}{\rho_s} \omega = \frac{\rho_l}{\rho_s} \int_{r_{\min}}^r f_V(r) dr \quad (5-1-25)$$

where  $\rho_s$  is the density of a dry porous body.

For calculation of a liquid flow density in a porous body the concept of a differential curve of surface porosity is introduced.

Let the cross sectional area be 1 cm<sup>2</sup>, part of which is occupied by capillaries, the remaining being solid material. If all the capillaries are filled with liquid, its specific content is

$$u = \frac{\int_0^L \rho_l A_l dL}{\rho_s \cdot 1 \cdot L} \quad (5-1-26)$$

where  $A_l$  is the area occupied by liquid or total area of the liquid filled holes per cm<sup>2</sup> of the body cross-section;  $L$  is the depth (thickness) of the layer.

If the number of cylindrical capillaries in a cross-section with a radius from  $r$  to  $r + dr$  is designated by  $dn_s$ , the surface porosity is then

$$\Pi_S = A_l = \int_{r_{\min}}^{r_{\max}} \pi r^2 \frac{dn_s}{dr} dr \quad (5-1-27)$$

where  $n_s$  is the number of capillaries per unit area of the cross-section.

If  $A_l$  is independent of  $L$ , that is the surface porosity is the same at any cross-section of the body which implies that the surface porosity is equal to the volume porosity ( $\Pi_V = \Pi_S$ ), then

$$u = \frac{\rho_l}{\rho_s} A_l = \frac{\rho_l}{\rho_s} \int_{r_{\min}}^{r_{\max}} \pi r^2 \frac{dn_s}{dr} dr \quad (5-1-28)$$

For many bodies the surface porosity changes along the coordinates, therefore alongside with a differential volume pore characteristic  $f_V(r)$ , a differential surface pore characteristic  $f_S(r)$  is used. It is defined by

$$f_S(r) = \frac{dA_l}{dr} = \pi r^2 \frac{dn_s}{dr} \quad (5-1-29)$$

## 5-2. THERMODYNAMICS OF SURFACE EFFECTS

Two phases in contact, for example two insoluble and nonwetting fluids ( $\alpha$  and  $\beta$ ) have a nonhomogeneous interface (Fig. 5-6 a). Properties of the interface are variable along the normal  $n$  to the interface. Gibbs suggested to replace the phase interface by a geometrical interface (Fig. 5-6 b). This model of two phases in contact allows one to use the thermodynamic formulas. It is assumed that properties of phase  $\alpha$  are invariable up to a geometrical interface. This assumption is also valid for phase  $\beta$ .

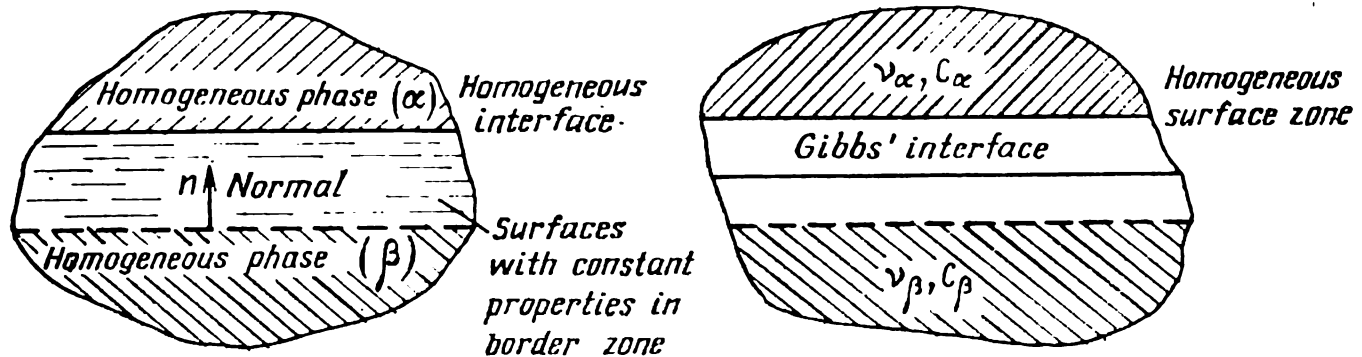


Fig. 5-6. Gibbs model for touching phases ( $v \equiv N$ )

In this model the volume  $V$  is the sum of two phases in contact ( $V = V_\alpha + V_\beta$ ). Bulk concentration of the soluble  $k$ -th species is denoted by  $C_k$  and the mole number by  $N_k$  ( $N_k = C_k V$ ).

Then

$$N_k = (N_k)_\alpha + (N_k)_\beta + (N_k)_S \quad (5-2-1)$$

where  $(N_k)_S$  is the mole number at the interface.

A similar relation holds for free Gibbs energy  $F$ :

$$F = F_\alpha + F_\beta + F_S \quad (5-2-2)$$



The Gibbs-Helmholz equation

$$dF = - S' dT + \sum_{k=1}^n \mu_k dN_k - P_\alpha dV_\alpha - P_\beta dV_\beta + \sigma dA_{\alpha\beta} \quad (5-2-3)$$

will also be used, where  $\sigma$ ,  $\mu$ ,  $S'$  are surface tension, chemical potential and entropy.

Use of relation (5-2-2) and the equalities

$$dF_\alpha = \sum_{k=1}^n \mu_k d(N_k)_\alpha - P_\alpha dV_\alpha$$

$$dF_\beta = \sum_{k=1}^n \mu_k d(N_k)_\beta - P_\beta dV_\beta$$

yields for isothermal conditions ( $dT = 0$ )

$$dF_S = \sum_{k=1}^n \mu_k dN_k - \sum_{k=1}^n \mu_k d(N_k)_\alpha - \sum_{k=1}^n \mu_k d(N_k)_\beta + \sigma dA_{\alpha\beta} \quad (5-2-4)$$

Then using relation (5-2-1), we obtain

$$dF_S = \sum_{k=1}^n \mu_k d(N_k)_S + \sigma dA \quad (5-2-5)$$

where  $A$  is the total surface area of  $S$ .

If the surface concentration of species  $k$  is designated by  $\Gamma_k [(N_k)_S = \Gamma_k A]$  then from equation (5-2-5) we get

$$dF_S = \sum_{k=1}^n \mu_k \Gamma_k dA + \sum_{k=1}^n \mu_k A d\Gamma_k + \sigma dA \quad (5-2-6)$$

since  $d(N_k)_S = \Gamma_k dA + A d\Gamma_k$ . From equation (5-2-6)

$$\left( \frac{\partial F_S}{\partial A} \right)_{T, V, N} = \sum_{k=1}^n \mu_k \Gamma_k + \sum_{k=1}^n \mu_k A \left( \frac{\partial \Gamma_k}{\partial A} \right)_{T, V, N} + \sigma \quad (5-2-7)$$

In general the surface concentration  $\Gamma_k$  varies along the surface; in a particular case  $\Gamma_k$  may be independent of  $A$

$$\left( \frac{\partial \Gamma_k}{\partial A} \right) = 0$$

Then

$$f_S = \frac{F_S}{A} = \sum_{k=1}^n \mu_k \Gamma_k + \sigma \quad (5-2-8)$$

For a single-component system composed of a pure liquid and its vapour, the interface may be placed so that  $\Gamma = 0$ , then

$$f_S = \sigma \quad (5-2-9)$$

However, in a general case the surface tension, numerically equal to reversible work of forming a unit area of liquid at extension, does not equal to the free surface energy  $f_s$ .

From equation (5-2-7) it follows that at  $\Gamma = 0$

$$\left(\frac{\partial F}{\partial A}\right)_{T, V, N} = \sigma \quad (5-2-10)$$

If  $\sigma$  is independent of  $A$  then

$$\left(\frac{\Delta F}{\Delta A}\right)_{T, V, N} = \sigma \quad (5-2-11)$$

If  $\sigma$  depends on  $A$ , as is the case with surface active materials, then

$$(\Delta F)_{T, V, N} = \int_{A_1}^{A_2} \sigma(A) dA \quad (5-2-12)$$

When considering capillary phenomena the curvature effect of the liquid surface on surface tension and location of the Gibbs surface should be accounted for. However, this effect should be considered only when the curvature radius is comparable with molecular dimensions.

Consider a system consisting of two phases  $\alpha$  and  $\beta$  in contact with solid surface  $S$ . The same Gibbs model will be used. The concentration of the  $k$ -th species at the three-phase contact line  $(N_k)_L$  is defined by the relation

$$N_k = (N_k)_\alpha + (N_k)_\beta + (N_k)_S + (N_k)_{S(\alpha\beta)} + (N_k)_{S(S\alpha)} + (N_k)_{S(S\beta)} + (N_k)_L \quad (5-2-13)$$

In the considered systems that part of free energy due to change of the length of the three-phase contact line as a result of change in surface areas may be neglected.

In the case of a smooth non-deformable solid and a constant contact angle, the thermodynamic equilibrium condition is defined by Gibbs' relation

$$\sum_{k=1}^3 \sigma_k(dL) dx \geq 0 \quad (5-2-14)$$

where  $dL$  is the element of length along the three-phase contact line;  $dx$  is the solid surface shift normal to element  $dL$ .

By using the relations

$$dA_{S\alpha} = dL dx; \quad dA_{S\beta} = -dL dx; \quad dA_{\alpha\beta} = dL dx \cos \theta$$

from (5-2-14) we obtain Young's equation

$$\sigma_{\alpha\beta} \cos \theta + \sigma_{S\alpha} - \sigma_{S\beta} = 0 \quad (5-2-15)$$

This equation is often derived from the force balance equation. However, the forces along the interface are not equal to  $\sigma_{S\alpha}$  and  $\sigma_{S\beta}$ .

Moreover, if a solid is partially soluble in either phase, adsorption results in considerable decrease of  $\sigma_{\alpha\beta}$ . Gibbs' concept of a model body is an unchangeable solid which is insoluble in liquid phases and whose surface tension is not affected by the liquid present above. The liquid surface tension  $\sigma'_{S\alpha}$  is then defined by the relation

$$\sigma'_{S\alpha} = \sigma_{S\alpha} - \sigma_S \quad (5-2-16)$$

Thus the surface tension  $\sigma_{S\alpha}$  at the solid-liquid interface is the sum of the two components  $\sigma'_{S\alpha}$  and  $\sigma_S$ . The presence of solid material affects the bulk compression forces which results in the component  $\sigma'_{S\alpha}$ , which Gibbs called the surface tension of a liquid in contact with a solid.

At the equilibrium state

$$\sigma'_{S\beta} + \sigma_S = \sigma'_{S\alpha} + \sigma_S + \sigma_{\alpha\beta} \cos \theta \quad (5-2-17)$$

hence

$$\sigma'_{S\beta} - \sigma'_{S\alpha} = \sigma_{\alpha\beta} \cos \theta \quad (5-2-18)$$

The contact angle  $\theta$  depends on the direction of motion of the three-phase contact line (contact angle hysteresis), surfactants, roughness and other factors.

In the case of a system consisting of a fluid ( $\alpha \equiv l$ ), saturated vapour ( $\beta \equiv v$ ) and solid ( $S$ ), according to Melrose's model a flat liquid-vapour interface coexists with a solid surface with a liquid spread. The spread may be eliminated by an inert mechanical bar which moves along the solid

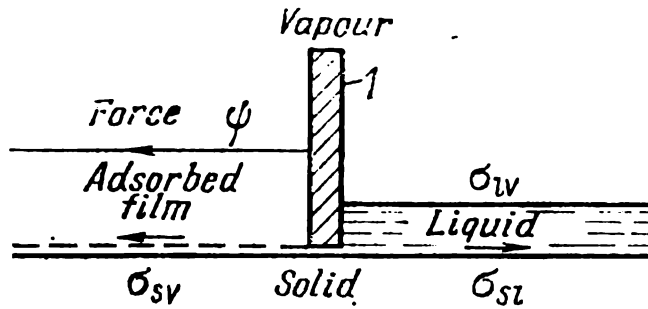


Fig. 5-7. Melrose model for liquid-vapour-solid equilibrium

1 — barrier moving freely without friction under the action of a force along the surface in the horizontal direction

surface without friction (Fig. 5-7). If the liquid is non-spreading, a force per unit length should be applied to the bar:

$$\psi = \sigma_{lv} + \sigma_{sl} - \sigma_{sv} \quad (5-2-19)$$

From equation (5-2-15) this force is equal to

$$\psi = \sigma_{lv}(1 - \cos \theta) \quad (5-2-20)$$

For the liquid to spread, vapour must be absorbed, which continues until force  $\psi$  disappears. At a state of equilibrium the location of the bar does not affect the free energy of the system since it is frictionless. Thus, if an adsorbed film is replaced by a thin liquid layer, free energy of the system does not change.

The film thickness depends on equilibrium conditions. At equilibrium the outer molecules of the film should have the same chemical potential as the vapour molecules and the liquid surface molecules. The thickness of adsorbed vapour increases until outer molecules are affected by attraction forces of the solid. Therefore, the tension of the upper film should be equal to that of the liquid-vapour interface and the tension of the lower film should be the same as that of the solid-liquid interface. Harkins called these films the double films.

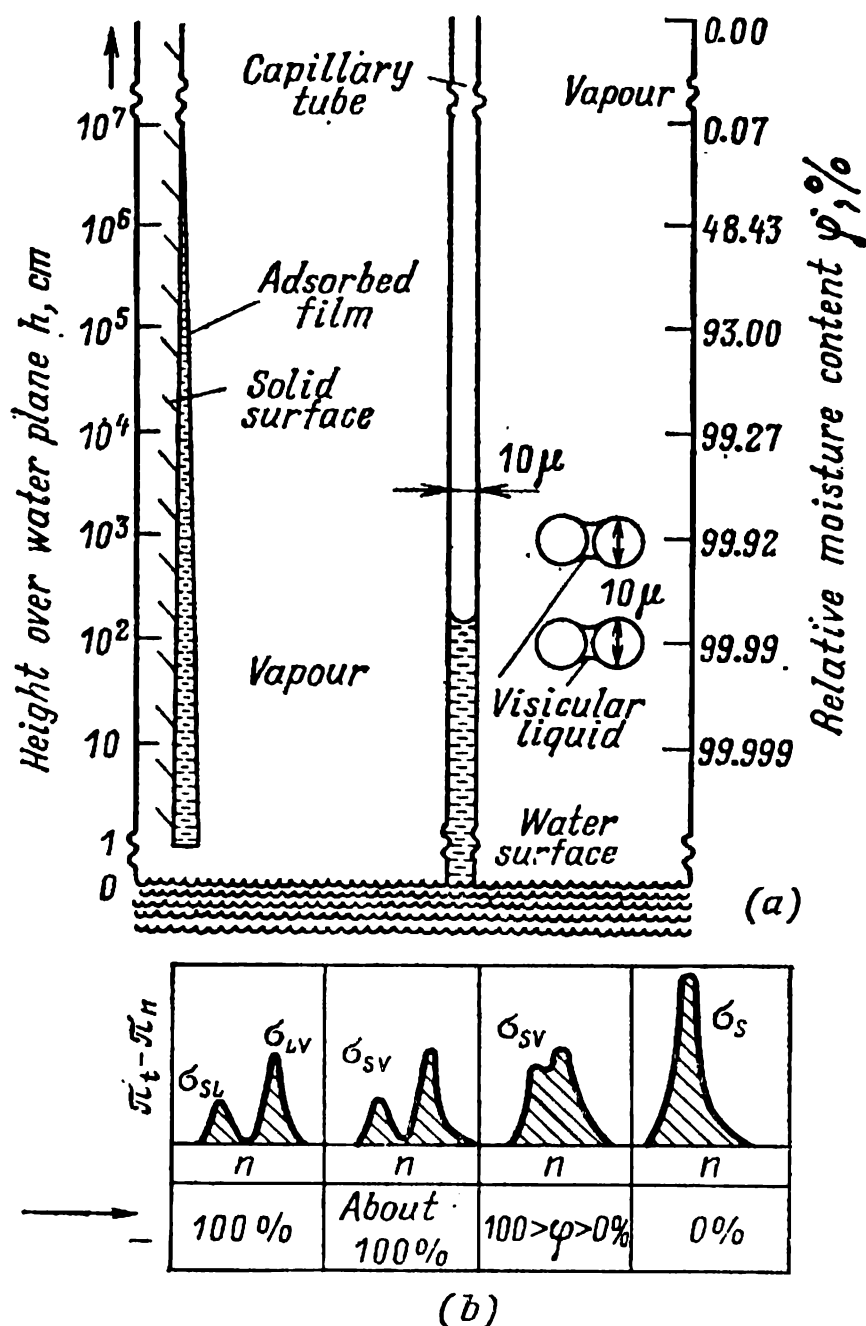


Fig. 5-8. Adsorbed equilibrium water film on the solid surface and capillary water suction (a) and anisotropy of surface tension (b) versus height  $h$  and relative pressure (relative humidity)  $\sigma = \int (\pi_t - \pi_n) dn$ ;  $n$  — distance along the normal to the surface

Figure 5-8 a shows a liquid film spread along a vertical solid surface. If vapour pressure at distance  $h$  from a free liquid surface is designated

by  $p_h$ , and that at the liquid surface by  $p_0$ , then

$$\varphi = 100 \frac{p_h}{p_0} = \exp \left( - \frac{\Delta \rho g h M}{\rho_l R T} \right) \quad (5-2-21)$$

where  $\Delta \rho = (\rho_l - \rho_v)$  is the liquid-vapour density difference;  $g$  is the acceleration due to gravity;  $M$  is the molecular mass;  $\varphi$  is the relative humidity or relative pressure of vapour;  $R$  is the universal gas constant.

At an infinitely large height ( $h = \infty$ ) vapour pressure  $p_h$  is zero ( $p_h = 0$ ), and surface tension of a solid relative to a vacuum is  $\sigma_s$ . Figure 5-8 *a* presents a schematic drawing of the change in film thickness with height, and as follows from formula (5-2-21), variation of relative humidity with height  $\varphi = f(h)$ .

Actual film thickness is unknown. For large  $\varphi$  the film thickness may be as large as 10-30 Å (4 to 12 molecular water layers), while values around 100 Å have been reported. Surface tension may be found in terms of the stress tensor  $\pi$

$$\sigma = \int (\pi_t - \pi_n) dn \quad (5-2-22)$$

where  $\pi_t$  and  $\pi_n$  are tangential and normal components of the stress tensor, respectively.

In Fig. 5-8 *b* the tension anisotropy effect is plotted versus relative humidity (height).

As the film thickness increases (the range of large  $\varphi$ ), two peaks appear in the tension anisotropy.

At the water surface  $\varphi = 100\%$  and adsorbed film is a double film as follows from the model depicted in Fig. 5-7.

The relation for adsorbed liquid mass  $m$  at a solid surface in a gravity field is obtained by Gibbs as

$$d\sigma' = mg dh \quad (5-2-23)$$

hence the excess mass  $m'$  of adsorbed film between the heights  $\Delta h = h_2 - h_1$  is

$$m'g = (\sigma')_{h_2} - (\sigma')_{h_1} \quad (5-2-24)$$

A more accurate calculation with respect to (5-2-22) results

$$m'g = \left[ \int (\pi_t - \pi_n) dn \right]_{h_2} - \left[ \int (\pi_t - \pi_n) dn \right]_{h_1} \quad (5-2-25)$$

where

$$\sigma'_h = \left[ \int (\pi_t - \pi_n) dn \right]_h - \left[ \int (\pi_t - \pi_n) dn \right]_{h=\infty} \quad (5-2-26)$$

Capillaries having a regular geometry (cylindrical and conical pipes, slots) are sometimes referred to as elementary capillaries. Capillary pressure  $p_c$  in such capillaries is defined by the Laplace formula

$$p_c = \sigma_{lv} \left( \frac{1}{r'_1} + \frac{1}{r'_2} \right) = 2\sigma_{lv} \left( \frac{\bar{1}}{r'} \right) \quad (5-2-27)$$

where  $r'_1$  and  $r'_2$  are principle curvature radii of a meniscus at any point;  $\left( \frac{\bar{1}}{r'} \right)$  is the mean meniscus curvature equal to

$$\left( \frac{\bar{1}}{r'} \right) = \frac{1}{2} \left( \frac{1}{r'_1} + \frac{1}{r'_2} \right) \quad (5-2-28)$$

In a state of equilibrium capillary pressure is balanced by gravitational pressure

$$h = \frac{p_c}{g\Delta\rho} = \frac{2\sigma_{lv}}{g\Delta\rho} \left( \frac{\bar{1}}{r'} \right) \quad (5-2-29)$$

For a cylindrical capillary ( $r'_1 = r'_2 = r'$ ) the lift will be

$$h = \frac{2\sigma_{lv}}{g\Delta\rho r'} = \frac{2\sigma_{lv} \cos \theta}{g\rho_l r} \quad (5-2-30)$$

since

$$\Delta\rho = \rho_l - \rho_v \approx \rho_l$$

Capillary radius  $r$  and meniscus radius  $r'$  are related by ( $r = r' \cos \theta$ ). In case of complete wetting ( $\cos \theta = 1$ ) the capillary wall will be covered by a film with thickness  $\delta$ . The capillary radius and the meniscus radius are then related by

$$r' = r - \delta - \frac{\delta}{\alpha} \quad (5-2-31)$$

where  $\alpha$  is a constant depending on the liquid and wall surface.

In Fig. 5-8 *a* the water height is shown in a cylindrical capillary with diameter of  $10\mu$ . Alongside, a liquid film of two spheres in contact,  $10\mu$  in diameter, at heights 100 and 1000 cm above the free water surface, is depicted. Adsorbed films are somewhat thinner at convex solid surfaces and somewhat thicker at concave surfaces compared to a film on a similar flat surface. As liquid spreads, the increase of free energy  $\Delta F$  is equal to

$$\Delta F = \sigma_{vl}(A_{lv} + A_{sv}) \quad (5-2-32)$$

By analogy with gravitation potential  $\psi_g(\psi_g = gh)$ , a concept of a capillary potential  $\psi_c$  is introduced by the relation

$$\psi_c = hg = \frac{p_c}{\Delta\rho} = \frac{2\sigma_{vl}}{\Delta\rho} \left( \frac{1}{r'} \right) \quad (5-2-33)$$

For wetting liquids  $\psi_c < 0$  since  $\left( \frac{1}{r'} \right)$  for a concave surface is negative and for nonwetting liquids ( $\cos \theta < 0$ )  $\psi_c > 0$ . Wetting liquid moves from the lower potential  $\psi_c$  to the higher one (liquid from a wide capillary is sucked by a narrow capillary) and nonwetting liquid moves from the higher to lower potential.

Relation (5-2-33) holds for a capillary with either end in contact with liquid. For a conical capillary with a mean meniscus curvature of  $\left( \frac{1}{r'} \right)_1$  at the narrow end and  $\left( \frac{1}{r'} \right)_2$  at the wide end, respectively, the capillary potential  $\psi_c$  is

$$\psi_c = \frac{2\sigma_{vl}}{\Delta\rho} \left[ \left( \frac{1}{r'} \right)_1 - \left( \frac{1}{r'} \right)_2 \right] \quad (5-2-34)$$

In such a capillary wetting liquid migrates towards the narrower end until the meniscus occupies the extreme position. In a cylindrical capillary ( $r'_1 = r'_2 = r'$ ),  $\psi_c$  is zero. A liquid column in such a capillary may be located anywhere and the liquid inside is stagnant. Capillary potential  $\psi_c$  is found from capillary pressure  $p_c$  or gravitational potential  $\psi_g$ . At equilibrium the gravitational and capillary potentials are equal ( $\psi_c = \psi_g$ ), which is achieved as follows. A capillary porous body is placed in contact with liquid. At equilibrium the liquid-content curve is plotted versus the height  $h$ ; this distribution describes the dependence of capillary potential  $\psi_c$  ( $\psi_c = hg$ ) on the moisture content.

Very often capillary pressure of a porous body is found from water displacement from the porous body by a different immiscible liquid, for example oil. A schematic diagram of the measuring unit is shown in Fig. 5-9. A porous body is first saturated with a wetting phase (water), then it is brought into contact with a fine-porous membrane which is constantly saturated with water. Water phase pressure is equal to gas pressure  $p_l$  in the left-hand chamber (Fig. 5-9). The nonwetting phase (oil) is in contact with the porous body, the pressure  $p_0$  acting on the corresponding piston is equal to gas pressure in the right-hand chamber. Here, the hydrostatic pressure is neglected and the pistons are assumed to be frictionless. The pressure difference  $p_0 - p_l$  is equal to capillary pressure  $p_c$  ( $p_c = p_0 - p_l$ ).

Saturation  $\omega_l$  is measured as the ratio of pore volume in the body filled with water to the whole pore volume. The quantity of water forced out is

$$\Delta V_l = V_s \Pi_v \Delta \omega_l \quad (5-2-35)$$

where  $V_s$  is the porous body volume and  $\Pi_v$  is the void fraction.

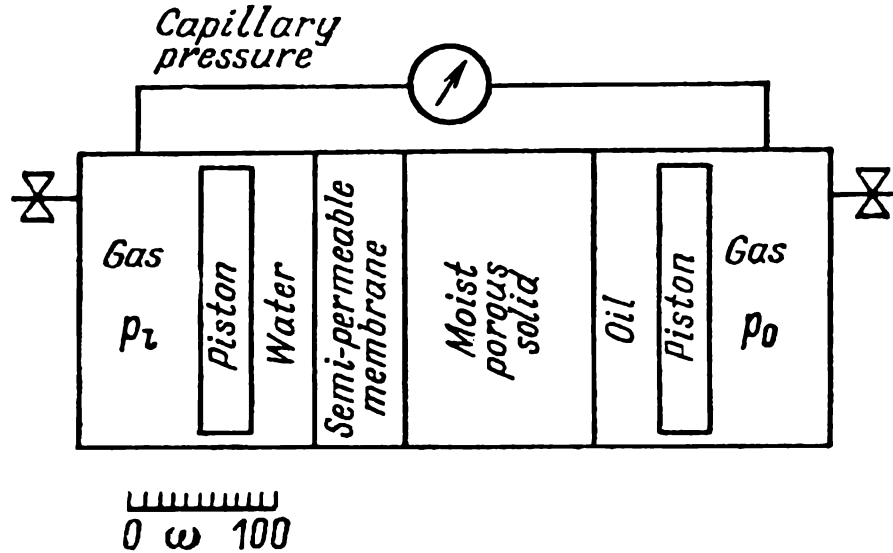


Fig. 5-9. Schematic of the instrument used to determine capillary pressure

External displacement work on a closed system to displace an elementary water volume  $dV_l$  from the body is  $(p_o - p_l) dV_l$ . The absolute value of this quantity is equal to free energy change  $dF$  in the system

$$dF = -p_s dV'_s - p_l dV'_l - p_o dV'_o + \sigma_{l0} dA_{l0} + \sigma_{s0} dA_{s0} + \sigma_{sl} dA_{sl} \quad (5-2-36)$$

but with the reverse sign. The total change of free energy  $dF_{tot}$  in the system and the environment is zero

$$dF_{tot} = 0 = (p_o - p_l) dV_l - p'_s dV'_s - p_l dV'_l - p_o dV'_o + \sigma_{l0} dA_{l0} + \sigma_{s0} dA_{s0} + \sigma_{sl} dA_{sl} \quad (5-2-37)$$

If compression work is neglected ( $dV'_s = dV'_l = dV'_o = 0$ ) and the total surface area of the body is assumed constant ( $dA_{s0} = -dA_{sl}$ ), then based on equation (5-2-15) we obtain

$$dF = - \int p_c dV_l = \sigma_{l0} d(A_{l0} + A_{s0} \cos \theta) \quad (5-2-38)$$

The relations derived allow experimental curves  $p_c = f(\omega)$  to be plotted and the change of free energy due to displacement or wetting to be calculated.

Before analysing the capillary pressure curves  $p_c = f(\omega)$  an infinitely slow displacement will be considered using a capillary model (Fig. 5-10a). Liquid is displaced in a reversible manner till the meniscus expands in tube  $A$ . The liquid level then quickly drops to the steady position  $B$ .



The procedure requires an ideally functioning valve. It is instantaneously closed as soon as redistribution of the liquid results in pressure increase. The valve opens again as the liquid configuration stabilizes and liquid pressure at the outer side of the valve equals the bulk pressure.

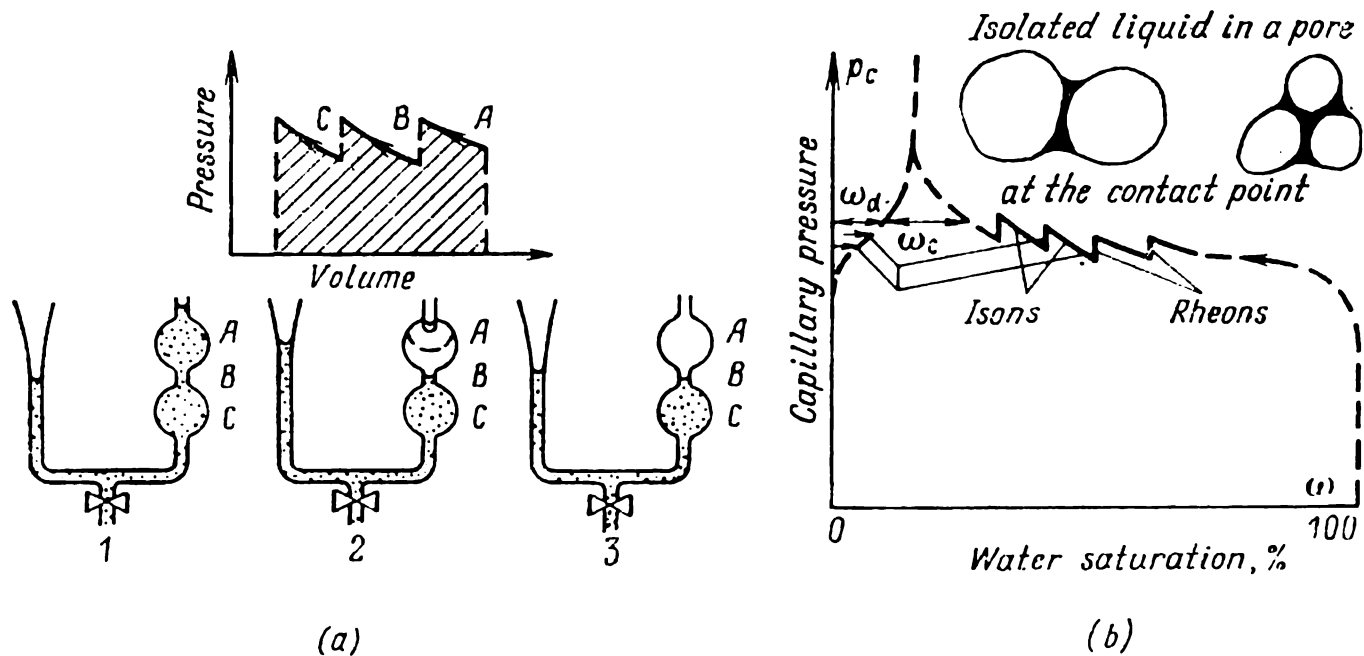


Fig. 5-10. Drainage mechanism

(a) in a capillary model; 1 — reversible displacement; 2 — redistribution at constant volume; 3 — reversible displacement; (b) drainage in a disperse material or packing

The presented plot of pressure versus liquid volume shows that the process consists of smooth reversible changes due to spontaneous pressure changes at constant saturation; reversible changes alternate with irreversible changes of the volume; and the whole process of liquid removal is of a quantum type.

Pressure rises slowly as dehydration proceeds until the unsteady configuration results in abrupt pressure variation at constant saturation. Such pressure variations have been called rheons by Melrose. At this moment the system cannot perform any work. The curve sections  $p_c = f(\omega)$  between rheons corresponding to reversible displacements are called isons. A rheon thus describes spontaneous liquid redistribution within a porous system. A liquid volume spontaneously displaced from a certain region may range between a part of a single pore and a pore assembly. For example, if the surface is rough, liquid displacement from a part of the pore is similar to the operation of a clock mechanism with a ratchet. A joint effect may result in liquid displacement out of several pores within one rheon. The rheon number along the curve  $p_c = f(\omega)$  is inversely proportional to the mean pore diameter raised to the third power and dependent on the sample size.

According to the second principle of thermodynamics, free energy of a system decreases due to spontaneous changes. As the changes occur at constant saturation, the free energy loss is

$$\Delta F = \sigma_{0l}(\Delta A_{0l} + \Delta A_{s0} \cos \theta) \quad (5-2-39)$$

Here the value in brackets should be negative. Isons may be described as displacement of one phase by another from the porous material at equilibrium. This displacement is smooth and reversible. Thus the relation

$$\Pi_V V_S \int_{\omega_{l1}}^{\omega_{l2}} p_c d\omega_l = \int \sigma_{l0} d(A_{l0} + A_{S0} \cos \theta) \quad (5-2-40)$$

holds.

During dehydration the work is done on a porous system, and free energy of the surface increases. The changes are isothermal and the heat released due to the work done is transferred to the environment.

In fine-disperse systems and capillary-porous bodies capillary pressure curves are irreversible, smooth and solid, as discontinuities (combinations of rheons and ions) are insignificant and cannot be practically observed (Fig. 5-11). That may be explained by the fact that the capacity of a system

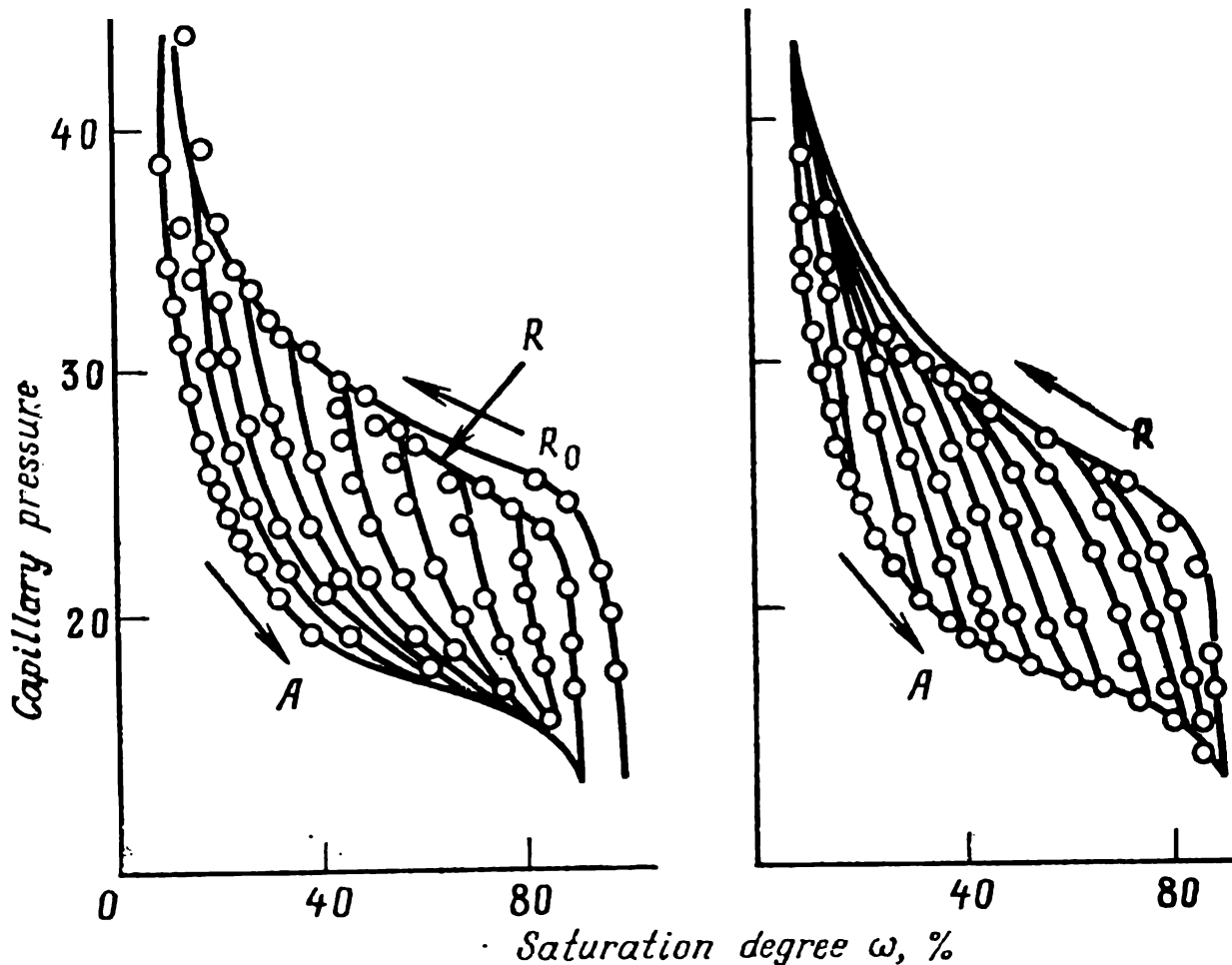


Fig. 5-11. Hysteresis of capillary pressure for micropores of diameter 48  $\mu\text{m}$  (a) and 60  $\mu\text{m}$  (b)

(a porous body) is large compared with the volume of liquid redistributed within the rheon.

In capillary pressure curves one may find:

- (1) limit saturation  $\omega_{li}$  (volume of a wetting phase at high capillary pressures) independent of further increase in pressure;
- (2) residual saturation  $\omega_{l0}$  at zero pressure;

(3) initial drainage curve  $R_0$  (displacement of a wetting phase from 100% saturation to  $\omega_{li}$ );

(4) suction curve  $A$  (rise of saturation from  $\omega_{l0}$  to  $\omega_{li}$ );

(5) secondary drainage curve  $R$ .

Experimental data available for many materials demonstrate that  $\omega_{li}$  is the same for initial and secondary drainage curves.

Curves of secondary drainage and suction make a hysteresis loop  $RA$  with all the subsequent reference points lying inside the loop. Involutives of the primary drainage curves which originate from the suction curve  $A$  may be found either at the curve intersection (Fig. 5-11b), or join on the secondary drainage curve in the region close to the intersection. The behaviour of the involutes of the suction curves is similar. Any point inside the hysteresis loop may be obtained in many ways.

The above behaviours of capillary pressure curves are in agreement with thermodynamic relations of wetting phase displacement.

Good agreement between experimental and calculated data is observed for disperse materials, the particles of which do not swell (no colloidal particles) and have a non-rough surface.

For many materials the curves  $p_c = f(\Omega)$  are similar, which led Leverett [5-7] to introduce a dimensionless function  $j(\Omega)$ , called the Leverett function

$$j(\Omega) = \frac{P_c}{\sigma_{12}} \sqrt{\frac{K_f}{\Pi}} \quad (5-2-41)$$

where  $K_f$  is permeability.

In Fig. 5-12 the Leverett function is plotted for loose sands. The figure shows that at  $\Omega \rightarrow \Omega_{il}$ , the capillary pressure  $p_c \rightarrow \infty$ . This limit saturation, sometimes called a residual saturation, characterizes a bound liquid. The limit saturation for many porous materials depends on permeability. For example, for sandstone the following empirical relation holds

$$\Omega_{il} = 0.7 - 0.3 \times 10^{-2} K_f^* \quad (5-2-42)$$

where  $K_f^*$  is the permeability (in millidarcy).

It should be pointed out that experimentation on curves  $p_c = f(\omega)$  is extremely difficult. A precise temperature control is necessary since slight variations affect capillary pressure values (Table 5-4). A wetting liquid should be very pure as small salt admixtures in water have a strong effect on capillary pressure. Curves  $p_c = f(\omega)$  allow calculation of a particle contact area, displacement work and a number of other quantities involved in a displacement process. These studies are of tremendous importance for filtration theory.

If it is assumed that the highest capillary lift depends on the widths of narrow passages, the maximum capillary potential then equals

$$\psi_c = h_{\max} g = \frac{2\sigma \cos \theta}{\rho_l B R} \quad (5-2-43)$$

where  $\delta = \delta_{lv}$ ;  $B$  is a constant depending on the type of arrangement and the way of liquid supply (capillary hysteresis effect).

If liquid is absorbed from below by a dry body, then the radius of the widest pore should be taken; if liquid is absorbed from above or if

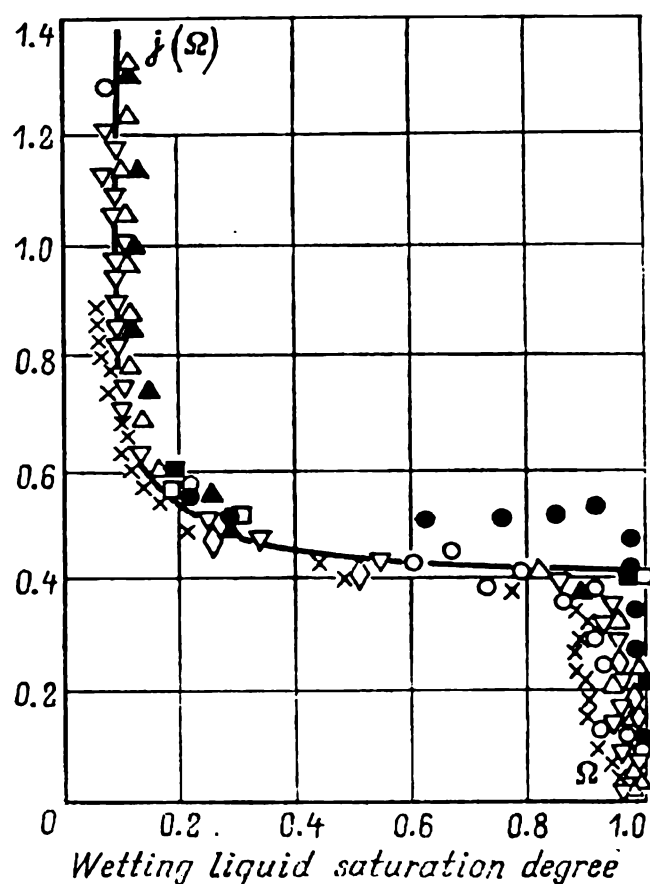


Fig. 5-12. The Leverette  $j$ -function for loose sands

the body is wetted beforehand, then the radius of the narrowest pore should be taken.

The maximum water lift in a dry body (in a high column consisting of spherical particles) at  $20^{\circ}\text{C}$  is: for a cubic arrangement  $h = 0.205R^{-1}$ ,

Table 5-4

Comparative Decrease of Capillary Pressure Depending on Relative Moisture Content, Water Meniscus Curvature, Temperature Oscillations and Salt Additions

Capillary pressure $p_c/g \Delta\rho$ , cm water column	Surface curvature $p_c/\delta_{lv}$ , $\text{cm}^{-1}$	Cylindrical capillary radius at $\theta = 0$ $2\delta_{lv}/p_c$ , cm	Relative moisture content $\varphi = p/p_0 \cdot 100, \%$	Equivalent temperature difference $\Delta T$ at $20^{\circ}\text{C}$	Equivalent concentration of NaCl, mole/l
0	0	$\infty$	100	0	0
10	$1.36 \times 10^2$	$1.47 \times 10^{-2}$	99.999	$1.6 \times 10^{-4}$	$3 \times 10^{-4}$
$10^2$	$1.36 \times 10^3$	$1.47 \times 10^{-3}$	99.99	$1.6 \times 10^{-3}$	$3 \times 10^{-3}$
$10^3$	$1.36 \times 10^4$	$1.47 \times 10^{-4}$	99.92	$1.3 \times 10^{-2}$	$2.4 \times 10^{-2}$
$10^4$	$1.36 \times 10^5$	$1.47 \times 10^{-5}$	99.27	$1.2 \times 10^{-1}$	0.22
$10^5$	$1.36 \times 10^6$	$0.47 \times 10^{-6}$	93.00	1.16	2.1

and for a hexagonal arrangement  $h = 0.52R^{-1}$ . In a wetted body the maximum water lift is  $h = 0.366R^{-1}$  ( $v = 90^\circ$ ) and  $h = 0.968R^{-1}$  ( $v = 60^\circ$ ), respectively.

As liquid is supplied, it is accumulated in the form of a biconcave lens at the contact point of spherical particles (Fig. 5-13a). The side surface

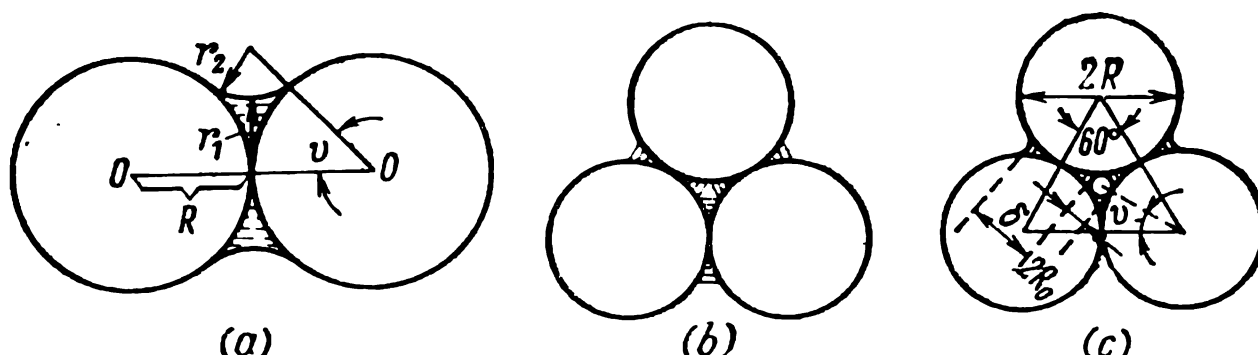


Fig. 5-13. Structures of capillary and film liquid in material pores  
(a) liquid cup; (b) transition from the Stokes state (entrapped water) to funicular state;  
(c) funicular state (entrapped air)

of the lens is convex-concave ( $r_1$  is the convex surface radius, and  $r_2$  is that of the concave surface).

The mean curvature of the surface is

$$\left(\frac{1}{r}\right) = \frac{1}{2} \left( \frac{1}{r_2} - \frac{1}{r_1} \right) \quad (5-2-44)$$

for a concave surface as  $r_2 < r_1$ . Such accumulation of liquid is referred to as a liquid cup and the liquid is called pendular. Capillary pressure of a pendular liquid with complete wetting ( $\cos \theta = 1$ ) is

$$p_c = 2\sigma \left( \frac{1}{r} \right) = \frac{\sigma}{R} K \quad (5-2-45)$$

where  $K$  is a constant whose value changes with angle  $v$ .

If the amount of liquid is small, it is distributed by discrete isolated annuli at the points of particle contact (isolated capillary state or entrapped liquid state). As more liquid is supplied, the cups become larger in size and finally join together. Joining occurs at  $v = 30^\circ$  (Fig. 5-13b, c). At this moment the capillary pressure  $p_{c1}$  at the most dense packing is  $4.1 \delta$  ( $p_{c1} = 4.1 \delta/R$ ), and the liquid content is 22.6% of the pore volume (for a cubic arrangement this figure is 18.3%).

Liquid spots are now in contact with each other and there exists a possibility of continuous transition from one point of a liquid film to another. As more liquid is supplied, necks of the liquid film thicken and finally the cell is completely filled with liquid. This is a capillary state when  $p_{c2} = 6.9 \delta/R$ . This value of  $p_{c2}$  corresponds to transition from pendular to a capillary state during liquid absorption.

During the reverse process (drainage), i.e. upon transition from a capillary state to a pendular state, menisci are formed first in upper pores, and then advance inside. Capillary pressure increases and as the meniscus enters the pore neck  $p_{c3} = 12.9\sigma/R$ . In the process of wetting in the transition state the capillary pressure was  $4.1 \delta/R$ . Therefore, the absorption and evacuation of capillary liquid are irreversible. With the decrease in the amount of liquid, the meniscus enters the following cell and expands. This expansion is instantaneous (unsteady-state) and the liquid redistribution is accompanied by a slight decrease of  $p_c$ . Such a variation in capillary pressure during dehydration may be experimentally observed.

A schematic of a very simple device is shown in Fig. 5-14. Glass beads pre-saturated by a volatile liquid were placed on a permeable membrane. The membrane was connected to a pressure meter. As liquid evaporated, the pressure decreased not continuously but in a stepwise manner. Therefore the displacement of liquid out of such disperse systems cannot be reversible and smooth because of random variations of liquid configuration.

Let us now return to the equilibrium state of vesicular and film liquid, and assume that menisci bounding pendular liquid elements are unclosed and the curvature of the connecting film is equal to the particle (sphere) surface curvature. The film spots in this case are small compared to particle and menisci sizes. If the neck thickness of a vesicular liquid element is denoted by  $\delta$  and the radius of liquid-free voids by  $R_0$  (see Fig. 5-13c), then

$$\delta = R - R_0; \quad R_0/R = \frac{2}{\sqrt{3}} - 1 \quad (5-2-46)$$

In this case, equilibrium between film and vesicular liquid will exist provided that  $p_c = \mathcal{P} = 6\sigma/R$ , i.e. when the film may develop a positive disjoining pressure  $\mathcal{P}$ .

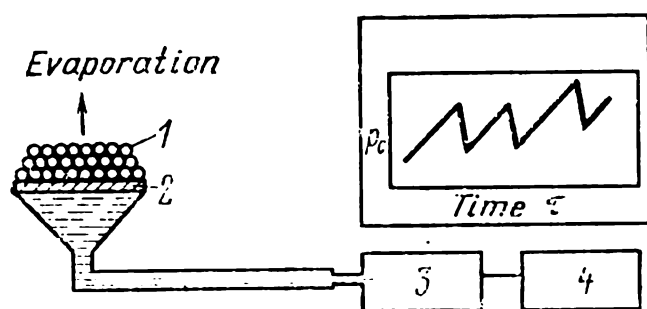


Fig. 5-14. Schematic of the instrument employed to measure pressure fluctuations

1 — packing consisting of glass-spheres ( $R = 1.5$  mm); 2 — highly permeable membrane made of sintered glass; 3 — sensor; 4 — pressure recorder

As the liquid quantity increases, film spots disappear and menisci join together. To maintain equilibrium between the film and meniscus the condition  $\mathcal{P} = \frac{\sigma}{R}(1 - \sqrt{3})^{-1}$  should be observed (negative disjoining pressure). Variation in the sign of  $\mathcal{P}$  is generally not found, i.e. equilibrium between the film and vesicular liquid is not possible. The limit capillary potential is called the critical capillary potential when film and vesicular

liquid exist. At the critical capillary potential the pores get filled with liquid in a stepwise manner.

Substitution of hydrostatic pressure drop by the capillary pressure

$$p_{cap} = \frac{\sigma \cos \theta}{R_s} \quad (5-2-47)$$

into the Darcy law (5-1-5) will result in the following formula for a permeability coefficient [5-9]

$$K_f = \frac{\sigma \cos \theta \int_{r_0}^r r^2 f(r) dr}{4r^2 f(r) \xi^2(\omega)} \quad (5-2-48)$$

where  $\xi(\omega)$  is the tortuosity factor dependent on the specific liquid content. The function  $f(r)$  is defined by

$$f(r) = \frac{\Pi p_{cap}^2}{2\sigma \cos \theta} \left( \frac{\partial p_{cap}}{\partial \omega} \right)^{-1} \quad (5-2-49)$$

Use of the relation between  $p_{cap}$  and  $\omega$  yields

$$f(r) = \frac{\xi(\omega) p_0 S_v}{2} \quad (5-2-50)$$

and the penetrability factor is

$$K_f = \frac{\sigma \cos \theta}{4r^2 \xi^2(\omega) S_v} \int_{r_0}^r r^2 S_v dr \quad (5-2-51)$$

$S_v$  stands for wetted surface area per unit volume. If  $f_s(r)$  is used to denote  $\xi(\omega) S_v$ , formula (5-2-51) becomes identical with the author's formula given in [5-10], which was obtained for a porous body model as a system consisting of capillaries. In this case  $f_s(r)$  denotes a differential curve of pore distribution along the radius. It is a function of the moisture content and tortuosity.

### (a) Properties of Thin Liquid Layers

In capillary-porous bodies or disperse systems the capillary walls have a sorbed liquid layer at a certain moisture content. These thin liquid layers at a solid surface have specific properties which will be discussed in greater detail.

Equality of chemical potentials and temperatures is a condition of thermodynamic equilibrium for any system. Suppose there is a thin liquid layer

on a solid surface. Chemical potential  $\mu_c$  in a boundary layer of liquid is not equal to the chemical potential in the bulk liquid  $M_\infty(\mu_c \neq \mu_\infty)$ . Then at isothermal conditions ( $T_c = T_\infty$ ) in the liquid boundary layer an additional pressure develops, which has been termed by P. B. Deryagin the disjoining pressure

$$\mathcal{P}_c = \frac{\mu_\infty - \mu_c}{V_m} \quad (5-2-52)$$

where  $V_m$  is the molar volume.

The disjoining pressure is determined by the pressure drop between liquid-solid body phases during transition through a flat interface from a thin liquid layer  $H$  to the bulk liquid adjacent to the thin layer. If the pressure drop is denoted by  $\Delta p_H$ , then

$$-\Delta p_H = \mathcal{P}_c \quad (5-2-53)$$

i.e. the pressure drop  $\Delta p_H$  is numerically equal to the disjoining pressure but has an opposite sign.

All the arguments are valid for a flat interface. In a general case the pressure drop at a liquid-vapour interface is

$$\Delta p = \Delta p_\sigma + \Delta p_H \quad (5-2-54)$$

where  $\Delta p_\sigma$  is the pressure drop due to the liquid surface curvature defined by the Laplace formula

$$\Delta p_\sigma = \sigma \left( \frac{1}{r_1} + \frac{1}{r_2} \right) \quad (5-2-55)$$

and the pressure drop  $\Delta p_H$  is determined by the value of disjoining pressure.

When a liquid flows at a high rate through a porous body, then apart from capillary and gravitational forces, inertia forces play a significant role. They may be estimated from the Reynolds number.

The behaviour of a porous body, through which liquid flows, may also be judged through the use of Bond (Bo) and Weber (We) numbers

$$\text{Re} = \frac{\rho v L}{\eta}; \quad \text{Bo} = \frac{\rho g L^2}{\sigma}; \quad \text{We} = \frac{\rho v^2 L}{\sigma} \quad (5-2-56)$$

where  $v$  is the liquid velocity and  $L$  is the characteristic dimension.

The Reynolds number characterizes the proportion of inertia and viscosity forces; the Bond number is the ratio of gravity to capillary forces, and the Weber number is the ratio of inertia to capillary forces.

A certain combination of We and Bo gives the well-known Froude number

$$\text{Fr} = \frac{v}{\sqrt{gL}} \quad (5-2-57)$$

which is a ratio of inertia to gravity forces.



Regions should be identified by dominating:

(1) inertia forces, (2) capillary forces and (3) gravitational forces. At small  $We$  and  $Bo$  capillary forces dominate in liquid motion (Fig. 5-15)

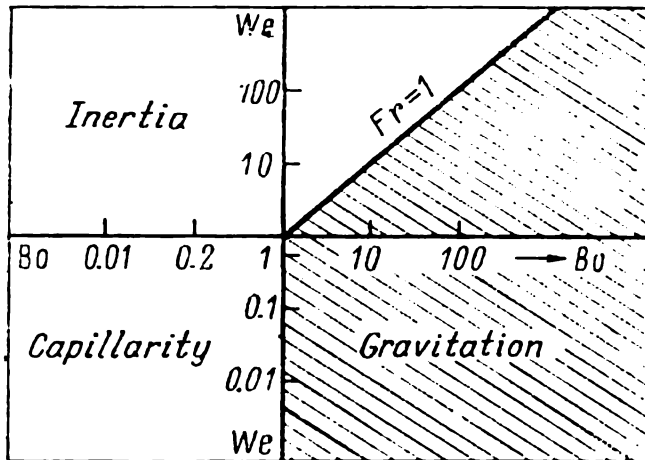


Fig. 5-15. Schematic showing ratio of inertia, gravity and capillary forces when liquid flows through porous material

and gravitation effects are negligible. At small  $Bo$  the Weber number determines the role of capillary and gravitational forces. At large  $Fr$  gravitation effects are negligible.

### 5-3. AVERAGING RULES

Real porous materials are complicated random structures. For a structural description of such materials a statistical method and geometrical modelling may be used. Generally two models are considered: Brener's periodical space model (Fig. 5-16a) and the skewed capillary model (Fig. 5-16b).

Brener's model is an infinite grid containing one or more solid particles. Liquid is the boundary for an elementary cell. Therefore the existence of a rigid framework with a negligible drag is admitted, which would support

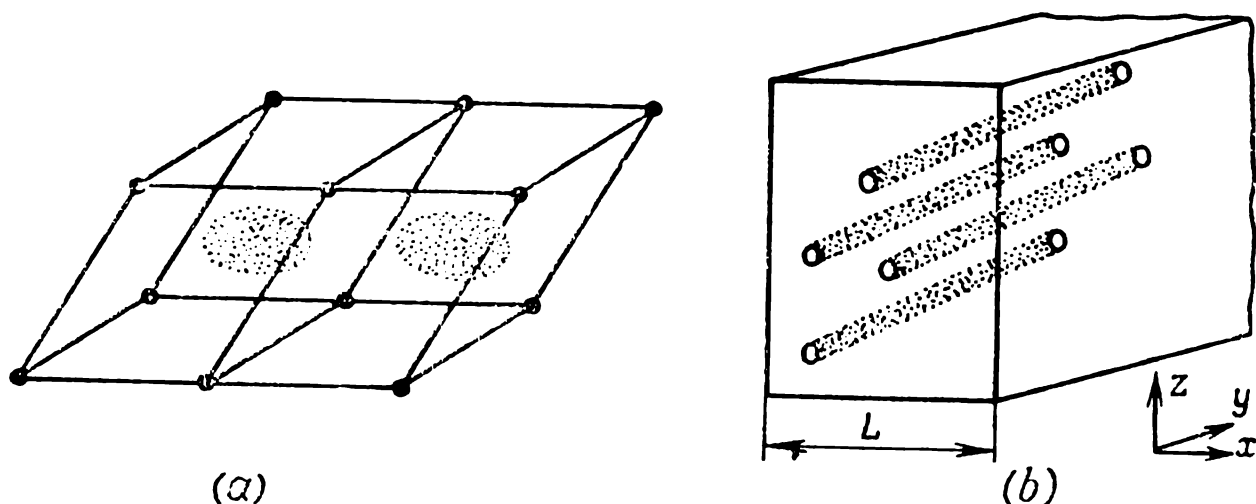


Fig. 5-16. Model of a porous material  
(a) periodic-space model; (b) inclined capillaries

elementary particles in space. A skewed capillary model may not be obtained from a straight capillary model by coordinate transformation as boundary conditions change with coordinate transformation.

The third method of investigation is a combination of statistical and geometrical modelling; it is based on correct averaging of transfer equations. These equations should hold for any geometry and be in conformity with a statistical model.

Consider the motion of an incompressible liquid ( $\text{div } \vec{v} = 0$ ), having constant viscosity ( $\eta = \text{const}$ ), in a porous material.

The liquid velocity vector  $\vec{v}$  and the location vector  $\vec{r}$  are related with the porous material. For a steady pressure the Navier-Stokes equation is simplified for the present case, as  $\partial \vec{v} / \partial \tau = 0$ . The pore distribution function  $f(r)$  is characterized by the relations

$$f(r) = \begin{cases} 1, & \text{if } r \text{ lies within the liquid region} \\ 0, & \text{if } r \text{ lies within the solid region} \end{cases}$$

All pores are assumed to be filled so that the void fraction

$$\Pi_v = \frac{V_f}{V} \quad (5-3-1)$$

where  $V_f$  is the liquid volume in pores;

$$\Pi_v = \frac{1}{V} \int_{(V)} f(r) dV \quad (5-3-2)$$

The averaging volume  $V$  is assumed constant, and  $V_f$  depends on the coordinates. If the function  $f(r)$  were known, the complete solution of the liquid motion problem would be obtained, but  $f(r)$  is unknown, therefore the averaged values are involved. Correct averaging is of great importance.

The following dimensions are peculiar for porous materials:

(1) pore dimension or grain diameter  $d$  within the range of which velocity  $\vec{v}$  changes considerably;

(2)  $l$  is a certain linear dimension of porous material, within the confines of which velocity  $\langle \vec{v} \rangle$  is averaged.

Mean volume (of a certain function  $\psi(\langle \psi \rangle)$ ), which characterizes liquid in a porous material, is defined by

$$\langle \psi \rangle = \frac{1}{V} \int_{V_f} \psi dV \quad (5-3-3)$$

Function  $\psi$  may be a tensor of any order, it is continuous for any value of  $V$ .

If  $l$  is the characteristic length of the averaged volume, there exists a constraint for relation (5-3-3)

$$d \ll l \quad (5-3-4)$$

Then an additional constraint is used which reads: a mean of a mean should also be a mean

$$\langle\langle\psi\rangle\rangle = \langle\psi\rangle \quad (5-3-5)$$

This is valid at

$$l \ll L \quad (5-3-6)$$

where  $L$  is the characteristic length of a porous material through which filtration occurs and within which the volume-averaged velocity changes considerably.

Let us consider averaging over area. The averaging volume should be chosen so that it would include the averaging area of the plane  $A_{(jk)}$  which should be constant. For example, if  $V$  is the volume of a cube and equals  $l^3$ , then  $A_{(jk)}$  is the area equal to  $l^2$  and parallel to the plane  $(jk)$ . If the coordinate origin lies at the center of the cube, then  $\bar{\psi}_0^{(j,k)}$  averaged over the surface at the origin will be

$$\bar{\psi}_0^{(j,k)} = \frac{1}{A_{(jk)}} \int_{(A_f)} \psi dA \quad (5-3-7)$$

where  $A_f$  is a part of the area  $A_{(j,k)}$  occupied by the liquid.

Averaged over the volume  $\langle\psi\rangle_0$  equals

$$\langle\psi\rangle_0 = \frac{1}{l} \int_{x_{(i)} = -\frac{1}{2}l}^{x_{(i)} = +\frac{1}{2}l} \bar{\psi}^{(j,k)} dx_{(i)} \quad (5-3-8)$$

Using the technique described in [5-6], we get

$$\langle\psi\rangle_0 = \bar{\psi}_0^{(j,k)} + \left(\frac{l}{L}\right)^2 \bar{\psi}_0^{(j,k)} \left(\frac{l}{L}\right)^2 \quad (5-3-9)$$

Thus, for the constraint (5-3-6) averaging over the area and the volume are the same.

This conclusion is also valid for a pore distribution function. With the assumption  $\psi = f(r)$ , we get

$$\langle f \rangle \approx \bar{f}^{(j,k)} \quad \text{at} \quad l \ll L. \quad (5-3-10)$$

Since  $\langle f \rangle = \Pi_v$ , then it follows from relation (5-3-10) that

$$\Pi_v \approx \bar{f}^{(1,2)} \approx \bar{f}^{(2,3)} \approx \bar{f}^{(3,1)} \quad (5-3-11)$$

Relation (5-3-10) is also valid for anisotropic porous materials.

The transfer equation involves gradients of characteristic functions. It is, therefore, necessary to find the relation between an averaged gradient of the function  $\psi(\langle \vec{\nabla} \psi \rangle)$  and a gradient of the averaged function  $(\vec{\nabla} \langle \psi \rangle)$ .

The relation was originally found by S. Whitaker [5-6] and it is of the form

$$\int_{V_f} \vec{\nabla} \psi dV = \vec{\nabla} \int_{V_f} \psi dV + \int_{A_i} \psi \vec{\delta}_n dA \quad (5-3-12)$$

where  $V_f$  is the liquid occupied volume bounded by surface  $A_f$ . If  $A_i$  and  $A_o$  are used to denote the liquid surface at the interface and the liquid inlet and outlet area, respectively, then

$$A_f = A_i + A_o \quad (5-3-13)$$

And an important relation is obtained

$$\langle \vec{\nabla} \psi \rangle = \vec{\nabla} \langle \psi \rangle + \frac{1}{V} \int_{A_i} \psi \vec{\delta}_n dA \quad (5-3-14)$$

As an example the continuity equation

$$\langle \vec{\nabla} \cdot \vec{v} \rangle = \frac{1}{V} \int_{V_f} \vec{\nabla} \cdot \vec{v} dV = \vec{\nabla} \cdot \langle \vec{v} \rangle + \frac{1}{V} \int_{A_i} \vec{v} \cdot \vec{\delta}_n dA = 0 \quad (5-3-15)$$

may be taken.

If the fixed porous material is immobile, i.e.  $\vec{v} = 0$  at the surface  $A_i$  (no slip), then

$$\vec{\nabla} \langle \vec{v} \rangle = 0 \quad (5-3-16)$$

Let us now consider the liquid motion equation. For the present case it may be written as

$$-\vec{\nabla} p + \eta \nabla^2 \vec{v} = 0 \quad (5-3-17)$$

where  $p$  is the total pressure including the gravitational pressure. In [5-12] it is shown that  $\vec{v}$  and  $\langle \vec{v} \rangle$  are related by

$$\vec{v} = \mathcal{M} \langle \vec{v} \rangle \quad (5-3-18)$$

where  $\mathcal{M}$  is the tensor transform which shows the relation between local velocity  $\vec{v}$  and the averaged volume  $\langle \vec{v} \rangle$ . This is a unique but not necessarily a linear transform. The transform matrix may be found from the equation

$$\nabla^4 \mathcal{M} = 0 \quad (5-3-19)$$

here at the surface  $A_i \mathcal{M} = 0$ , and at  $\vec{r}_0 \langle \mathcal{M} \rangle = v$

$$\vec{\nabla} \langle p \rangle = -\eta \mathcal{K}^{-1} \langle \vec{v} \rangle \quad (5-3-20)$$

$$\mathcal{K}^{-1} = -\frac{1}{V} \int_{A_i} \vec{\delta}_n \int_0^{S(r)} \vec{\lambda} (\nabla^2 \mathcal{M}) dS dA \quad (5-3-21)$$

where  $\lambda$  is the vector of a tangent to an arbitrary curve in the liquid region.

Let us assume that  $\vec{\nabla} \langle p \rangle = 0$  at  $\langle \vec{v} \rangle = 0$ , then it may be shown that  $\mathcal{K}^{-1}$  has an inversion and may be written

$$\langle \vec{v} \rangle = -\frac{1}{\eta} \mathcal{K} [\vec{\nabla} \langle p \rangle] = -\frac{1}{\eta} \mathcal{K} \{ \vec{\nabla} [\langle p \rangle + \rho \langle \varphi \rangle] \} \quad (5-3-22)$$

The averaged gravitational constant

$$\langle \dot{\varphi} \rangle = -\frac{1}{V} \int_{V_f} (\vec{r} \cdot \vec{g}) dV = -\Pi \vec{r} \cdot \vec{g} \quad (5-3-23)$$

Since  $\vec{g}$  is constant;  $\vec{r}$  is the location vector characterizing the center of gravity of the liquid volume.

The averaged pressure used in equation (5-3-22) is defined by the relation

$$\langle p \rangle = \Pi \langle p \rangle_f \quad (5-3-24)$$

Using the gradient quantities averaging rules, we obtain

$$\langle v \rangle = \frac{1}{\eta} \mathcal{K} \{ \Pi [\vec{\nabla} \langle p \rangle_f - \rho \vec{g}] + [\langle p \rangle_f - \rho \vec{r} g] \vec{\nabla} \Pi \} \quad (5-3-25)$$

If porosity  $\Pi$  is independent of the coordinates, the second term in the right-hand side of formula (5-3-25) equals zero ( $\vec{\nabla} \Pi = 0$ ). Estimation of tensor  $\mathcal{K}$  gives the order

$$\mathcal{K} = 0(d^2) \quad (5-3-26)$$

If  $\mathcal{K}^*$  denotes the dimensionless tensor ( $\mathcal{K}^* = \mathcal{K}/d^2$ ), then we get

$$\langle v \rangle = -\frac{\Pi d^2}{\eta} \mathcal{K}^* [\vec{\nabla} \langle p \rangle_f - \rho \vec{g}] \quad (5-3-27)$$

Tensor  $\mathcal{K}^*$  depends on the porous material structure. The scalar components of tensor  $\mathcal{K}^*$  should be found experimentally. In accordance with Koseni's geometrical capillary model  $\mathcal{K}^*$  is proportional to  $\Pi^2/(1 -$

—  $\Pi$ )<sup>2</sup>. For the periodic space model  $\mathcal{K}^*$  is symmetrical even for anisotropic porous materials, precisely

$$K_{ij} = \left( \frac{d^2}{32} \right) \lambda_i \lambda_j \quad (5-3-28)$$

where  $\lambda$  are the scalar components of a unit vector along capillaries.

For anisotropic porous materials  $\mathcal{K}$  is a function of  $\vec{\xi}$  ( $\vec{\xi}$  is the unit vector of orientation showing the dominant direction). For an isotropic material  $\vec{\xi} = 0$ , for a model with skewed-capillaries  $\vec{\xi}$  is parallel to capillaries. If the particles in Brenner's model were ellipsoids, then vector  $\vec{\xi}$  would have been parallel to the principle axis. If the porous material is symmetrical with respect to one plane, then

$$\mathcal{K} = \begin{pmatrix} K_{11} & 0 & 0 \\ 0 & K_{22} & K_{23} \\ 0 & K_{32} & K_{33} \end{pmatrix} \quad (5-3-29)$$

since  $K_{12} = K_{21} = K_{31} = K_{13} = 0$ . If a porous material is symmetrical relative to two orthogonal planes, then in addition to the earlier said  $K_{23} = K_{32} = 0$ , i.e.

$$\mathcal{K} = \begin{pmatrix} K_{11} & 0 & 0 \\ 0 & K_{22} & 0 \\ 0 & 0 & K_{33} \end{pmatrix} \quad (5-3-30)$$

It is clear that if a porous material is symmetrical about two orthogonal planes, it should also be symmetrical relative to a porous orthogonal plane. Materials for which tensor  $\mathcal{K}$  is determined by major values of  $K_{11}$ ,  $K_{22}$ ,  $K_{33}$  are called orthotropic. If  $K_{22} = K_{33}$ , the materials are called transverse isotropic materials.

In [5-6] it has been shown that tensor  $\mathcal{K}$  may be expressed as

$$K_{ij} = B'_{ij} + B''_{ijk} \lambda_k + B'''_{ijmn} \lambda_m \lambda_n \quad (5-3-31)$$

where tensors  $B$  are polynomials of  $\vec{\xi}$ , and  $\vec{\lambda}$  is a unit vector of orientation.

In this case the Darcy law is of the form

$$\langle \vec{v} \rangle = - \frac{\Pi d^2}{\eta} (B_{(1)}^* \delta + B_{(2)}^* \vec{\lambda} \vec{\lambda}) [\vec{\nabla} \langle p \rangle_f - \rho \vec{g}] \quad (5-3-32)$$

where superscript  $*$  refers to a dimensionless quantity and  $\delta$  is a unit tensor.

For the skewed-capillary model  $B_{(1)}^* = 0$ . Thus, for a detailed description of liquid filtration through porous materials experimental determination of four scalar quantities ( $B_{(1)}^*$ ,  $B_{(2)}^*$ ,  $\lambda_1$ ,  $\lambda_2$ ) is necessary. Comparison of formulas (5-3-32) and (5-3-27) gives

$$B_{(1)}^* \delta_{ij} + B_{(2)}^* \lambda_i \lambda_j = K_{ij}^* \quad (5-3-33)$$

hence

$$B_{(2)}^* = (\delta_{ij} K_{ij}^* - 3B_{(1)}^*) \quad (5-3-34)$$

Relations (5-3-33) and (5-3-34) are valid for transverse isotropic bodies. Anisotropy is unequivocally defined by the orientation unit vector  $\vec{\xi}$  when the insensitivity principle for a porous structure holds for the Darcy law.

There are two volume average theorems for a gradient of  $B$  and for the divergence of this quantity [5-11] ( $B$  may be a vector or tensor)

$$\langle \vec{\nabla} B \rangle = \vec{\nabla} \langle B \rangle + \frac{1}{V} \int_{A_i} B \vec{n}^1 dA \quad (5-3-35)$$

$$\langle \text{div } B \rangle = \text{div } \langle B \rangle + \frac{1}{V} \int_{A_i} B \vec{n}^1 dA \quad (5-3-36)$$

Relation (5-3-35) is called the theorem for the volume average of a gradient, and relation (5-3-36) is the theorem of the volume average of divergence.

Application of the average theorems to the Navier-Stokes equation [5-11] gives

$$\begin{aligned} \frac{\partial \langle \rho \vec{v} \rangle}{\partial \tau} + \text{div } \langle \rho \vec{v} \vec{v} \rangle &= \text{div } \langle \vec{\mathcal{P}} \rangle \\ &+ \rho \langle \vec{F} \rangle + \frac{1}{V} \int_{A_i} \vec{\mathcal{P}} \cdot \vec{n}^1 dA_i \end{aligned} \quad (5-3-37)$$

where  $\mathcal{P}$  is the stress tensor,  $\vec{F}$  is the external force;  $A_i$  is a part of an arbitrary closed surface enveloping volume  $V$  and coincides with the pore surface. This motion equation in a porous material differs from a similar equation of continuum by an additional term (the third term in the right-hand side).

For a particular case of steady motion ( $\partial \rho \vec{v} / \partial \tau = 0$ ) and an incompressible liquid ( $\text{div } \vec{v} = 0$ ) equation (5-3-37) may be simplified by introducing the following relations

$$\vec{F} = - \vec{\nabla} \Phi, \quad \langle p \rangle = \langle p_0 \rangle + \rho \langle \Phi \rangle \quad (5-3-38)$$

where  $\Phi$  is the external force potential;  $\langle p \rangle$  is the local pressure averaged over the volume with due regard for the external force effect; then

$$\overline{\nabla}(\langle p - p_0 \rangle) - \text{div} \langle \mathcal{P} - \rho \delta \rangle - \frac{1}{V} \int_{S_f} [\mathcal{P} - (p_0 - \rho \Phi) \delta] \bar{n}^1 dS = 0 \quad (5-3-39)$$

where  $p_0$  is the constant initial pressure.

The last term in (5-3-39) represents the force exerted by liquid at the pore walls (over and above the hydrostatic or interaction force) calculated per unit volume. This force results from motion:

$$g = - \frac{1}{V} \int_{S_f} [\mathcal{P} - (p_0 - \rho \Phi) \delta] \bar{n}^1 dS \quad (5-3-40)$$

In this case the equation of motion is of the form

$$- \text{div} (\langle \mathcal{P} - p \delta \rangle) + g = K \langle v \rangle \quad (5-3-41)$$

where  $K$  is the resistance to liquid motion in the porous material.

If the coefficient of viscosity  $\eta$  is constant, then

$$\text{div} (\langle \mathcal{P} - p \delta \rangle) = \eta \text{div} (\overline{\nabla} \langle v \rangle) \quad (5-3-42)$$

hence we obtain the modified Darcy equation

$$\overline{\nabla}(\langle p - p_0 \rangle) - \eta \text{div} (\overline{\nabla} \langle v \rangle) + R \langle v \rangle = 0 \quad (5-3-43)$$

where  $R$  is the drag coefficient for nonoriented porous structures defined by

$$g = R[\langle v \rangle - \langle u \rangle] \quad (5-3-44)$$

in which  $\langle u \rangle$  is the local velocity of the porous body averaged over the volume.

Equation (5-3-43) was first derived by G. C. Slattery and was used for the solution of a number of particular problems. Applying the Fourier-Kirchhoff equation separately to the porous framework and to the liquid contained it, G. Slattery obtained the following heat transfer equation [5-11]

$$[\Pi \rho_f c_f + (1 - \Pi) \rho_s c_s] \frac{\partial \langle T \rangle^m}{\partial \tau} + \rho_f c_f \text{div} [\Pi \langle T \rangle^m \langle v \rangle_f] \quad (5-3-45)$$

$$= \lambda_f \text{div} [\overline{\nabla} \{ \Pi \langle T \rangle^m \}] + \lambda_s \text{div} \{ \overline{\nabla} [(1 - \Pi) \langle T \rangle^m] \} + \text{div} \bar{q}^*$$



where  $\Pi$  is the porosity,  $\langle T \rangle^m$  is the average temperature defined by

$$\langle T \rangle^m = \frac{1}{1 - \Pi} \langle T \rangle^s = \frac{1}{\Pi} \langle T \rangle^{(f)} \quad (5-3-46)$$

The superscript  $f$  refers to liquid, and  $s$  to solid.

The heat flux vector  $\bar{q}^*$  is equal to

$$\bar{q}^* = \rho_f c_f \{ \langle T \rangle^{(f)} \langle v \rangle^{(f)} - \langle Tv \rangle^f \} + (\lambda_f - \lambda_s) \frac{1}{V} \int_{A_i} \langle T \rangle^m \bar{n}^1 dA \quad (5-3-47)$$

If the porosity is independent of coordinates, then equation (5-3-45) is simplified

$$[\Pi \rho_f c_f + (1 - \Pi) \rho_s c_s] \frac{\partial \langle T \rangle^m}{\partial \tau} \quad (5-3-48)$$

$$+ \Pi \rho_f c_f \operatorname{div} [\langle T \rangle^{(m)} \langle v \rangle^{(f)}] = - \operatorname{div} \bar{q}$$

where the vector  $\bar{q}$  is

$$\bar{q} = - [\Pi \lambda_f + (1 - \Pi) \lambda_s] \operatorname{grad} \langle T \rangle^{(m)} + \bar{q}^* \quad (5-3-49)$$

Equation (5-3-48) differs from the ordinary Fourier-Kirchhoff equation for porous materials by the additional heat flux vector  $\bar{q}^*$ .

According to formula (5-3-47) the first term in curly brackets is an analogue of turbulent heat flux as the latter is defined by

$$\bar{q}_t = c \rho (\overline{Tv} - \bar{T} \bar{v}) \quad (5-3-50)$$

Thus, in porous materials an additional mode of heat transfer exists due to the complicated mechanism of liquid motion. This mode, however, exists when the fluid motion through a porous material is sufficiently intense, for example, in filtration processes.

#### 5-4. THERMODYNAMIC PROPERTIES OF MOISTURE TRANSFER

Moist materials are colloidal capillary-porous bodies. Moisture in such materials is held by different kinds of forces.

In the author's monograph [5-12] the thermodynamic properties of colloidal, capillary-porous and colloidal capillary-porous bodies have

been considered in detail. In a thermodynamic and molecular equilibrium state a moist material contains a definite amount of moisture which is called equilibrium or hygroscopic moisture content. The pressure of liquid vapours present in the material (material pressure)  $p_m$  equals partial vapour pressure  $p_v$  of moist air. The relative air humidity  $\varphi$  ( $\varphi = p_v/p_s$ ) equals relative material pressure ( $\varphi = p_m/p_s$ ). Here  $p_s$  is the saturated vapour pressure at a given temperature.

The equilibrium moisture content of a body  $u_e$  depends on relative air humidity and air temperature:  $u_e(\varphi, T)$ .

The relation between  $u_e$  and  $\varphi$  at a constant temperature is called a sorption or desorption isothermal line depending on the way in which equilibrium is attained (sorption or desorption). Sorption and desorption isotherms are S-shaped and do not coincide over the whole range of  $\varphi$  except at two points, i.e. when  $\varphi = 0$  and  $\varphi = 1$ .

So far no strict theory has been developed which could provide explanation to sorption and desorption hysteresis. The shape of desorption and sorption isothermal curves depends on bond forms of moisture with moist material. From the analysis of sorption and desorption isothermal curves the following conclusions can be made.

1. The initial section  $\varphi = 0$  to 0.1 of isothermal curve  $\varphi = f(W_e/W_m)$  facing the moisture content axis has convexity characteristic for monomolecular adsorption; liquid absorption is accompanied by appreciable heat liberation.

2. In the section where  $\varphi = 0.1$  to 0.9 the bulge of isothermal curve faces the air humidity axis, a peculiarity of polymolecular adsorption, liquid absorption is also accompanied by heat liberation but the amount of heat liberated is considerably less than in the first case.

3. In the section where  $\varphi = 0.9$  to 1.0 liquid is absorbed without heat liberation and is mainly capillary.

Further absorption of liquid above the maximum hygroscopic moisture content takes place by direct contact of the liquid with material. During absorption microcapillaries and pores get filled and osmotic liquid absorption through semi-penetrable cells of closed walls takes place. The properties of this absorbed liquid do not differ from those of free liquid, and the liquid vapour pressure, in particular, is practically equal to the saturated vapour pressure of free liquid ( $\varphi = 1$ ).

The given procedure is not complete, but the analysis of vast experimental data on heat and mass transfer coefficients in terms of the mass content of absorbed material provides support to this procedure. Three types of bonds are distinguished by the bond energy: chemical, physico-chemical and physico-mechanical (this classification was suggested by academician

P. A. Rebinder). Usually the last two types are considered because a chemical bond is strongest and does not break even at 120-150°C. For cuprous sulfate crystalline hydrates the bond energy at 25°C is about  $(8-3) \times 10^{10}$  Erg/mol.

Adsorption bond of substance with a porous body, as well as the osmotic absorption of liquid is grouped with a physico-chemical bond. To this must be classed immobilization of liquid, i.e. of liquid present in the cell due to the formation of colloid material. Liquid contained in capillaries and wetting liquid are bound by physico-mechanical forces.

The amount of work done  $A$  to separate one mole of water in an isothermal reversible process is proportional to the logarithm of dimensionless pressure ( $\log_e \varphi$ )

$$A = -RT \log_e \varphi \quad (5-4-1)$$

Using the Gibbs-Helmholtz equation we obtain the following expression for the evaporation heat  $Q$  of bound water

$$Q = RT^2 \left[ \frac{\partial}{\partial T} \log_e p_u - \frac{\partial}{\partial T} \log_e p_s \right] = Q_1(u) - Q_0 \quad (5-4-2)$$

where  $Q_1(u) = RT^2 \partial/\partial T (\log_e p_u)$  is the heat of evaporation of water from moist material,  $Q_0 = RT^2 \frac{\partial}{\partial T} (\log_e p_s)$  is the heat of evaporation of free water. The condition  $Q > 0$  ( $Q_1(u) > Q_0$ ) implies that pressure  $p_u$  of liquid vapour in the material is less than the pressure  $p_s$  of the free liquid vapour ( $p_u < p_s$ ), and the rate of growth of the curve  $\log_e p_u = f(T)$  is higher than that of the curve  $\log_e p_s = f(T)$ .

Over a small interval of temperature it may be considered that  $Q = \text{const.}$  Then

$$p_u = p_s \exp \left( -\frac{Q}{RT} \right) \quad (5-4-3)$$

An adsorption bond depends on the specific mass content (moisture content) and may be found from sorption or desorption isotherms  $u_t = f(\varphi)$  (see Fig. 5-17). At  $\varphi = 0.25$ , for example, the adsorption bond energy is about  $4 \times 10^{10}$  Erg/mole.

Liquid contained in capillaries and bound by menisci is a free liquid except for the very thin layer of liquid adsorbed on capillary walls. A decrease in vapour pressure over the meniscus characterizes the bond energy of free liquid which is a result of the curved meniscus effect rather than of

interaction with the solid surface. Therefore, the capillary bond energy is independent of the wall material under conditions of complete wetting and of the amount of liquid contained in capillaries.

The capillary bond energy  $A(r)$  is

$$A(r) = \frac{2\sigma}{r} V_0 \quad (5-4-4)$$

where  $V_0$  is the specific volume.

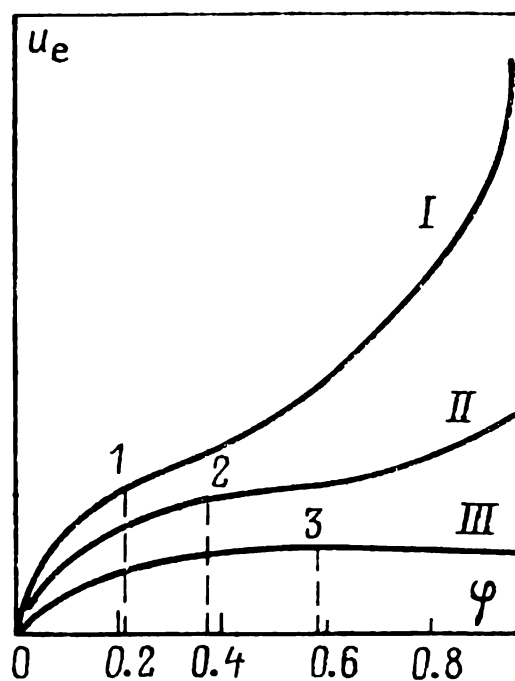


Fig. 5-17. Schematic showing sorption isotherms of steam for hydrophilic (II) hydrophobic (III) and intermediate (I), surfaces.

We shall give some data on capillary bond energy for water: at  $2r = 10^{-3}$  cm  $A(r) = 5.3 \times 10^3$  Erg/mole at  $2r = 10^{-6}$  cm  $A(r) = 5.3 \times 10^9$  Erg/mol.

The vapour pressure  $p_u$  of a capillary-porous material is generally determined from Thomson's formula

$$\log_e \varphi = - \frac{2\sigma \rho_v \cos \theta}{\rho_f p_s r_m} \quad (5-4-5)$$

where  $r_m$  is the mean capillary radius. Here a correction should be made for the adsorbed vapour layer thickness at the capillary wall.

Formula (5-4-5) is valid for a capillary radius over the range  $0.5 \times 10^{-7} < r < 10^{-5}$  cm. The lower limit is defined by the size of body molecules, the radii of which equal to about  $10^{-8}$  cm. For capillaries of radius more than  $10^{-5}$  cm the vapour pressure over the meniscus is practically equal to the saturated vapour pressure over a flat surface.

The works of M. F. Kazansky and his followers have revealed three forms of capillary moisture: capillary vesicular water in coarse pores and capillary water in microcapillaries (see Fig. 5-18).

Based on the analysis of desorption isotherms and drying thermograms a classification of capillaries into macrocapillaries ( $r > 10^{-5}$  cm) and microcapillaries ( $r < 10^{-5}$  cm) has been accepted. Some authors regard capil-

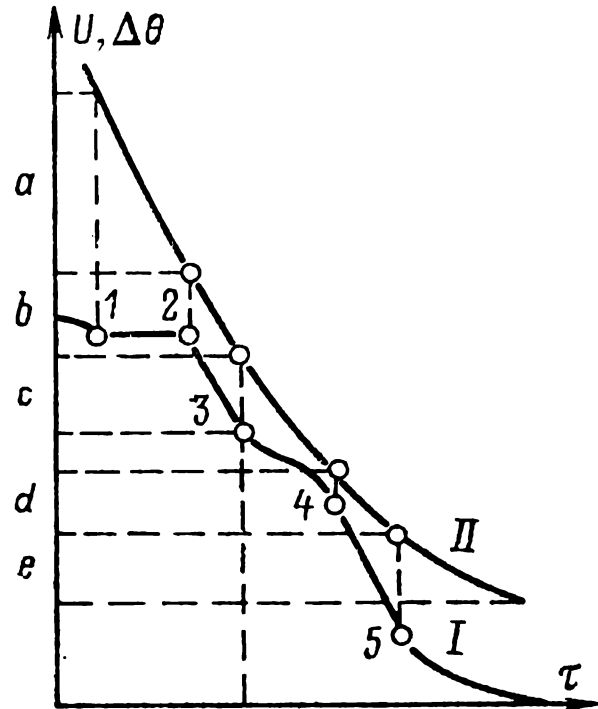


Fig. 5-18. Drying thermogram (I) and curve (II) of drying kinetics of capillary-porous materials of various types

laries of radius less than  $10^{-6}$  cm ( $r < 0.01 \mu\text{m}$ ) as microcapillaries. In Table 5-5 are presented the characteristics of macro- and microcapillaries of some materials [5-2].

Table 5-5

#### Properties of Some Capillary-Porous Materials

Material	Density kg/m <sup>3</sup>	Pore volume, m <sup>3</sup> /m <sup>3</sup>	Micropores ( $r < 0.01 \mu\text{m}$ )	
			Volume, m <sup>3</sup> /m <sup>3</sup>	Volume fraction
Porcelain	2530	0.031	0.0034	0.109
Kaolinite	2370	0.369	0.126	0.341
Pine charcoal	1720	0.883	0.354	0.412
Birch charcoal	1720	0.936	0.656	0.700
Active carbon	1780	0.870	0.856	0.986

The vapour pressure of liquid over a thin film is defined by formula (5-2-21). The earlier given formulas imply that the relation between the inverse moisture content and  $\log_e \varphi$  is linear. As demonstrated by B. A. Posnov [5-13] this relation is well supported by experimental data. Posnov's formula is of the form

$$\frac{1}{W_e} = \frac{1}{W_m} + B \log_e \varphi \quad (5-4-6)$$

where  $W_m$  is the maximum hygroscopic moisture content of the material;  $B$  is a factor dependent on temperature  $B(T)$ . Formula (5-4-6) is valid over the range of  $\phi$  from 0.1 to 1.0. Maximum hygroscopic moisture content  $W_m$  is inversely proportional to the absolute temperature ( $W_m \sim T^{-1}$ ).

In Table 5-6 the values of  $W_m$  and  $B$  are furnished for certain materials at different temperatures.

### (a) *Moisture-Transfer Potential*

The kinetics of heat and mass transfer in capillary-porous bodies depends on transfer-potential difference. The concept of heat (temperature) transfer potential was introduced long ago and it has received rigorous substantiation in thermodynamics. The concept of moisture transfer potential in wet bodies has recently been introduced on the basis of thermodynamic analogy of heat and mass transfer.

Unlike molecular physics, thermodynamics deals with macroscopic properties of a body or a system of bodies and the processes of their interaction neglecting microscopic properties. This fact is of great importance when investigating moisture transfer in capillary-porous bodies where the molecular processes are extremely complicated. Application of thermodynamic methods does not, however, eliminate the use of the molecular-kinetic method. Thermodynamics and molecular-kinetic theory should supplement each other, one and the same test material should serve as an object of comprehensive analysis. Transfer of moisture is inseparable from heat transfer and the heat and mass transfer processes should be treated as interrelated. Therefore, the use of the methods and the concepts which are successfully used in heat transfer phenomena is quite natural.

The chemical potential, dependent on temperature and partial vapour pressure, is known to be a vaporous moisture transfer potential in humid air. In the hygroscopic state the vaporous moisture potential may, therefore, be expressed in terms of moisture content and the temperature of the solid. In the moist state region the chemical potential calculated per unit mass of absorbed water is equal to the chemical potential of free water, i.e. it is a constant (the variation in vapour pressure over the macrocapillary meniscus surface with varying capillary radius is almost equal to zero). The experimental methods of determining single moisture transfer potential for a material containing any amount of moisture are of great interest both for the hygroscopic and moist states.

Like a thermal potential (temperature) a moisture transfer potential will be introduced. To this end the following experimental facts will be considered. At thermodynamic equilibrium, say, at hygrometric equilibrium, moisture in a body or a system of bodies is distributed in a definite manner. As the total moisture mass increases, moisture content also rises. The moisture transfer potential is a function of moisture content and external parameters which at thermodynamic equilibrium should be uniform over the body or a system of bodies.

Table 5-6

**Maximum Reduced Hygroscopic Moisture Content and Factor B**

Material	$t, ^\circ\text{C},$	$W_m, \%$	$B, 1/\%$
Asbestos fibre	24	1.00	0.21
Asbestos paper	24	4.54	0.58
Hemspoint	24	13.0	0.16
Laquered paper	24	8.33	0.46
Wrapping paper	—	17.2	0.10
Writing paper	24	17.5	0.12
Packing paper (kraft)	24	17.2	0.98
Hygroscopic cotton wool	—	28.6	0.03
Glass wool	—	0.62	4.92
Felt (wool)	40	3.31	0.83
	20	3.50	0.75
	0	3.50	0.71
	-20	3.70	0.59
	-40	4.00	0.58
Gypsum slag concrete	20	11.5	0.11
Chasovyar clay	—	11.1	0.21
Wood	20	23.8	0.09
Pre-dried wood	20	22.2	0.15
Pine	40	13.15	0.23
	20	14.28	0.20
	0	14.28	0.17
	-20	14.28	0.15
	-40	14.70	0.13
Jute	24	25.00	0.08
Milled pine wood	—	13.7	0.08
Kaolin	24	1.43	0.65
Rubber	24	1.61	1.15
Ceramics	20	1.56	4.07
Kizelgur	24	4.00	0.71
Clay brick	20	0.59	13.0
Red brick	40	1.00	10.90
	20	1.11	7.70
	0	1.25	6.00
	-20	1.33	4.28
	-40	1.54	3.54
Sun-dried brick	40	5.55	0.41
	20	5.71	0.33
	0	6.09	0.29
	-20	6.28	0.24
	-40	6.57	0.20
Sand-lime brick	40	1.82	1.93
	20	1.88	1.55
	0	2.00	1.30
	-20	2.15	1.07
	-40	2.38	0.95
Glue	—	14.7	0.09
Coke	—	2.85	1.02
Flax (linen)	24	11.75	0.15
Flax thread	—	13.9	0.07
Soap	—	40.0	0.12

Table 5-6 (continued)

Material	$t, ^\circ\text{C}$	$W_m, \%$	$B, 1/\%$
Foam concrete	40	2.90	1.08
	20	2.94	0.88
	0	3.07	0.70
	-20	3.33	0.63
	-40	3.64	0.52
Foam plaster	60	25.2	0.003
Foam silicate	40	1.88	1.85
	20	2.00	1.46
	0	2.08	1.21
	-20	2.22	1.02
	-40	2.38	0.83
Pre-dried foam glass	20	1.33	10.0
Nitrocellulose silk	--	17.8	0.07
Raw silk	24	19.2	0.08
Wool	—	31.2	0.67
Glass wool	24	4.34	3.18
Slag concrete	40	2.90	1.08
	20	2.94	0.88
	0	3.07	0.70
	-20	3.33	0.63
	-40	3.64	0.52

In Figure 5-19 are shown the distributions of a thermal potential and the enthalpy of two bodies (lead and iron), and the distributions of similar equilibrium parameters for a peat slab and filter paper sheets. It is evident from the figure that lead and iron have the same temperature of  $50^\circ\text{C}$ . The enthalpies at the contact line of these bodies are different, namely, the enthalpy of iron is 5.5 kcal/kg and that of lead is 1.5 kcal/kg. Figure 5-19 shows that the distribution of moisture content in peat and paper is uniform. In a paper sheet it amounts to 0.5 kg/kg and in peat — 2.1 kg/kg. At the contact line of peat and paper sudden variation in moisture content similar to the enthalpy drop takes place. Further it will be demonstrated that the mass transfer potential for both materials is similar and equals 180 mass-transfer degrees. The moisture content growth in all parts of the system implies that the potential function either increases or decreases monotonically.

Potential  $\theta(u, T)$  can always be chosen so that  $\theta$  would increase with  $u$ , and for all potential scales  $\partial\theta/\partial u > 0$ .

### (b) Moisture Capacity of a Moist Body

Let us introduce a specific isothermal moisture capacity concept similar to that of specific heat  $c$

$$c_m = \left( \frac{\partial u}{\partial \theta} \right)_T \quad (5-4-7)$$

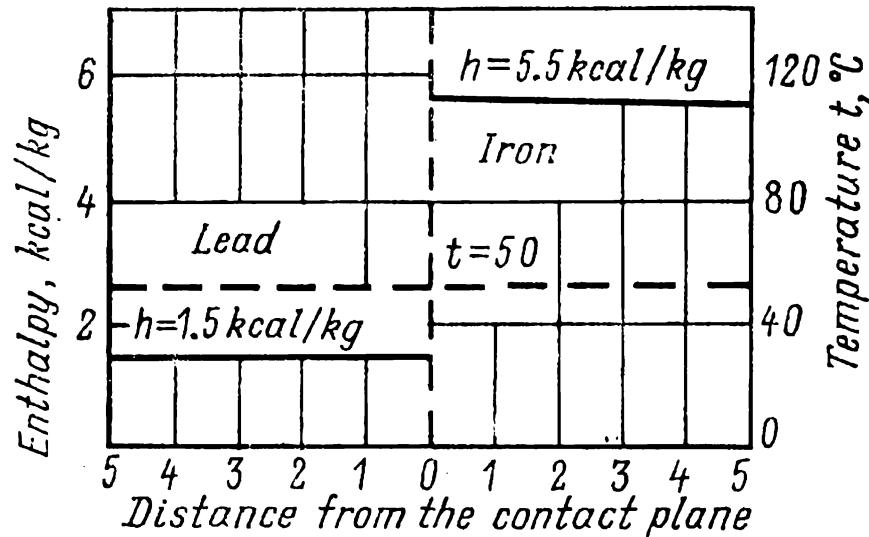


where  $(\partial u/\partial \theta)_T$  is a partial derivative of the specific moisture content with respect to the moisture transfer potential at constant temperature.

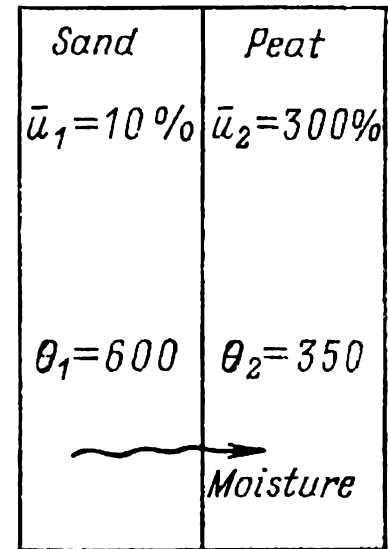
If the specific moisture capacity  $c_m$  is constant, then from relation (5-4-7) it follows that

$$u = b + c_m \theta \quad (5-4-8)$$

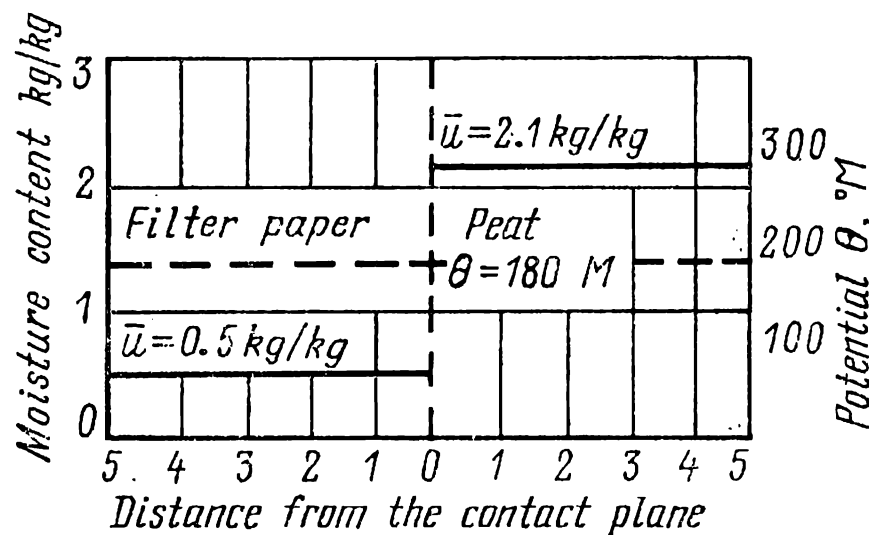
where  $b$  is a constant.



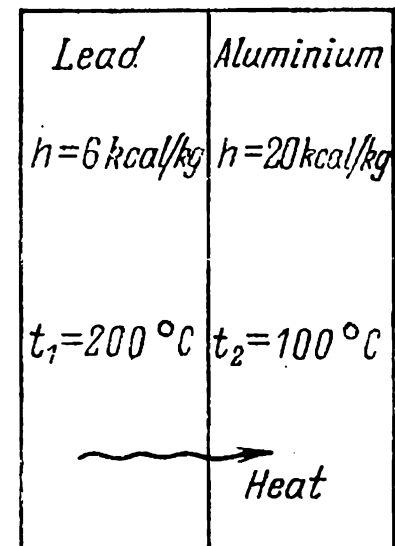
(a)



(b)



(c)



(d)

Fig. 5-19. Distribution of enthalpy and moisture content and heat and moisture transfer potentials in a system of two touching materials (a, b). Transfer of heat and mass from the material with a smaller content to that with a larger content (c, d).

Distance in cm

The moisture-transfer potential at constant temperature and moisture capacity is a linear function of moisture content. The specific moisture capacity is measured in  $\text{kg/kg} \cdot ^\circ\text{M}$ , where  $^\circ\text{M}$  denotes mass-transfer (moisture-transfer) degree.

It should be pointed out that in the case of molar transfer of a capillary liquid the capillary potential  $\psi$  is the moisture-transfer potential. By definition a capillary potential is negative and moisture is transferred from the lower capillary potential to the upper one similar to heat transfer in the range of negative temperatures expressed in centigrades ( $t < 0^\circ\text{C}$ ). At  $u = 0$  the capillary potential is maximum ( $\psi_{\max}$ ) and for a certain maximum moisture content it equals zero. Thus, for a capillary potential the constant appearing in (5-4-8) equals the product of maximum capillary potential by specific moisture capacity. In case of molecular moisture transfer (selective diffusion) the osmotic pressure  $p$ , for which the derivative  $\partial p/\partial u$  is negative, is a transfer potential.

The moisture transfer potential  $\theta$  will be assumed positive ( $\partial u/\partial \theta > 0$ ) and the constant  $b$  over the range of small moisture-content values will be assumed equal to zero with the assumption that the moisture transfer potential of a perfectly dry body equals zero. In this case the moisture transfer potential of free liquid equals the difference between maximum capillary potential  $\psi_{\max}$  and the capillary potential for a given moisture content  $\psi$  ( $\theta = \psi_{\max} - \psi$ ). The moisture transfer potential gradient will equal to capillary potential gradient with opposite sign ( $\nabla \theta = -\nabla \psi$ ). A similar relation holds also for diffusion transfer of liquid.

The described liquid transfer mechanism applies also to moisture transfer within a capillary pore, which is enclosed by liquid menisci, when liquid evaporates from one meniscus of the pore and vapour condenses at the other. In this case it is necessary that temperature drop along the capillary pore in the direction of transfer should be infinitesimal, i.e. evaporation and condensation would proceed at the same temperature. It is quite natural that the amount of liquid that evaporates from one meniscus must be equal to the amount of condensed vapour at the opposite meniscus. Such a process of vapour transfer within a closed capillary pore is equivalent to liquid transfer from the thermodynamic point of view\*.

For a constant specific moisture capacity, in the range of small moisture contents, from formula (5-4-8) we get

$$u = c_m \theta \quad (5-4-9)$$

i.e. the moisture transfer potential  $\theta$  scale is similar to centigrade scale for positive temperatures.

According to the accepted scale the moisture transfer is directed from the higher potential to the lower potential. At thermodynamic equilibrium the potentials of bodies in contact are equal ( $\theta_1 = \theta_2$ ) and their specific moisture contents are different.

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\* In the filtration theory such "closed" pores do not contribute to the effective porosity and in the calculation of liquid motion they are neglected. This is a principle difference between the filtration theory and that of moisture transfer in drying processes [5-12].

From formula (5-4-9) it follows that at  $\theta_1 = \theta_2$

$$\frac{u_1}{u_2} = \frac{c_{m1}}{c_{m2}} \quad (5-4-10)$$

i.e. the specific moisture content ratio of the bodies is proportional to the ratio of their moisture contents. For one and the same body  $c_{m1} = c_{m2}$  and  $u_1 = u_2$ , i.e. the moisture content distribution in the body at the equilibrium state will be uniform (see Fig. 5-19). The amount of moisture  $\Delta M_B$  transferred from one body to the other when it comes in contact with bodies having different potentials is

$$\Delta M_B = M_0 \Delta u = c_m M_0 (\theta_2 - \theta_1) \quad (5-4-11)$$

where  $\theta_1$  and  $\theta_2$  are moisture-transfer potentials before and after the moisture transfer;  $M_0$  is the mass of a perfectly dry body.

Formula (5-4-11) is similar to the elementary heat-transfer formula

$$\Delta Q = c_q M_0 (t_2 - t_1) \quad (5-4-12)$$

where  $c_q$  is the specific heat of the body.

The product of specific moisture capacity  $c_m$  by the mass of a perfectly dry body  $M_0$  will be referred to as the moisture capacity

$$C_m = c_m M_0 = c_m \rho_0 V_0 \quad (5-4-13)$$

The product  $c_m \rho_0$ , equal to the ratio  $C_m/V_0$ , is the volumetric moisture capacity of the body.

Relations (5-4-9) through (5-4-13) are valid for bodies having constant moisture capacity. If the moisture capacity changes with moisture-transfer potential  $\theta$ , and consequently, with moisture content  $u$ , then in these relations  $c_m$  is the mean specific moisture capacity over the moisture transfer range  $\Delta\theta = \theta_2 - \theta_1$ .

### (c) *Experimental Scale of Moisture-Transfer Potential*

To make use of the derived relations an experimental scale of moisture-transfer potential  $\theta$  is necessary. Unlike experimental thermodynamics, where the heat-transfer potential (temperature) is directly measured, in the present case the moisture content is determined directly by drying the body to a perfectly dry state. For better understanding of the principle of potential  $\theta$  scale an analogy will be used.

First we describe the calorimetric experiment for determining heat capacity. The test body heated to a definite temperature is put into liquid (water) present in the calorimeter. Heat transfer between the body and liquid will result in thermal equilibrium. By measuring the rise in liquid tempera-

ture the heat capacity of the body is determined. The specific heat capacity of liquid (water), in this case, is taken equal to one. The specific heat content (enthalpy) of the reference (calorimetric) liquid will equal to the temperature of the body and liquid, and the specific heat of the test body will equal to the ratio of specific enthalpies of the body and reference liquid. In this experiment the heat capacities of the body and liquid are assumed constant.

Thus, the most simple method for making the potential  $\theta$  scale resides in the determination of specific moisture contents of a system of two (test and reference) bodies in contact with each other and at thermodynamic equilibrium. Here it is necessary that direct contact should ensure free moisture transfer between the bodies. If the moisture content of a body is more than the maximum sorption moisture content ( $u > u_{cm}$ ), the moisture transfer takes place when the capillary porous bodies come in direct contact with each other. If the moisture content is less than the maximum sorption one ( $u < u_{cm}$ ), the necessary condition for thermodynamic equilibrium to set in is constant temperature and surrounding air humidity ( $T = \text{const}$ ,  $\phi = \text{const}$ ). In this case moisture is transferred by sorption and desorption in vapour-air mixture, and direct liquid transfer through the surfaces in contact also occurs.

Like calorimetric liquid cellulose is used as a reference solid, because it is highly hygroscopic (its maximum sorption moisture capacity at 25°C is about 0.28 kg/kg) and easily becomes wet.

Now we describe a very simple experiment. A test body (for example, quartz sand) is placed in a hollow cylinder open at the both ends and half filled with filter paper sheets of known moisture content. The cylinder is then sealed at both ends, weighed to determine the density of the test body and put in a thermostat\*. After a certain time interval the moisture content distribution in the reference and test bodies is measured by conventional methods. The moisture content of the test and reference materials at the contact boundary may readily be found from the graphs shown in Fig. 5-19 (for peat  $u_2 = 2.1$  kg/kg, and for reference filter paper  $u_1 = 0.5$  kg/kg). The moisture transfer potential  $\theta$  may be found from the value of the specific moisture content of the reference material provided its specific moisture capacity is given. Unlike the specific heat of a reference calorimetric liquid, the specific isothermal moisture capacity of a reference material  $(c_{mr})_T$  is taken not equal to one, but equal to 1/100 of the maximum sorption moisture content

$$(c_{mr})_T = \frac{1}{100} u_{cm} \quad (5-4-14)$$

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\* If the moisture content of a body is less than the maximum sorption moisture content, the cylinder with open ends is placed in a desiccator of known air humidity ( $T = \text{const}$ ,  $\phi = \text{const}$ ).

i.e. potential  $\theta$  of  $100^\circ\text{M}$  corresponds to a specific moisture content  $u_{cm}$  of a reference body. This selection of specific moisture capacity is based on the following considerations:

(1) maximum sorption moisture content at a certain temperature is a constant independent of the process by which it is achieved (sorption and desorption isotherms coincide at  $\varphi = 1$ );

(2) use of different reference bodies at the maximum hygroscopic point ( $\varphi = 1$ ) yields one and the same potential  $\theta$  equal to  $100^\circ\text{M}$ . This allows introduction of a unified experimental scale of potential  $\theta$ , which eliminates some difference in reference bodies.

The moisture-transfer potential  $\theta$  is thus determined from the moisture content  $u_r$  of the reference body for which a constant specific moisture capacity of  $0.01 u_{cm}$  is assumed, where  $u_{cm}$  is the maximum sorption moisture content of the reference body at  $t = 25^\circ\text{C}$ :

$$\theta = \frac{u_r}{(c_{mr})_T} = \left( \frac{u_r}{u_{cm}} \right)_T \cdot 100 \quad (5-4-15)$$

For a filter paper at  $25^\circ\text{C}$   $u_{cm} = 0.277$ . Then the mass-transfer potential  $\theta$  for a system of bodies in contact (see Fig. 5-19)

$$\theta = \frac{0.5}{0.277} 100 = 180^\circ\text{M} \quad (5-4-16)$$

This potential corresponds to the moisture content of peat, which equals  $2.1 \text{ kg/kg}$ .

The moisture capacity of a body can serve as a qualitative characteristic of moisture bond with wet materials. Its value depends on the reference material. Cellulose filter sheets were used as reference material. The choice was based on the following considerations:

(1) cellulose contains all forms of moisture bonds (adsorption, capillary and osmotic);

(2) equilibrium moisture content varies almost uniformly with temperature at different air humidities  $\varphi$ .

In Figure 5-20 the isotherms of filter paper sorption at temperatures from  $22$  to  $80^\circ\text{C}$  are plotted based on N. V. Artsikhovskaya's data. The relative equilibrium moisture content seems to be practically independent of temperature, and is a single-valued function of air humidity (small variations in  $u_p/u_c$  with temperature are random and depend on the experimental procedure). This is a very important result as it makes it possible to draw a conclusion that moisture transfer potential in a hygroscopic range depends only on  $\varphi$ . The relation  $\theta = f(\varphi)$  is shown in the same figure as a plot obtained by averaging sorption isotherms for different temperatures. Hence, a mass-transfer potential  $\theta$ , measured by a reference body (cellulose), in a hygroscopic range characterizes isothermal mass transfer of vapour and liquid in capillary-porous bodies.

From a simple thermodynamic analysis, supported by G. A. Maksimov's experiments, it follows that the equilibrium moisture content of a body for definite values of  $t_c$  and  $\varphi$  is independent of the fact whether the test material was in contact with other bodies or it alone attained equilibrium. It is assumed that the contact does not result in any chemical reactions or changes in the structure of materials.

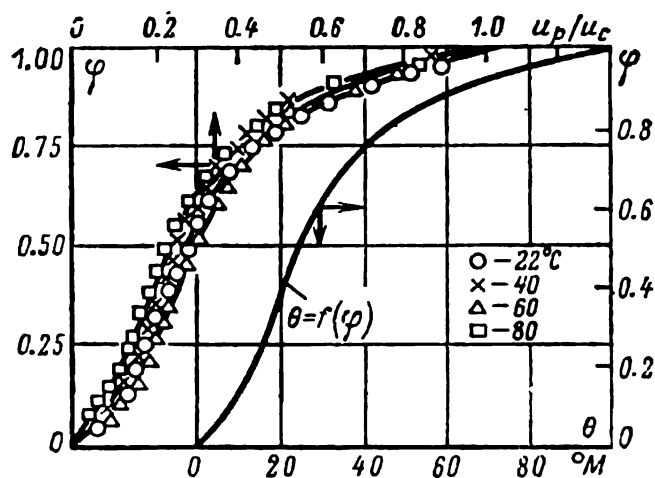


Fig. 5-20. Plot of relative equilibrium moisture content  $u_p/u_c$  of filter paper with moisture transfer potential  $\theta$  versus air humidity  $\varphi$

Thus, a moisture transfer potential of  $100^\circ\text{M}$  corresponds to the maximum sorption moisture content of any material, and the potential from  $0^\circ$  to  $100^\circ\text{M}$  corresponds to a hygroscopic state. The moisture transfer potential  $\theta$  ( $0 < \theta < 100$ ) is thus determined from the humidity of air with which the body is in equilibrium, i.e. humid air serves as the standard for the experimental scale of moisture transfer.

Air humidity logarithm, equal to the logarithm of the ratio of water vapour pressure over the body to the vapour saturation pressure over the free water surface, is directly proportional to the free isothermal bond energy of the given amount of moisture  $u_p$  present in the body. Moisture transfer potential  $\theta$  is thus the function of isothermal free bond energy of moisture with capillary-porous material.

It is of interest to compare the potential  $\theta$  with the moisture-transfer potential used in agrophysics. In the works of american scientists the quantity  $pF$ , equal to the logarithm of suction force  $F$ , has been taken as the moisture transfer potential in soils and grounds. In a moist state range this value is proportional to a capillary pressure found from tensiometric experiments. In a hygroscopic range the suction force  $F$  is determined from the relative air humidity conforming to the equilibrium moisture content of the body

$$F = \frac{RT}{M_1} \log_e \varphi \quad (5-4-17)$$

where  $M_1$  is the molecular mass and  $R$  is the universal gas constant. The suction force  $F$  in both the cases is measured in pressure units (cm of water column).

Special experiments were carried out to find the potential  $pF$  for a reference material (filter paper) in the moist-state range, and in the hygroscopic range this potential was calculated from formula (5-4-12). In the moist-state range  $pF$  was found to be a linear function of moisture content. Thus, in this range the potentials  $pF$  and  $\theta$  are related by a simple relation

$$pF = 3.66 - 1.5 \times 10^{-3} \theta \quad (5-4-18)$$

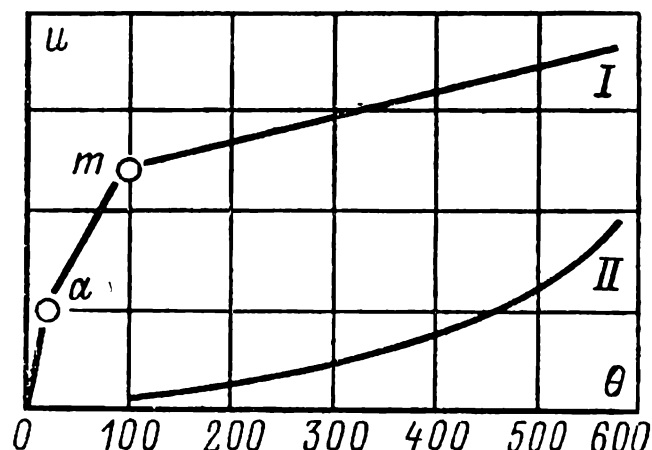
at  $\theta > 100^\circ\text{M}$ .

In the hygroscopic range the relation between  $pF$  and  $\theta$  has a complicated form\*, but it must be mentioned that potential  $\theta$  is unambiguously defined by air humidity  $\varphi$ . With a single-valued relation  $\theta = f(\varphi)$ , it is possible to find moisture transfer potential  $\theta$  from the experimental data on equilibrium moisture contents (sorption and desorption isotherms) for different materials, and the specific isothermal moisture capacity may be found from the curves  $u_p = f(\theta)$ .

Thus, in a hygroscopic range ( $0 < \theta < 100^\circ\text{M}$ ) the moisture transfer potential and specific moisture capacity are found from sorption or desorption isotherms. In the moist-state range ( $\theta > 100^\circ\text{M}$ ) these parameters are determined from the moisture content of the reference body in contact with the test material at thermodynamic equilibrium.

Experiments with different materials have revealed that in most cases the dependence  $u = f(\theta)$  has the appearance of a broken line for colloid capillary-porous bodies (Fig. 5-21), and only for coarse disperse materials the dependence  $u = f(\theta)$  represents a curve with its convex facing the axis  $\theta$ .

Fig. 5-21. Typical curve of moisture content  $\bar{u}$  versus moisture transfer potential  $\theta$ ,  $^\circ\text{M}$  for colloid capillary-porous materials (I) and coarse disperse materials (II)



The presence of discontinuity at point  $a$  in the hygroscopic range ( $0 < \theta < 100^\circ\text{M}$ ) may be an explanation of transition from one bond form to another, for example, from polymolecular adsorption bound water to capillary moisture. Moisture capacity of the material increases as we pass from free water to bound moisture.

\* If for a filter paper Posnov's formula is used as a first approximation and  $Bu_p$  is assumed to be independent of temperature, we obtain  $\theta = 100 - Bu_p \times 100 \log_e \varphi$ . Thus, the quantity  $(100 - \theta)$  is proportional to  $\log_e \varphi$ , and the potential  $pF$  to  $\log_e \left( \frac{RT}{M} \log_e \varphi \right)$ , herefore the relation between  $pF$  and  $\theta$  is not linear.

With moisture capacity known the moisture content may be calculated by the following formulas

$$u = c'_m \theta; 0 < \theta < \theta_a \quad (5-4-19)$$

$$u = c''_m(\theta - \theta_a) + c'_m \theta_a, \theta_a < \theta < 100^\circ\text{M} \quad (5-4-20)$$

$$u = c'''_m(\theta - 100) + c''_m(100 - \theta_a) + c'_m \theta_a \text{ at } \theta > 100^\circ\text{M} \quad (5-4-21)$$

where  $c'_m$ ,  $c''_m$ ,  $c'''_m$  are the specific isothermal moisture capacities in the corresponding ranges of  $\theta$ ; their values for some materials are furnished in Table 5-7.

Table 5-7

Isothermal Specific Moisture Capacities of Some Moist Materials at 25°C

Material	$c'_m \cdot 10^2$	$c''_m \cdot 10^2$	$c'''_m \cdot 10^2$	Material	$c'_m \cdot 10^2$	$c''_m \cdot 10^2$	$c'''_m \cdot 10^2$
Foam concrete $\rho_0 = 700$	0.140	0.040	—	Red brick	0.004	0.009	—
Peat slabs	0.674	0.052	—	Quartz sand:			
Wood (pine)	0.380	0.270	0.21	100 < $\theta$ < 300	—	—	0.007
Fibrolite	0.130	0.090	—	300 < $\theta$ < 550	—	—	0.025
Red clay	0.180	0.180	0.03	550 < $\theta$ < 700	—	—	0.90
Lowland peat	—	1.200	0.51				

In most cases the potential  $\theta$  varies slightly with temperature. But for a number of capillary-porous bodies the relation between  $\theta$  and  $T$  is close to the linear one. For example, for quartz sand, according to the data reported by F. M. Polonskaya [5-14], the temperature coefficient of the moisture-transfer potential  $(\partial\theta/\partial T)u = 0.6^\circ\text{M}/^\circ\text{C}$  at  $200 < \theta < 300^\circ\text{M}$ ;  $10^\circ < t < 80^\circ\text{C}$

#### (d) Thermogradient Coefficient

Thermogradient coefficient  $\delta_u$  is classed with thermodynamic parameters. It is defined as the ratio of moisture content drop  $\Delta u$  to temperature drop  $\Delta T$  at a steady state with no moisture transfer

$$\delta_u = \left( \frac{\Delta u}{\Delta T} \right)_{j=0} \quad (5-4-22)$$

The formula of moisture conduction law is

$$j = -\lambda_m \nabla \theta - \frac{\lambda_m}{c_m} \delta_u \nabla T \quad (5-4-23)$$

where  $\lambda_m$  is the coefficient of moisture conduction.



From formula (5-4-23) we get an expression for the thermogradient coefficient  $\delta_\theta$  based on the mass-transfer potential difference.

The term  $c_m$  is introduced into (5-4-23) to have same dimensions ( $\text{deg}^{-1}$ ) for  $\delta_u$  and  $\delta_\theta$ . Then

$$\delta_\theta = c_m \left( \frac{\Delta\theta}{\Delta T} \right)_{j=0} \quad (5-4-24)$$

Since the mass-transfer potential depends on moisture content and temperature, then we get

$$d\theta = \frac{1}{c_m} du + \left( \frac{\partial\theta}{\partial T} \right)_u dT \quad (5-4-25)$$

where  $\theta'_T = (\partial\theta/\partial T)_u$  is the temperature coefficient of the moisture-transfer potential. Using relation (5-4-24), we get an expression for the moisture diffusion law

$$j = -a_m \rho_0 \vec{\nabla} u - a_m \rho_0 \delta_u \vec{\nabla} T \quad (5-4-26)$$

where  $\delta_u$  is the thermogradient coefficient equal to

$$\delta_u = \delta_\theta + c_m \theta'_T \quad (5-4-27)$$

If  $\delta_\theta = 0$ , i.e. nonisothermal transfer of moisture is described only by the moisture transfer potential gradient ( $\delta_\theta = 0$ ), then

$$\delta_u = c_m \theta'_T \quad (5-4-28)$$

If moisture transfer is caused not only by  $\vec{\nabla}\theta$ , but also by  $\vec{\nabla}T$ , this is observed in the moist state range, then expression (5-4-27) should be used. For a number of capillary-porous materials  $c_m \theta'_T \ll \delta_\theta$ , then we have

$$\delta_u = \delta_\theta = \delta \quad (5-4-29)$$

The thermogradient coefficient of capillary-porous bodies increases with moisture content, reaches a certain maximum value, and then decreases. The behaviour of  $\delta_u$  depends on the capillary-porous structure of the body and the distribution of capillary moisture in it. Entrapped air may possibly affect liquid distribution as temperature decreases. At the heated parts of the porous body the entrapped air expands and forces out the liquid to cold part.

A similar calculation, performed as per the above described procedure, shows that as the maximum moisture content is attained, the coefficient  $\delta_u$  equals zero since at this state all the pores are filled with liquid and there is no entrapped air.

For typical colloid bodies the curve  $\delta_u = f(u)$  has a maximum, at certain moisture contents the curve  $\delta_u(u)$  crosses the moisture content axis (coefficient  $\delta_u = 0$ ).

In the range of large moisture contents the value of  $u_0$ , corresponding to  $\delta_u = 0$ , shows the boundary between osmotic and adsorption moistures. In the range of small moisture contents the value  $u_0$  characterizes thermo-diffusion and effusion vapour transfers in macrocapillaries.

A thermogradient coefficient of colloid capillary-porous bodies is plotted in Fig. 5-22. It is seen from the figure that the curve  $\delta_u = f(u)$  has a maximum which shifts as the temperature increases towards smaller moisture contents.

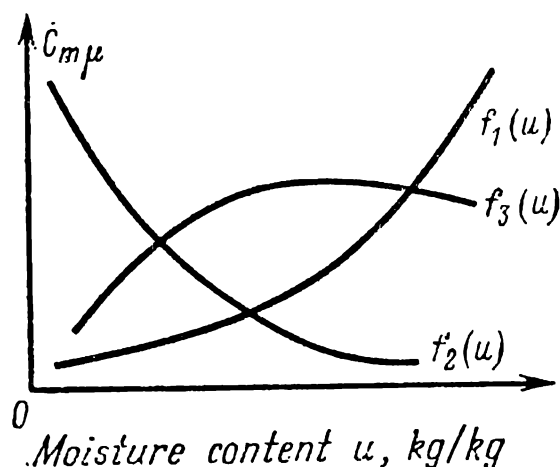


Fig. 5-22. Schematic shapes of curves  $C_{m\mu} = f_1(u)$ ,  $\left(\frac{\partial \mu}{\partial T}\right)_u = f_2(u)$  and  $\delta_u = f_3(u)$  for colloid capillary-porous materials

As in case of a colloid body, for pine wood the coefficient  $\delta_u$  equals zero at large moisture contents. Maxima of the curves  $\delta_u = f(u)$  shift to the right as the temperature decreases. The coefficient  $\delta_u$  decreases as the temperature increases. This is also characteristic of osmotic moisture except for small moisture contents when the coefficient  $\delta_u$  grows with temperature [5-10].

#### (e) Application of Chemical Potential to the Calculation of Moisture Capacity and the Thermogradient Coefficient

It has been shown earlier that the chemical potential  $\mu_1$  is the transfer potential of vaporous moisture. If we assume that vaporous moisture as a first approximation follows the law of ideal gases, then the chemical potential will be a function of partial pressure  $p_1$ , and temperature  $T$ , e.g.  $\mu_1 = f(p_1, T)$ . In the range of hygroscopic state the partial vapour pressure is a function of temperature and the moisture content  $p_1 = f(u, T)$ . For this reason the chemical potential  $\mu_1$  will also be a function of the moisture content and temperature  $\mu_1 = f(u, T)$ .

In the range of hygroscopic state liquid moisture is bound by adsorption forces (mono- and polymolecular adsorption moisture), capillary forces (microcapillary moisture) and diffusion osmotic forces (osmotic moisture bonds).

In these cases the chemical potential  $\mu_2$ , which is also a function of the moisture content and temperature of the body  $\mu_2 = f(u, T)$ , may as a first approximation be taken as a liquid moisture transfer potential.

Thus, in a hygroscopic range the chemical potential  $\mu$ , which is a function of temperature and moisture content of the material, may be taken as a moisture transfer potential. In the moist state range the chemical potential cannot be used as a moisture transfer potential\*. It has been demonstrated in [5-15] that the quantity  $\overline{\nabla}(\mu/T)$ , which is the motive force  $X_m$  of mass transfer in irreversible thermodynamics, is approximately proportional to the logarithm of relative air humidity  $\varphi$  in

$$X_m = T\overline{\nabla} \left( \frac{\mu}{T} \right) = RT\overline{\nabla} \log_e \varphi \quad (5-4-30)$$

Taking into consideration the fact that the bond energy  $E$  of moisture with material equals

$$E = \left( \frac{\partial F}{\partial m} \right)_T = -RT \log_e \varphi \quad (5-4-31)$$

it may be concluded that  $E$  is the moisture transfer potential. It should be pointed out that this conclusion was first drawn in [5-16] based on the analysis of experimental curves showing distribution of relative air humidity in a boundary layer upon drying gypsum plates. By taking the chemical potential  $\mu$  as the moisture transfer potential and using the relation

$$\mu = RT \log_e \varphi \quad (5-4-32)$$

the values of  $\mu$  (J/mol) were calculated for different  $\varphi$  and  $T$  [5-13]. Then with the use of sorption and desorption isotherms  $u = f(\varphi, T)$ , the values of chemical potentials for various materials were calculated as a function of moisture content and temperature  $\mu = f(u, T)$ . Using the plots  $\mu = f(u, T)$ , the temperature coefficient of chemical potential  $(\partial\mu/\partial T)_u$  and the isothermal specific moisture contents  $c'_m$  were calculated from

$$c'_m = \left( \frac{\partial u}{\partial \mu} \right)_T \quad (5-4-33)$$

by the graphical differentiation method. The results are available in [5-15].

In this case the thermogradient coefficient will be

$$\delta_u = c'_m \left( \frac{\partial \mu}{\partial T} \right)_u \quad (5-4-34)$$

The values of specific moisture capacities, temperature coefficients, chemical potentials  $(\partial\mu/\partial T)_u$  and thermogradient coefficients for some materials are listed in [5-12]\*\*.

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\* The chemical potential may in principle be applied to all the ranges of moist state, but its practical use is difficult in the moist state range (*editor's note*)

\*\* These coefficients are applicable for equilibrium conditions, therefore they are valid only at very small drying rates (*editor's note*)

An analysis of available experimental data reveals that specific mass capacity  $c_{m\mu}$  (mol/J) is slightly dependent on temperature. So as a first approximation it may be considered that isothermal moisture capacity is a single-valued function of moisture content. The schematic of the curve  $c_{m\mu} = f(u)$  is shown in Fig. 5-22.

The temperature coefficient of chemical potential  $(\partial\mu/\partial T)_u$  decreases considerably as  $u$  rises. The available experimental data reveal that  $(\partial\mu/\partial T)_u$  is practically independent of temperature. In most cases the thermogradient coefficient  $\delta_u$  increases with moisture content. Curves  $\delta_u = f(u)$  for wood have a maximum, curves for different materials have a similar shape.

Comparing the values of  $\delta_u$ , calculated by formula (5-4-22), with the experimental values of  $\delta_u$ , when  $\delta_u$  was calculated directly from the moisture content drop, we find that they are in good agreement.

This once again confirms that the thermogradient coefficient is a thermodynamic characteristic and may be calculated from sorption and desorption isotherms.

## 5-5. MOLECULAR-KINETIC METHOD

In this section an attempt is made to describe in detail the mechanism of vapour transfer in a free-molecular flow in a thin capillary (microcapillary) tube with due regard for evaporation both from the liquid meniscus and from the lateral surface of the capillary with a temperature gradient along the capillar wall [5-16]. The analysis of the mechanism has shown that within a capillary-porous material, partially filled with liquid, evaporation and condensation processes may occur. The dimensionless parameters  $s$  and  $l$  are the main characteristic values. The former is the product of the dimensionless heat of liquid-vapour phase transition by the relative temperature drop along the capillary, and the latter is the ratio of capillary length to the radius (dimensionless length of the capillary).

Rough estimation of  $s$  for liquid evaporation into vacuum ( $T = 220^\circ$ ;  $\Delta T = 10^\circ\text{C}$ ) yields  $s \simeq 1.3$ . Then for  $l = s$  and at  $l > 1$  (i.e. when the capillary length is greater than the radius), vapour condensation may occur at the capillary meniscus, while near the exit evaporation may proceed. However, if  $l < 1$  (the length of the capillary is less than its radius) evaporation takes place at both capillary ends under like conditions ( $l = s$ ). Thus, the pore structure ( $l$ ) affects considerably the mechanism of vapour migration. The value  $l$  is always positive, while  $s$  may be negative as well depending on the sign of the temperature gradient  $\Delta T$  along the capillary ( $s \leq 0$ ). If  $s < 0$ , then with  $l = -s$ , the mechanism of vapour migration differs from that at  $l = s$ . Hence, the direction of the temperature gradient has an effect on evaporation and condensation processes in the capillary. This phenomena is observed only at nonisothermal conditions and may be referred to as the thermal effect of vaporous moisture migration.

Successive condensation and evaporation in a pore or capillary on liquid menisci were discussed earlier, but these processes were assumed to occur at isothermal conditions. In the present case moisture evaporation and condensation in a capillary are determined by the temperature drop along its length and on the geometry of pores and capillaries. This is the main point in the mechanism of moisture transfer in the capillary under consideration.

It was found through a number of investigations that surface diffusion of vapour or gas contributed much to the transfer process. Therefore, the velocity of vapour migration along the capillary depends not only on the linear diffusion velocity but also on the time of molecular adsorption on the surface of the capillary wall, i.e. on the velocity of surface diffusion. The obtained relationships may be used to analyse the mechanism of drying moist materials and transpiration cooling of heat-protecting materials.

To conclude, it is shown how the approximate theory of the mean free path length may be used for investigation of the kinetics of vapour transfer through a highly disperse porous body when the capillary model is not exact.

#### (a) Microcapillary Vapour Flow with Phase Conversions

We shall consider a cylindrical capillary with a flat bottom ( $X = 0$ ) on one side, with evaporation (sublimation) proceeding both on the lateral surface and at the bottom. A constant temperature gradient is prescribed along the capillary. The number of molecules evaporating from a unit area per unit time is

$$J(\alpha, T) = \alpha \frac{p_e(T)}{(2\pi mkT)^{1/2}} \quad (5-5-1)$$

where  $m$  is the molecular mass,  $k$  is the Boltzman constant,  $p_e$  is the pressure of saturated vapours,  $\alpha$  is the evaporation (condensation) coefficient.

It is assumed that flow of molecules leaving a unit surface contains  $\alpha$  fraction of evaporating molecules and  $1 - \alpha$  fraction of reflected molecules. In [5-17] diffuse and specular reflections are considered.

1. *With diffuse reflection*, for a flow of molecules from a unit area of the lateral surface of a capillary the following equation is obtained (the flow is based on the value  $j(1, T_0) = j_0$ ):

$$I(x) = \alpha \frac{p_e(T)}{p_e(T_0)} \left( \frac{T_0}{T} \right)^{1/2} + (1 - \alpha) \left[ \int_0^1 I(\xi) K_1(|x - \xi|) d\xi + I_0 K(x) \right] \quad (5-5-2)$$

The first summand herein describes evaporating molecules, the second, those getting onto unit area of the lateral surface in the vicinity of the point  $x$

from the remaining lateral surface (integral summand) and from the bottom  $[I_0 K(x)]$  and then reflected

$$I_0 = \alpha_0 + 2l(1 - \alpha_0) \int_0^1 I(\xi) K(\xi) d\xi, \quad l = \frac{L}{R} \quad (5-5-3)$$

where  $R$  is the radius of capillary and  $L$  is the length of capillary.

The expressions for the functions  $K(x)$  and  $K_1(x)$  are given in [5-18]

$$K(x) = \frac{l^2 x^2 + 2}{2(l^2 x^2 + 4)^{1/2}} - \frac{l}{2} x, \quad K_1(x) = -\frac{dK(x)}{dx} \quad (5-5-4)$$

For a nonisothermal capillary surface, if the surface temperature varies by the law

$$T(x) = T_0(1 + Ax) \quad (5-5-5)$$

and  $|A| \ll 1$ , the pressure  $p_e$  may be expressed as follows

$$p_e(x) = B \exp \left\{ -\frac{Q}{kT(x)} \right\} = B \exp \left\{ -\frac{Q}{kT_0} (1 - Ax) \right\} \quad (5-5-6)$$

In a number of works (for example, [5-19]) it is shown that for comparatively short capillaries the functions  $K(x)$  and  $K_1(x)$  may be substituted by exponents with sufficient degree of accuracy. Thus

$$K(x) = \frac{1}{2} \exp \{-lx\}$$

Using approximate expressions for  $K$  and  $K_1$ , we substitute (5-5-6) into (5-5-2). By differentiating (5-5-2) twice and combining the relation obtained with the initial one, we get the following differential equation

$$\frac{d^2 I}{dx^2} - c^2 I = b \exp \{sx\} \quad (5-5-7)$$

where

$$s = \frac{QA}{kT_0}, \quad c^2 = \alpha l^2, \quad b = \alpha (s^2 - l^2)$$

Here  $Q$  is the heat quantity necessary for evaporation of one molecule.

Solution of (5-5-7) for  $s \neq \pm c$  is of the form

$$I = a_1 \exp \{cx\} + a_2 \exp \{-cx\} + \frac{b}{s^2 - c^2} \exp \{sx\} \quad (5-5-8)$$

The coefficients  $a_1$  and  $a_2$  are sought by substituting (5-5-8) into (5-5-2) (assuming that  $\alpha_0 = \alpha$ ) and equating the coefficients at  $\exp \{lx\}$  and

$\exp \{-lx\}$ :

$$\begin{aligned}
 a_1 \frac{\exp\{c\}}{c-l} - a_2 \frac{\exp\{-c\}}{c+l} &= -\frac{b}{s^2-c^2} \frac{\exp\{s\}}{s-l} \\
 a_1 \left[ -\frac{1}{c+l} + \frac{1-\alpha}{c-l} (\exp\{c-l\} - 1) \right] &+ a_2 \left[ \frac{1}{c-l} - \frac{1-\alpha}{l+c} \right. \\
 &\times (\exp\{-l+c\} - 1) \left. \right] = \frac{b}{(s^2-c^2)(s+l)} - \frac{\alpha}{l} \\
 &- \frac{b(1-\alpha)}{(s^2-c^2)(s-l)} (\exp\{s-l\} - 1)
 \end{aligned} \tag{5-5-9}$$

In case of  $s = \pm c$  (that for  $\alpha = 1$  may be written as  $\frac{Q}{kT_0} \frac{\Delta T}{T_0 l} = \pm 1$ ), the solution of equation (5-5-7) is expressed as follows

$$I = a'_1 \exp\{cx\} + a'_2 \exp\{-cx\} + \frac{b}{2s} x \exp\{sx\} \tag{5-5-10}$$

where  $a'_1$  and  $a'_2$  as well as  $a_1$  and  $a_2$  in (5-5-8) are determined from the system of algebraic equations obtained by substitution of (5-5-10) into (5-5-2).

For the flow of molecules at the capillary exit we find

$$N = j_0 \pi R^2 \left[ I_0 K_2(1) + 2l \int_0^1 I(x) K(1-x) dx \right] \tag{5-5-11}$$

where

$$\frac{dK_2(x)}{dx} = -2l K(x)$$

In a particular case of  $\alpha = 1$  and  $s \neq c$  for approximate  $K(x)$  and  $K_2(x)$  we obtain from (5-5-2) and (5-5-11)

$$N = j_0 \pi R^2 \exp\{-l\} \left[ 1 + \frac{l}{s+l} (\exp\{s+l\} - 1) \right] \tag{5-5-12}$$

It follows from (5-5-12) that at  $l \gg s$  the flow is expressed by

$$N = j_0 \pi R^2 \left( 1 - \frac{s}{l} \right) \exp\{s\} \tag{5-5-13}$$

i.e.  $N$  tends to  $j_0 \pi R^2 \exp\{s\}$  independent of  $l$ .

At  $\alpha = 1$  and  $s = 0$  (isothermal case) the value of flow independent of  $l$  is found from (5-5-2) and (5-5-11)

$$N_l = j_0 \pi R^2 \quad (5-5-14)$$

The same relationship is arrived at with exact expressions for the functions  $K(x)$  and  $K_2(x)$ . A similar result is qualitatively explained in [5-20].

At  $\alpha \neq 1$  the expressions for  $I(x)$  and  $N$  become rather cumbersome. Numerical calculations were performed for ice at  $T_0 = 213^\circ\text{K}$ , with approximate functions  $K(x)$ ,  $K_1(x)$  and  $K_2(x)$  being used.

The calculations showed that for  $\alpha \neq 1$  and  $\Delta T = 0$  the flow at the capillary exit at small  $l$  increases with  $l$ , and then practically becomes independent of  $l$  (Fig. 5-23). In case of  $\alpha \neq 1$  and  $\Delta T \neq 0$  the behaviour of the curves characterizing the flow dependence on  $l$  was more complicated.

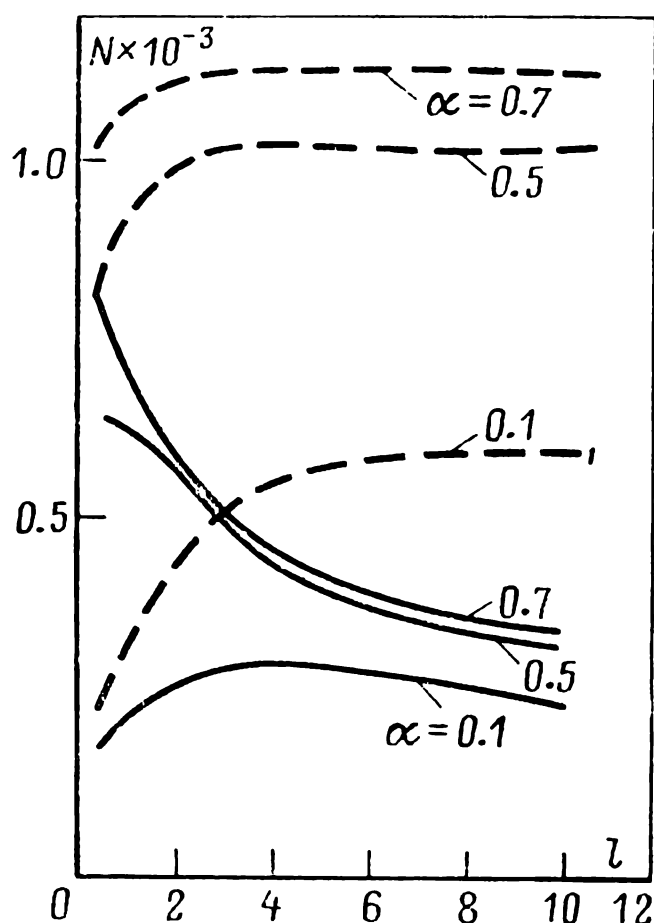


Fig. 5-23. Plot of flow  $N$  at the exist of capillary versus  $l$  for the diffuse reflection scheme ( $T_0 = 213 \text{ K}$ ,  $l = 10^{-1} \text{ cm}$ )

Dashed line,  $\Delta T = 0$ ; solid line,  $\Delta T = -10$

At the lateral surface the net dimensionless flow of molecules is defined by

$$-J(x) = [I^-(x) - I(x)] \simeq \alpha[I^-(x) - \exp \{sx\}] \quad (5-5-15)$$

where  $I^-$  is a flow of molecules getting onto unit lateral surface.  $I^-$  is sought from the equation

$$I^-(x) = \int_0^1 I(\xi) K_1(|x - \xi|) d\xi + I_0 K(x) \quad (5-5-16)$$

Upon determining  $I$  from (5-5-2), we shall calculate  $I^-$  from (5-5-16) and then the net molecular flow from (5-5-15).



Write the flow  $J(x)$  for a particular case of  $\alpha = 1$  omitting the cumbersome general formula. At  $s \neq \pm l$

$$J(x) = \frac{1}{2} \exp \{-lx\} \left[ \frac{2\beta^2}{\beta^2 - 1} \exp \{(\beta + 1)lx\} - \frac{1}{\beta - 1} \exp \{l(2x + \beta - 1)\} - \frac{1}{\beta + 1} \right] \quad (5-5-17)$$

where  $\beta = \frac{s}{l}$ , and the positive value of  $J$  corresponds to evaporation.

It follows from (5-5-17) that the behaviour of  $J(x)$  is determined by  $l$  and  $\beta$ . Thus, for  $1 < \beta < \exp \{l(\beta - 1)\}$  at  $x = 0$   $J < 0$ , at  $x = 1$   $J > 0$ . As to the case of  $\beta = 0$ , i.e. of an isothermal capillary, evaporation occurs along its full length and the flow  $J$  attains its maximum value at  $x = 1$ .

For  $s = l$  we have

$$J(x) = \frac{1}{2} \exp \{-lx\} \left[ \left( \frac{3}{2} - l(1 - x) \right) \exp \{2lx\} - \frac{1}{2} \right] \quad (5-5-18)$$

For  $s = -l$

$$J(x) = \frac{1}{2} \exp \{-lx\} \left[ \frac{1}{2} (1 + \exp \{2l(x - 1)\}) - lx \right] \quad (5-5-19)$$

From (5-5-18) it is seen that for  $l < 1$   $J(0) > 0$  and  $J(1) > 0$ , i.e. evaporation takes place at both ends of the capillary and at  $l > 1$   $J(0) < 0$ ,  $J(1) > 0$ , i.e. with increasing  $x$  transition from condensation to evaporation occurs.

Expression (5-5-19) indicates that for all  $l$  at  $x = 0$  evaporation occurs, for  $x = 1$  evaporation takes place at  $l < 1$  and condensation, at  $l > 1$ .

When evaporation proceeds only at the bottom with diffuse reflection of molecules from the lateral surface, the flow  $I$  may be obtained from (5-5-2) assuming  $\alpha = 0$ .

2. We shall elucidate the effect of *specular reflection* of molecules on the net flow at the capillary exit.

The number of molecules that evaporated from the lateral surface and left the capillary after multiple specular reflections, is written as follows [5-21]:

$$N_w = 2 \pi R^2 l \int_0^1 j(\alpha, T) S_1(1 - x) dx \quad (5-5-20)$$

where

$$S_1(x) = K(x) + \sum_{n=1}^{\infty} (1 - \alpha)^n \left[ K\left(\frac{x}{n+1}\right) - K\left(\frac{x}{n}\right) \right]$$

For the number of molecules, which evaporated from the bottom, and then repeatedly reflected from the walls and left the capillary, we obtain (by assuming  $\alpha_0 = \alpha$ )

$$N_b = \pi R^2 j(\alpha, T_0) S_2(1) \quad (5-5-21)$$

where

$$S_2(x) = K_2(x) + \sum_{n=1}^{\infty} (1 - \alpha)^n \left[ K_2\left(\frac{x}{n+1}\right) - K_2\left(\frac{x}{n}\right) \right]$$

Besides, the account should be taken of the molecules which left the capillary after they had evaporated from the lateral surface, got to the bottom and reflected specularly from it. They are allowed for approximately through substitution by the diffusely reflected molecules

$$N'_w = 2\pi R^2 l S_2(1) (1 - \alpha) \int_0^1 j(\alpha, T) S_1(x) dx \quad (5-5-22)$$

The flow at the capillary exit is determined by summation of (5-5-20) through (5-5-22).

It follows from the calculation results for diffuse and specular reflections that at small  $l$  the values of exit flows are similar. In case of specular reflection with increasing  $l$  the rate of flow increase is higher than with diffuse reflection.

The case of the bottom evaporation with specular reflection from the bottom and diffuse reflection from the side surface is considered further.

### (b) *Surface Diffusion Effect on Free-Molecular Vapour Flow Through a Capillary*

Experimental investigation of free molecular gas flows in capillaries has revealed the difference between the measured conductivity and that calculated by the Knudsen formula. One of the reasons for this difference is surface diffusion [5-22]. This is explained by the fact that at low density the molecules adsorbed on the surface may be considered as two-dimensional gas so that in the presence of the density gradient, a two-dimensional diffusion flow takes place which is described by

$$J_s = - D_s \frac{dn}{dX} \quad (5-5-23)$$

where  $D_s$  is the surface diffusivity.

Some works are known in which due consideration has been given to the surface diffusion effect. Thus, in [5-23] the problem is considered on the Knudsen flow in a long tube with surface diffusion flow, wherein the effect of surface diffusion due to linear dependence of the surface density on the coordinate shows itself through contribution of this flow to the Knudsen flow.

In other works an integro-differential equation has been solved for the surface density of molecules, which in the extreme case of no surface diffusion becomes the integral Klausung equation. Particularly, in [5-24], the solution of this equation is built by the iteration method for the case of a short capillary  $\left(l = \frac{L}{R} \ll 1\right)$  and in [5-18] and [5-25] a similar problem is solved numerically for any  $l$ . In these studies no investigation is, however, made of the effect of different terms in the integro-differential equation on the capillary conductivity.

This subsection deals with an approximate analytical investigation of a free-molecular vapour flow with regard for surface diffusion in a cylindrical capillary bounded by a flat surface of evaporating liquid from one side ( $X = 0$ ) and connected with the tank filled with the vapours of this liquid, from the other ( $X = L$ ). The liquid surface is characterized by the specular reflection coefficient  $\sigma$ .

An integro-differential equation for the molecular surface density is of the form:

$$\begin{aligned} \frac{1}{L^2} \frac{d}{dx} \left[ D_s(x) \frac{dn}{dx} \right] = \frac{n(x)}{\tau(x)} - \int_0^1 \frac{n(\xi)}{\tau(\xi)} [K_1(|x - \xi|) \\ + \sigma K_1(x + \xi)] d\xi - \frac{N_0}{n_0} K(x) - \frac{N_1}{n_0} [K(1 - x) + \sigma K(1 + x)] \end{aligned} \quad (5-5-24)$$

where  $n$  is the dimensionless surface density of molecules based on the density  $n_0 = n(0)$ ,  $\tau$  is the time of adsorption;  $N_0$  and  $N_1$  are flows of molecules diffused from the unit surface area of the bottom and of those which have passed from the tank through a unit area of the open end, respectively.

The term in the left-hand side of equation (5-5-24) is specified by the surface diffusion of molecules. The integral term in the right-hand side corresponds to the flow of molecules falling per unit area of the capillary side surface from the remaining surface as a result of desorption by the cosine law (the first summand) and of specular reflection from the bottom (the second summand).

The first term out of the integral sign describes the flow of molecules diffusing from the bottom and falling on per unit surface area of the capillary, the second and third terms characterize flows of molecules entering the capillary through the open end and falling onto the wall directly and after specular reflection from the bottom.

We shall formulate the boundary conditions for integro-differential equation (5-5-24). It should be noted that surface diffusion takes place not only in a capillary, but on an inner surface of the tank as well. In this case it is assumed that at a distance of the order of  $X = \frac{1}{\sqrt{E_1}}$  adsorption

equilibrium is achieved on the tank surface and the density of molecules (dimensional) is defined by

$$n_e = N_1 \tau_1 \quad (5-5-25)$$

Then for the open end of the capillary, with account taken of the conjugation conditions (equality of densities and flows) on the capillary-tank interface, the following boundary condition of the third kind may be written

$$\frac{dn}{dx} = L \sqrt{E_1} (n_e - n) \quad (5-5-26)$$

where

$$E_1 = \frac{1}{\tau_1(1) D_s(1)}$$

For correct formulation of the boundary condition at  $x = 0$  the values characterizing kinetics of two-dimensional evaporation should be prescribed [5-26]. Since no such data are available in the literature, the use is made of the following condition

$$n(0) = 1 \quad (5-5-27)$$

The flows  $N_0$  and  $N_1$  were calculated by the formulas

$$N_0 = (1 - \sigma) \frac{P_e}{(2\pi mkT_0)^{1/2}}, \quad N_1 = \frac{P_1}{(2\pi mkT_1)^{1/2}}$$

1. *Isothermal case.* Assuming  $\tau = \text{const}$ ,  $D_s = \text{const}$  and substituting  $K(x)$  by the exponent, from (5-5-24) we get

$$G_L \frac{d^2 n}{dx^2} = n(x) - \frac{l}{2} \int_0^1 n(\xi) [\exp\{-l|x - \xi|\} + \sigma \exp\{-l(x + \xi)\}] d\xi \quad (5-5-28)$$

$$- \frac{N_0 \tau}{2n_0} \exp\{-lx\} - \frac{N_1 \tau}{2n_0} [\exp\{-l(1 - x)\} + \sigma \exp\{-l(1 + x)\}]$$

where

$$G_L = \frac{1}{EL^2}$$

It follows from (5-5-28) that the larger the dimensionless parameter  $G_L$ , the greater the contribution of the surface diffusion into the distribution of  $n$  over the capillary surface. In particular, at  $G_L \gg 1$  the density  $n$  is found by the solution of equation

$$\frac{d^2 n}{dx^2} = 0 \quad (5-5-29)$$

with boundary conditions (5-5-26) and (5-5-27), i.e. it varies linearly

$$n = 1 + Bx, \quad B = \frac{L\sqrt{E}(n_e - 1)}{1 + L\sqrt{E}} \quad (5-5-30)$$

On the other hand, in the case of  $G_L \ll 1$  equation (5-5-28) has a minor parameter at the major derivative. Therefore, at any infinitesimal values of  $G_L$  at the capillary ends the regions (boundary layers) exist, where the differential term in equation (5-5-28) should be accounted for, i.e. if the minor parameter method is used the disturbed solution will not be equally applicable for the whole range of  $x$ . This nonuniformity is known [5-27] to be displayed when the perturbation parameter is a ratio of two lengths. In the present problem  $G_L^{1/2}$  is a ratio of the mean square displacement of a particle over the surface  $\frac{1}{\sqrt{E}}$  to the capillary length  $L$ .

To solve equation (5-5-28) it is differentiated twice with respect to  $x$ . Combination of the expression obtained with initial equation (5-5-28) yields

$$\frac{d^4 n}{dx^4} - c_1^2 \frac{d^2 n}{dx^2} = 0, \quad c_1^2 = l^2 + G_L^{-1} \quad (5-5-31)$$

Solution of the above equation is written as follows

$$n = a_1 \exp \{-c_1 x\} + a_2 \exp \{-c_1(1-x)\} + a_3 x + a_4 \quad (5-5-32)$$

By substituting (5-5-32) into (5-5-31) and equating the coefficients at  $\exp \{-lx\}$  and  $\exp \{lx\}$ , we obtain two equations for the determination of  $a_1, a_2, a_3$  and  $a_4$ . Two other equations are sought from boundary conditions (5-5-26) and (5-5-27). Hence, to calculate four unknown coefficients a system of four linear equations should be solved

$$\left. \begin{aligned} & a_1 l \left[ \frac{1}{l - c_1} + \frac{\sigma}{l + c_1} (\exp \{-(c_1 + l)\} - 1) \right] + a_2 l \left[ \left( \frac{1}{l + c_1} + \frac{\sigma}{c_1 - l} \right) \exp \{-c\} - \frac{\sigma}{c_1 - l} \exp \{-l\} \right] + \frac{a_3}{l} [-1 - \sigma + \sigma \exp \{-l\} (l + 1)] + a_4 (1 - \sigma + \sigma \exp \{-l\}) \\ & = \frac{\tau}{n_0} (N_0 + \sigma N_1 \exp \{-l\}) \\ & \frac{la_1}{l + c_1} \exp \{-c_1\} + \frac{la_2}{l - c_1} + a_3 \frac{l + 1}{l} + a_4 = \frac{N_1 \tau}{n_0} \\ & a_1 \exp \{-c_1\} (1 - c_1 G_L^{1/2}) + a_2 (1 + c_1 G_L^{1/2}) + a_3 (1 + G_L^{1/2}) + a_4 = n_e \\ & a_1 + a_2 \exp \{-c_1\} + a_4 = 1 \end{aligned} \right\} \quad (5-5-33)$$

To investigate the solution obtained, numerical calculations were performed. Of particular interest is the case of  $G_L \ll 1$  (note that for  $G_L = 0$  solution of equation (5-5-28) is a linear function). As it could be expected, for such values of  $G_L$  the existence of boundary layer at the capillary ends is clearly manifested and in the middle the curve  $n(x)$  becomes a linear function (Fig. 5-24) whose slope coincides with that of the straight

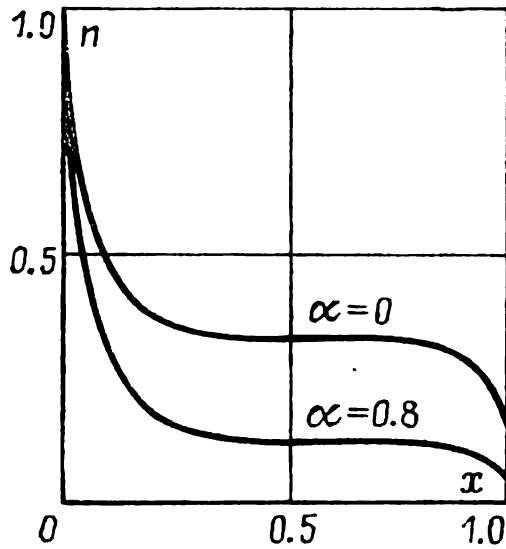


Fig. 5-24. Plot of surface density of molecules versus longitudinal coordinate

line which is a solution of equation (5-5-24) at  $G_L = 0$ . The explanation is that in this range the right-hand side of equation (5-5-28) makes a major contribution to its solution. This peculiarity of relation between  $n$  and  $x$  at  $G_L \ll 1$  is seen in the figure, where numerical results are presented for water with the following parameters:  $G_L = 4 \times 10^{-3}$ ;  $l = 0.1$ ;  $\sigma = 0$  and  $0.8$ ;  $T = 300^\circ\text{K}$ ;  $p_e = 3430 \text{ N/m}^2$ ;  $p_1 = 490 \text{ N/m}^2$ .

As to the flows in the case of  $G_L$  being of the order of unity, the curve  $n(x)$  cannot be divided into three zones.

The net flow of molecules at the capillary exit is equal to the sum of surface diffusion flow  $N_s(1) = 2\pi R J_s(1)$  and the flow of molecules  $N(1)$  emitting through the open end of the capillary. It should be kept in mind that at  $G_R \gg 1$ ,  $N_s \gg N$  and at  $G_R \ll 1$ ,  $N_s \ll N$  (since  $G_R = \frac{1}{ER^2}$  characterizes the ratio of conductivities due to surface and Knudsen diffusion).

The expression for  $N$  is of the form

$$N = \pi R^2 \{N_0 K_2(1) - N_1 [1 - \sigma K_2(2)]\} + \frac{2\pi R^2 n_0 l}{\tau} \int_0^1 n(x) [K(1-x) + \sigma K(1+x)] dx \quad (5-5-34)$$

Here the first term describes a flow of molecules which diffused from the bottom and covered the distance  $L$  without collisions. The second term represents a flow of molecules entering the capillary through the open

end. The third term stands for a flow of molecules which reached the bottom without collisions, reflected specularly from it and then emitted through the open end. The integral term characterizes the flows of molecules desorbed by the internal surface of the capillary and left it without collision against the wall and after specular reflection from the bottom.

The comparison of the approximate analytical solution of equation (5-5-24) and its numerical solution for the exact function  $K(x)$  has shown that there is but slight difference between the values of  $n(x)$  corresponding to exact and approximate expressions for  $K(x)$ ; this difference increases with decrease of  $G_L$  and increase of  $l$ . As for the approximate values of  $N$ , they differ from one another in a greater degree than those of  $n(x)$  do.

In (5-5-34), in order to decrease the calculation errors, an exact expression for the function  $K(x)$  should be used rather than the approximate one. For  $G_R \ll 1$ , however, integration of the terms with  $a_1$  and  $a_2$  yields a small contribution into the value of  $N$ . Therefore, to simplify the integration, in (5-5-34) an approximate exponential expression for  $K(x)$  may partially be used (in the summands including  $a_1$  and  $a_2$ ).

With no surface diffusion, we have for  $N$  relationship (5-5-34) wherein  $n(x)$  is the solution to integral equation (5-5-28) at  $G_L = 0$  with an exponential kernel, i.e. it is a linear function. This allows easy calculation of the integral term in (5-5-34) with exact kernel  $K(x)$ . Specifically, in the case of  $L > R$  and  $N_1 = 0$  we obtain the value accurate to  $R^2/L^2$

$$N = \frac{7}{3} \frac{R}{L} \frac{N_0}{1 - \sigma + 2 \frac{R}{L}} \left( 1 - \frac{6\sigma}{7} \frac{R}{L} \right) \quad (5-5-35)$$

This expression is a generalization of the known Knudsen formula [5-28] for the case of evaporation from a finite length capillary. For comparison, it should be pointed out that the exact Knudsen formula ( $L/R \gg 1$ ,  $\sigma = 0$ ) includes coefficient  $8/3$ , and when the exponential kernel is used in (5-5-34), coefficient 2.

2. *Nonisothermal case.* The values  $\tau$  and  $D_s$  are related with the surface temperature as follows [5-22]

$$\tau = \tau_0 \exp \left\{ \frac{U}{kT(x)} \right\}, \quad D_s = D_{s0} \exp \left\{ - \frac{\varepsilon}{kT(x)} \right\} \quad (5-5-36)$$

Assume that  $T = T_0(1 + Ax)$  and  $|A| \ll 1$ . Then linearization and substitution of the variable  $n_1(x) = n(x) \exp \left\{ - \frac{u}{kT(x)} \right\}$  reduce equation for  $n_1$  to a form similar to that in the case of an isothermal capillary except that in the left-hand side a term containing the first variable  $\frac{dn_1}{dx}$  appears.

Herein, the ratio of the coefficients at the first and second derivatives equals

$$\gamma = \frac{\Delta T}{T_0} \frac{\varepsilon - 2U}{kT_0} \quad (5-5-37)$$

where  $U$  and  $\varepsilon$  are the adsorption and activation energies of one molecule, respectively.

Since from an approximate analysis [5-22] follows that  $\frac{1}{2} \leq \frac{\varepsilon}{U} \leq \frac{2}{3}$ , and  $\frac{U}{kT_0} \gg 1$ , the order of magnitude of  $\gamma$  is equal to  $\frac{\Delta T}{T_0} \frac{U}{kT_0}$ . If  $\gamma \approx 1$ , then at  $G_L(0) \ll 1$ , i.e. in case of short boundary layers the term containing the second derivative is much greater than the summand with the first derivative. Hence, the solution of the equation for  $n_1$  can be presented in the form

$$n_1(x) \simeq \frac{n_i(x)\tau_0}{\tau_i}$$

i.e.

$$n(x) \simeq n_1(x) \exp \left\{ \frac{U}{kT(x)} \right\} = \frac{n_i(x)\tau(x)}{\tau_i} \quad (5-5-38)$$

everywhere but on the nearest vicinity to the open end (where the temperature  $T_1$  effect manifests itself due to the boundary condition (5-5-26) at the exit) (subscript  $i$  refers to the isothermal case).

The flow  $N$  at the capillary exit is expressed by the formula differing from (5-5-34) only by  $1/\tau(x)$  included in the subintegral expression. Because of relationship (5-5-38), nonisothermity has almost no effect on the integral for  $|A| \ll 1$ . The first summand in (5-5-34) depends only on  $T_0$ . Two other summands containing the flow  $N_1$  depend on  $T_1^{-1/2}$ , but since  $T_1 = T_0(1+A)$  and  $|A| \ll 1$  this dependence is not significant. At low temperature drop the flow  $N$  is, therefore, independent of the nonisothermity, while the surface densities  $n(x)$  for  $A = 0$  and  $A \neq 0$  are essentially different. Hence, in a nonisothermal case the net molecular flow  $N + N_s$  differs from the appropriate flow in an isothermal capillary (since  $N_s = -\frac{2\pi D_s(1)n_0}{l} \frac{dn}{dx}$

[1-2].

The numerical solution to equation (5-5-24) for the case of ice evaporation has shown the validity of the above considerations, particularly, of expression (5-5-38). As to the cases of  $G_L(0) \approx 1$ , i.e. the absence of boundary layers, and of  $|A| \approx 1$ , i.e. high-temperature drops, equation (5-5-24) was solved numerically to determine  $n(x)$  and  $N$ .



(c) *The Kinetics of Vapour Transfer Through a Porous Body*

We shall consider a parallel plane layer of a highly dispersed porous body bounded from below by evaporating liquid and by its vapours from above. The system is under isothermal conditions.

The transient vapour flow regime which is often encountered in engineering processes is the most complicated for theoretical description. Consider a vapour flow through a dry layer of a porous body by representing the porous body-gas system as a binary mixture. The molecules of one of the species are fixed, their sizes and mass greatly exceed the sizes and mass of the second species (real gas) [5-29]—[5-31].

Let us take account of the fact that the possibility of covering the distance  $X$  by a molecule without collision is  $\exp \{-(X/\lambda)\}$  [5-38]. In what follows it is assumed that the free molecular path length is constant. In the model under consideration  $\lambda$  is of the form [5-32, 5-33]

$$\frac{1}{\lambda} = \frac{1}{\lambda_{11}} + \frac{1}{\lambda_{12}} \quad (5-5-39)$$

where  $\lambda_{11}$  is the free path length of real molecules relative to the molecules themselves;  $\lambda_{12}$  is the free path length of real molecules relative to molecules at rest. When determining  $\lambda_{12}$  account should be taken of the proper volume of the molecules at rest and the possibility of their mutual overlapping.

Let us consider the interface between evaporating liquid and porous body. The evaporating liquid occupies the portion  $\Pi$  of the whole surface area. The part of the surface  $1 - \Pi$  is occupied by solid particles of the porous body and reflects diffusely all the molecules falling on it.

The total flow of molecules, passing through the dry layer of porous material in the direction of the axis  $X$ , equals the difference between the number of molecules flying upwards and those flying downwards. At steady state this flow does not depend on the coordinate and, with the above in view, is of the form

$$\begin{aligned} N = & \left\{ \frac{2\Pi\alpha p_e}{(2\pi mkT)^{1/2}} + 4(1 - \alpha\Pi) \left[ L \int_0^1 \frac{n'(\xi)v}{4\lambda} E_2 \left( \xi \frac{L}{\lambda} \right) d\xi \right. \right. \\ & \left. \left. + \frac{p(1)}{(2\pi mkT)^{1/2}} E_3 \left( \frac{L}{\lambda} \right) \right] \right\} \times E_3 \left( x \frac{L}{\lambda} \right) + L \int_0^x \frac{n'(\xi)v}{2\lambda} E_2 \left[ \frac{L}{\lambda} (x - \xi) \right] d\xi \\ & - L \int_0^1 \frac{n'(\xi)v}{2\lambda} E_2 \left[ \frac{L}{\lambda} (\xi - x) \right] d\xi - \frac{2p(1)}{(2\pi mkT)^{1/2}} E_3 \left[ \frac{L(1 - x)}{\lambda} \right] \quad (5-5-40) \end{aligned}$$

where  $n'$  is the volume density of molecules and  $v$  is the mean thermal velocity of molecules.

$$E_n(x) = \int_0^1 \mu^{n-2} \exp \left\{ -\frac{x}{\mu} \right\} d\mu$$

Here  $\mu$  is the cosine of the angle between the velocity vector and the normal to plane  $YZ$ .

In equation (5-5-40) the first four terms express the number of molecules passing upwards through the plane  $YZ$  at a distance  $X$  from the liquid surface. They include evaporated molecules, those scattered in the layer between the evaporating surface and the plane  $YZ$ , the molecules, which scattered in the layer of thickness  $L$ , fell onto the boundary surface and diffusely reflected from it, and the molecules from the volume over the porous body diffusely reflected from the boundary surface. Last two terms represent the molecules that have come from the gas over the porous body, and those which collided in the porous material of the thickness  $L-X$ .

In a general form equation (5-5-40) does not lend itself to solution. Therefore, the functions  $E_2$  and  $E_3$  are substituted by exponential functions similarly to [5-34].

By differentiating equation (5-5-40) twice and combining the obtained expressions with the initial one, we obtain a differential equation for the molecular density. From the equation derived and (5-5-40) we obtain the expression for a flow of molecules.

$$N_1 = \frac{4\lambda\Pi\alpha}{(2\pi mkT)^{1/2}} \frac{p_e - p(1)}{4\lambda + 3L\alpha\Pi} \quad (5-5-41)$$

At  $\frac{\lambda}{L} \rightarrow \infty$  from (5-5-41) we find an expression for evaporation from a free surface

$$N_0 = \alpha\Pi \frac{p_e - p}{(2\pi mkT)^{1/2}} \quad (5-5-42)$$

At  $\frac{\lambda}{L} \rightarrow 0$  we get an ordinary expression for a diffusion flow [if in the given diffusional approximation it is assumed that  $n_e \approx n(0)$ ]:

$$N_1 = -\frac{1}{3} \omega \lambda \frac{n'(1) - n'(0)}{L} \quad (5-5-43)$$

where  $n'$  is the volumetric molecular density,  $\omega$  is the thermal molecular velocity.

In a transient case, a flow of particles depends on the coordinate and time. Consider the case of  $\alpha\Pi = 1$ . Under the zeroth initial conditions with the use of the Laplace transformation and the assumptions made for the solution of equation (5-5-40), it is easy to show that the equation for the flow of molecules in dimensionless coordinates ( $t_1 = \frac{t}{t^*}$ ,  $t^*$  is the characteristic time of the problem) is of the form

$$-\frac{4}{9} \left( \frac{\lambda}{L} \right)^2 \frac{\partial^2 N}{\partial x^2} + \left( \frac{\lambda}{ct^*} \right)^2 \frac{\partial^2 N}{\partial t_1^2} + N + 2 \frac{\lambda}{ct^*} \frac{\partial N}{\partial t_1} = -\frac{1}{3} \omega \frac{\lambda}{L} \frac{\partial n'}{\partial x} \quad (5-5-44)$$

Since  $\frac{\lambda}{L}$  and  $\frac{\lambda}{ct^*}$  are usually small, we retain in the equation the terms of the first order of smallness. Then equation (5-5-44) takes the form

$$N = -2 \frac{\lambda}{ct^*} \frac{\partial N}{\partial t^*} - \frac{1}{3} \omega \frac{\lambda}{L} \frac{\partial n'}{\partial x^*} \quad (5-5-45)$$

where the first term on the right-hand side accounts for the finite velocity of mass propagation  $c$ .

## 5-6. HEAT CONDUCTION IN CAPILLARY-POROUS AND DISPERSE MATERIALS

Heat conduction in capillary-porous and disperse materials may be investigated in two ways. The first one resides in the experimental investigation of thermophysical properties with the use of known methods based on the solution of heat-conduction problems for various geometries.

Although theoretical prerequisites in most cases are observed with certain approximations, this method makes it possible to determine effective values of thermophysical characteristics with sufficient accuracy.

Reliable measurements of thermal conductivity of capillary-porous and disperse materials are required not only for obtaining qualitative data, but also for better understanding of basic phenomena of heat transfer in such systems, since a well set up experiment makes it possible to study the dependence of thermal properties of materials on temperature, pressures and the type of gas present in pores, porosity, grain sizes of the solid phase and other factors.

A purely empirical approach to the problem of heat transfer in capillary-porous and disperse systems is inadequate, as for the study of other phenomena. The emphasis given to purely quantitative values, numerous empirical and semi-empirical relations obtained without theoretical analysis and without any consideration of mutual effects of various factors does not lead to proper understanding of fundamental heat transfer processes taking place in heterogeneous systems.

Therefore, the second approach finds wide application. It is based on the consideration of real structure of a capillary-porous or disperse material, the simulation of such a system and subsequent calculation of its effective thermal conductivity for real working conditions of the material.

Despite great diversity of real heterogeneous porous systems as regards their chemical composition, porosity, pore and particle sizes, their orientation with respect to the heat flux, and the complexity of theoretical analysis and mathematical description of thermal processes which take place in such materials; at present analytical correlations are available enabling one to calculate with certain accuracy the effective thermal conductivity of heterogeneous porous systems. It should however be pointed out that no reliable relationships are available as yet, which may be universal for the calculation of effective thermal conductivity of particular classes of capillary-porous and disperse materials. Further studies in this field might lead to such relationships that would make it possible to cut down the need of performing complicated, time and money consuming experiments.

The generalized conductivity principle [5-35], [5-36] is a widely used and effective method for theoretical study of heat conduction in capillary-porous and disperse materials. This principle is based on the analogy between the differential equations of steady heat flows, electric current, electric and magnetic inductions and mass flows. Such an analogy allows the fundamental relations of electrostatics and electrodynamics to be used for calculating the thermal conductivity of a system.

It should be pointed out that use of generalized conductivity principle proves to be the most effective principle, if it involves consideration of material structure, estimation of system porosity, particle and pore sizes, particle contact methods, evaluation of the effect of conductive, convective and radiative components on the effective thermal conductivity of the system.

In disperse materials heat may be transferred by:

- (1) heat conduction in the solid particles;
- (2) heat conduction in the gas that fills material pores;
- (3) heat conduction in gaseous microgaps between particles;
- (4) heat conduction from one particle to another at the points of contact (contact heat conduction);
- (5) convection of the gas in which lies the porous material;
- (6) particle to particle radiation.

Even the simple enumeration of all the modes of heat transfer in disperse systems clearly demonstrates that effective thermal conductivity of such a system is a very complicated function of temperature, gas pressure,

chemical composition of the material and gas, porosity, shapes and sizes of pores and particles, degree of blackness, and the temperatures of boundary surfaces, accommodation coefficients of the particle surface with respect to the filler gas and various other factors.

Compiling, analysis and solution of equations, which take into account all the modes of heat transfer and all the above listed factors, meet with serious difficulties. Therefore, all the relations obtained so far have been derived on the basis of certain general and particular assumptions, on the basis of simplified physical picture of the considered process and idealized concept about the structures of materials.

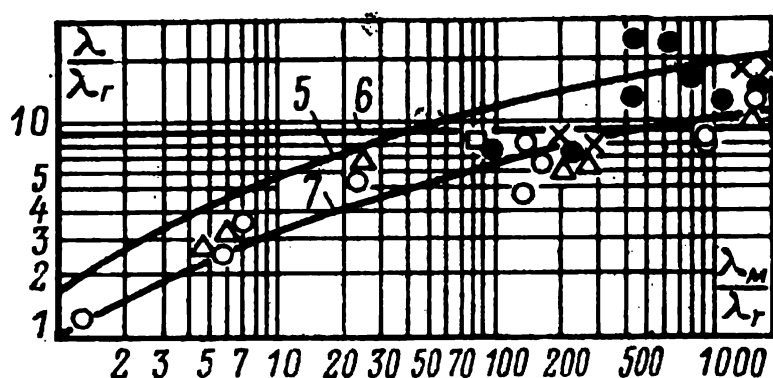
One of the most common assumptions used in the analysis of processes of heat transfer in disperse materials at definite conditions is the concept about additivity of different mechanisms of heat transfer in such materials and because of this about the possibility of neglecting one or the other mode of heat transfer under definite conditions. An analysis of various theoretical studies, on one hand, and of vast experimental data, on the other hand, enables one to explain in general the following purely qualitative relations of heat transfer in capillary-porous and disperse materials:

1. Thermal conductivity of solid particles has no dominating effect on the thermal conductivity of porous materials, its contribution, especially in nonmetallic materials, is minor [5-37], [5-38].

2. Effective thermal conductivity of capillary-porous and disperse materials strongly depends on the thermal conductivity of the filler gas (Fig. 5-25).

Fig. 5-25. Plot of granular bed heat-conduction coefficients versus heat-conduction coefficients of the grain material and filler gas

○ — after data of [5-80]; × — after data of [5-81]; △ — after data of [5-82]; ● — after data of [5-76]; 5 — according to formula given in [5-50]; 6 — according to formula given in [5-52]; 7 — according to formula given in [5-67]



3. Thermal conductivity of gaseous microgaps between solid particles produces considerable effect on the effective thermal conductivity of materials.

4. Contact heat conduction between particles plays a significant role and dominates at moderate temperatures and low pressures of gas; at normal pressures it may be neglected.

5. Effective thermal conductivity of capillary-porous and disperse materials starts depending on the filler gas pressures at definite Knudsen numbers (Figs. 5-26 through 5-29).

6. At deep vacuum the effective thermal conductivity of disperse materials at high temperature is almost fully caused by radiative heat transfer; at high temperatures the radiant component of thermal conductivity dominates at all pressures of the filler gas (Figs. 5-30 through 5-32).

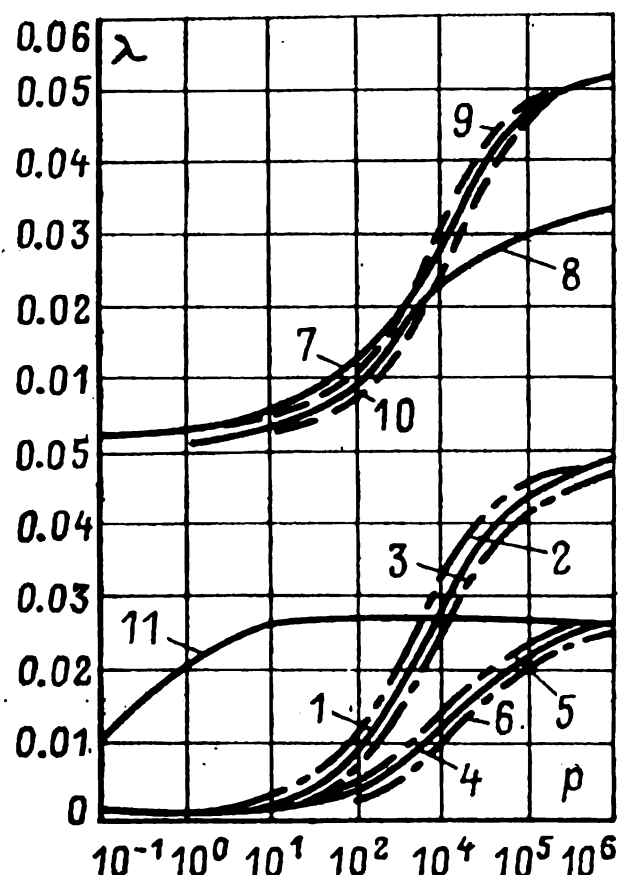


Fig. 5-26. Plot of thermal conductivity of disperse materials versus pressure [5-77]

1 — magnesia (experiment),  $T = 185\text{K}$ ,  $d = 35\ \mu\text{m}$ ; 2, 3 — the same, calculated by formula given in [5-57];  $a = 1$ , and  $a = 0.7$ ,  $\gamma = 3.04 \times 10^4\ \text{N/m}^3$ ,  $\Pi = 0.95$ ; 4 — silica gel, experiment,  $d = 1.5\ \mu\text{m}$ ,  $\gamma = 10^8\ \text{N/m}^3$ ; 5, 6 — the same, prediction,  $a = 1$  and  $a = 0.7$ ,  $\gamma = 2.4 \times 10^4\ \text{N/m}^3$ ,  $\Pi = 0.95$ ; 7, 8 — perlite, experiment,  $d = 0.015\ \text{mm}$ ,  $\gamma = 3.7 \times 10^8\ \text{N/m}^3$  and  $d = 0.3\ \text{mm}$ ,  $\gamma = 1.7 \times 10^8\ \text{N/m}^3$ ; 9, 10 — the same, calculated,  $d = 0.015$ ,  $1.0\ \text{mm}$  and  $a = 0.7$ ,  $\gamma = 3.7 \times 10^4\ \text{N/m}^3$ ,  $\Pi = 0.84$ ; 11 — air, experiment,  $T = 330\ \text{K}$  [5-79]

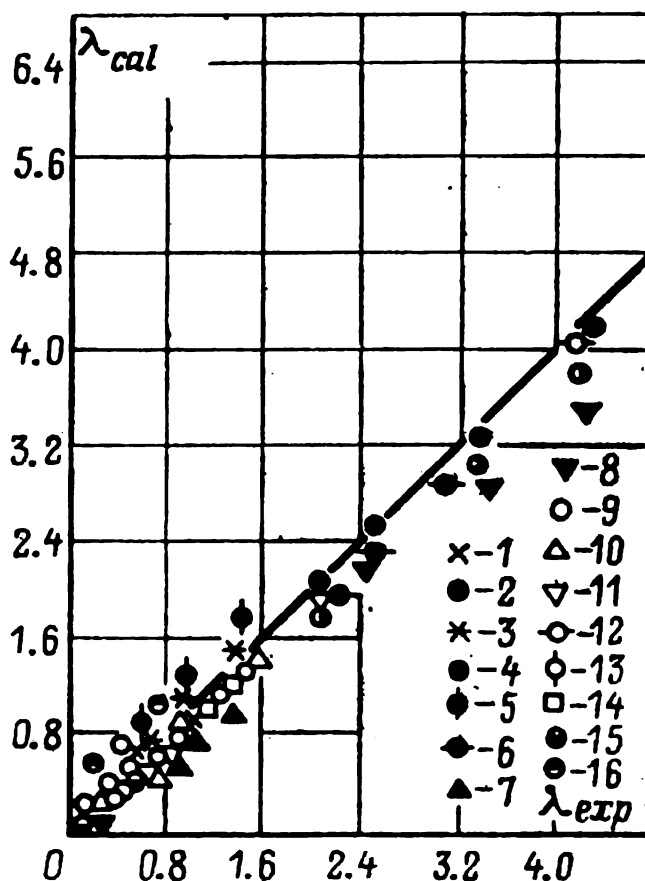


Fig. 5-27. Correlation of calculated and experimental values of effective heat-conduction coefficients of packings [5-55]

In helium: 1, 2 — slag and cast iron pellets of 2.0 and 3.0 mm fractions, respectively [5-55]; 3 through 8 — slag and cast iron pellets of 2.0-3.0 mm fractions (according to formulas of [5-67], [5-52], [5-40] with consideration for  $\lambda_p$ ); in carbon dioxide: 9 through 16 — slag and cast iron pellets of 2.0-3.0 mm fractions (according to formulas of [5-55], [5-67], [5-74] with consideration for  $\lambda_p$ ); 13 — slag pellets of 2.0-3.0 mm fractions (formulas of [5-40] with due regard for  $\lambda_p$ ); 14 — cast-iron shot of 2.0-3.0 mm fractions (formulas of [5-59] with  $\lambda_p$  included)

7. Convective transfer of heat may be neglected in the great majority of disperse and capillary-porous materials, because the very small sizes of pores and microgaps at particle junctions prevent the formation of convective currents due to temperature gradient. It is known that convective heat transfer in pores may be neglected at  $\text{GrPr} < 10^8$  [5-10]. This holds

for particle diameters not exceeding 4-6 mm. Most of the real disperse systems involve particles of smaller diameters.

8. At high temperatures the particle diameter has appreciable effect on the effective thermal conductivity of a disperse system, as it defines "the number of screens". Under similar conditions systems with larger diameter particles have large thermal conductivity, because the "number of screens" in this case is less (Figs. 5-33 to 5-35).

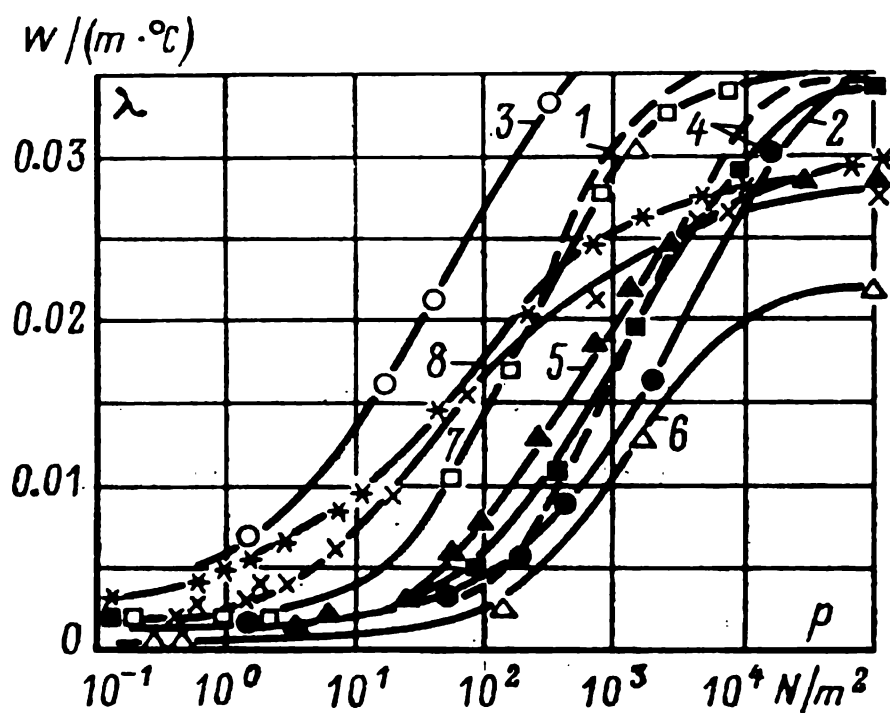


Fig. 5-28. Plot of heat-conduction coefficients of fibrous materials versus air pressure [5-67]

Curves	$\gamma$ , kg/m <sup>3</sup>	$T$ , K	$d$ , $\mu$ m
1	74	338	2.58
2	174	338	0.69
3	155	338	15.2
4	250	338	3.6
5	64	297	1.15
6	240	192	1.15
7	160	192	8.5
8	15	192	10

Tremendous number of relations derived for the determination of effective thermal conductivity of disperse systems may conventionally be divided into three groups: relations applicable for the estimation of effective thermal conductivities of disperse materials at pressures close to atmospheric, at low and moderate temperatures; relations which account for the dependence of thermal conductivity of a gas on pressure and, therefore, these may be used for calculating the effective thermal conductivity of

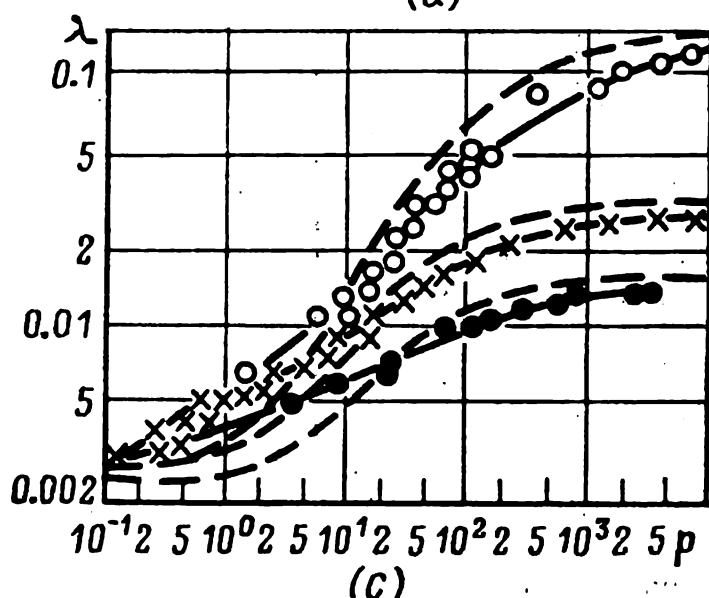
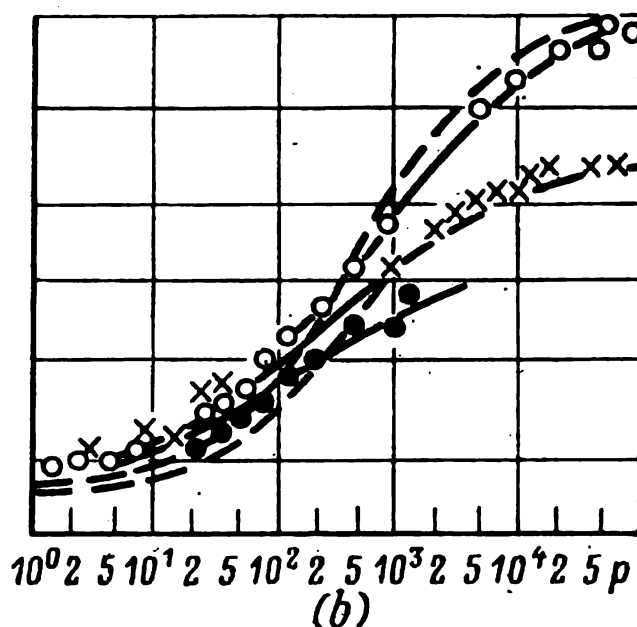
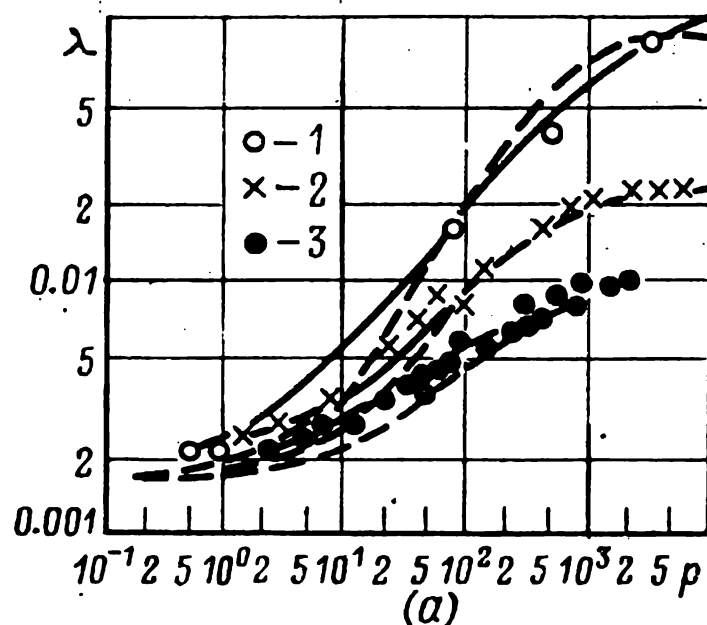
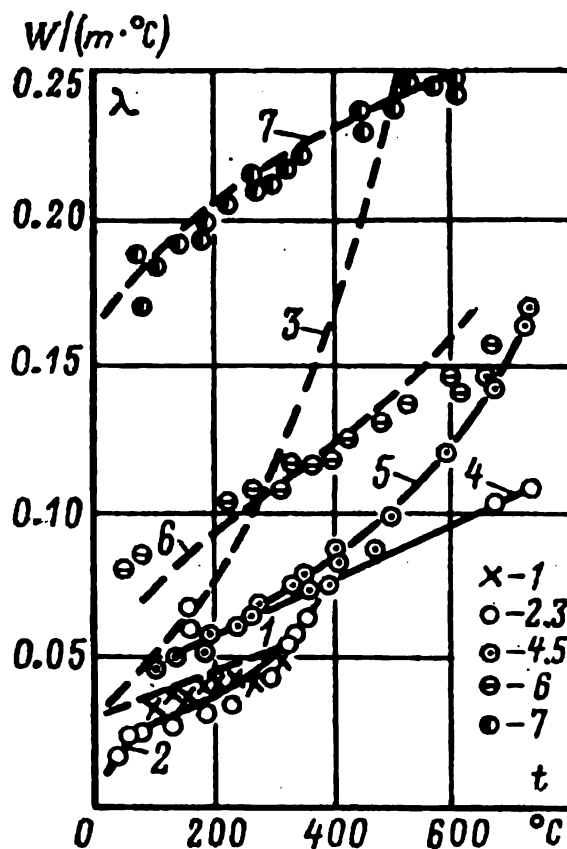


Fig. 5-29. Apparent heat-conduction coefficient of mipore (a), silica gel (b) and mineral wool (c) ( $\text{W/m} \cdot ^\circ\text{C}$ ) according to [5-83], at the boundary wall temperatures 290 and 90 K versus filler gas pressure,  $\text{N/m}^2$  (solid curves show experimental data, dashed lines correspond to values calculated by the formula of [5-83])

1 — helium; 2 — air; 3 — krypton

Fig. 5-30. Heat-conduction coefficient of powders at atmospheric pressure [5-64]

1 — lamp soot,  $\gamma = 210 \text{ kg/m}^3$ ,  $d = 0.1 \mu\text{m}$ ; 2 — aerogel,  $\gamma = 93 \text{ kg/m}^3$ ,  $d = 0.025 \mu\text{m}$ ; 3 — aerogel,  $\gamma = 50 \text{ kg/m}^3$ ,  $d = 0.025 \mu\text{m}$ ; 4 — perlite from Aragats,  $\gamma = 118 \text{ kg/m}^3$ ; 5 — perlite from Aragats,  $\gamma = 70 \text{ kg/m}^3$ ; 6 — magnesia,  $\gamma = 400 \text{ kg/m}^3$ ,  $d = 30 \mu\text{m}$ ; 7 — quartz sand,  $\gamma = 1770 \text{ kg/m}^3$ ,  $d = 120 \mu\text{m}$ ;  $\lambda_M \approx 0.3 \text{ W/(m} \cdot ^\circ\text{C)}$ ; dashed line corresponds to the results calculated by formula of [5-64]





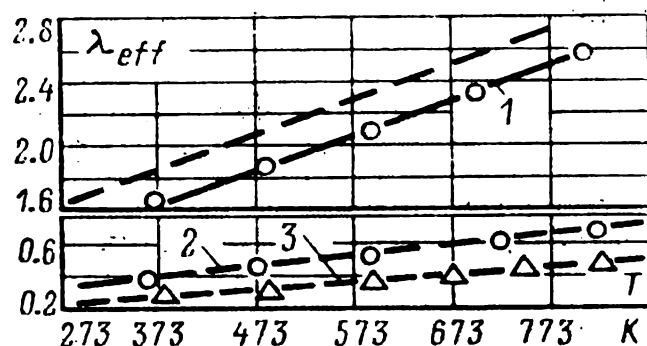


Fig. 5-31. Plot of  $\lambda_{eff}$  versus temperature [5-55]

1, 2, 3 — MgO ( $d = 0.205$  mm;  $H = 0.42$ ) in helium, air and argon, respectively [5-76]; dashed line, calculation by the formula of [5-55]

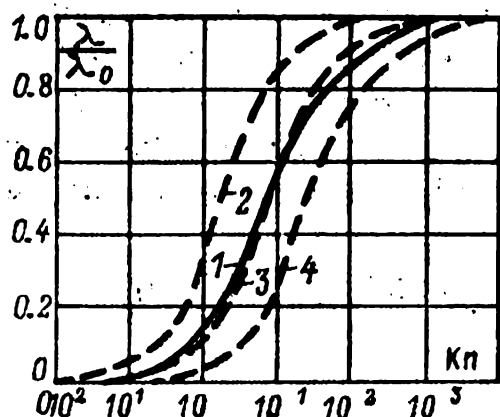


Fig. 5-33. Plot of dimensionless heat-conduction coefficient of helium versus Knudsen number and accommodation coefficient [5-77]

1 — experimental data [5-78] at  $t = 40^\circ\text{C}$ ,  $\delta = 37.5$  mm,  $\Delta t = 13-60^\circ\text{C}$ , surfaces are chrome-plated; 2, 3, 4 — calculation by the formula of [5-77],  $a_1 = a_2 = 1$ ; 0.4 and 0.1

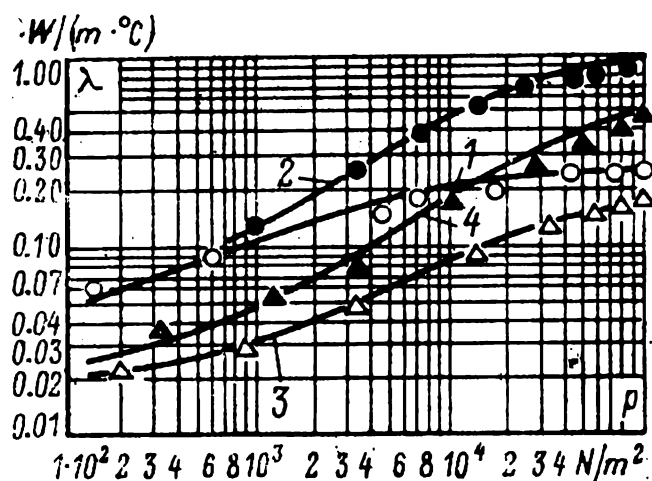


Fig. 5-32. Plot of heat-conduction coefficients of powder carborundum versus gas pressure [5-67]

1 —  $d = 0.194$  mm (air as filler gas); 2 —  $d = 0.194$  mm (helium); 3 —  $d = 0.27$  mm (air); 4 —  $d = 0.027$  mm (air); dots indicate experimental data, lines correspond to values calculated by the formula of [5-67]

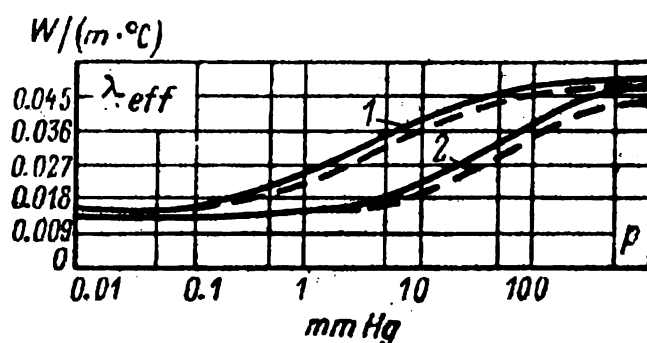
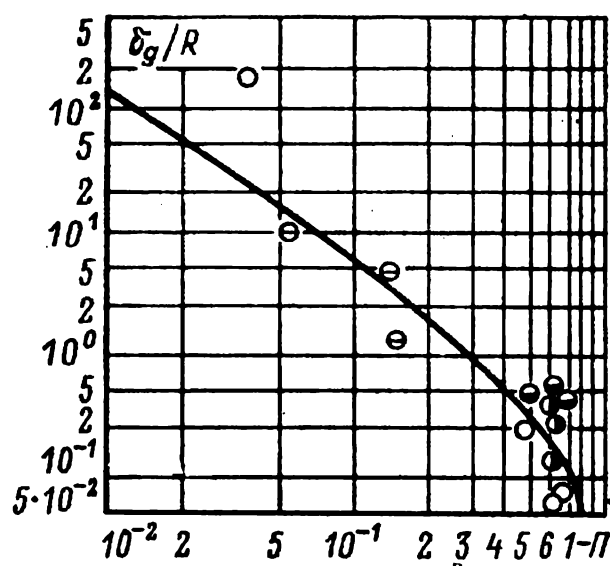


Fig. 5-34. Plot of  $\lambda_{eff}$  versus the filler gas pressure [5-55]

1, 2 — glass beads of diameters 470 and 29  $\mu\text{m}$ , respectively [5-75]; dashed lines — calculation by the formula of [5-55]

Fig. 5-35. Ratio of characteristic gas gap to a particle radius [5-64]

○ — processed experimental data [5-81]; ⊖ — the same [5-67], [5-80], [5-83]; ⊙ — the same [5-82]; ⊗ — the same [5-69]; ● — the same [5-82]; solid curve — values calculated by the formula of [5-65]



materials in a rarefied medium; relations accounting for radiant heat transfer and used to calculate effective thermal conductivity at high temperatures.

As a simple disperse system, a system consisting of alternately arranged flat layers of solid material and gas is considered. The layers may be both normal and parallel to the heat flux [5-39]. In the former case the effective thermal conductivity of the system will be minimum and is defined by

$$\lambda_{eff}^{\min} = \frac{\lambda_d \lambda_c}{\Pi \lambda_d + (1 - \Pi) \lambda_c} \quad (5-6-1)$$

In the latter case the effective thermal conductivity of the system will be maximum:

$$\lambda_{eff}^{\max} = \Pi \lambda_c + (1 - \Pi) \lambda_d$$

where  $\lambda_d$  and  $\lambda_c$  are thermal conductivities of disperse and continuous phases, respectively;  $\Pi$  is the porosity. These relations on account of their exclusive use in describing limiting cases and of ignoring the nature of particle contacts and material structure can serve only to denote that range in which lie the values of effective thermal conductivity of real disperse systems.

In the USSR the V. Z. Bogomolov formula is widely used [5-40]

$$\lambda_{eff} = 3\pi \lambda_d \log_e \frac{0.43 + 0.31\Pi}{\Pi - 0.26} \quad (5-6-2)$$

This formula has been derived for packed hexagonal arrangements of particles with porosity equal to or close to 26% and does not take account of thermal conductivity of particles. Besides this, as shown in [5-41] the numerical factor appearing in the relation is overestimated by a factor of two. But the formula in the above given form gives better agreement with experimental data.

Maxwell was the first to indicate one of the first trends of the use of generalized conductivity principle [5-35]. He calculated the electric field of the system composed of isotropic material infused with foreign particles of the spherical form. The relation obtained by Maxwell

$$\lambda_{eff} = \lambda_c \left[ \frac{\lambda_d + 2\lambda_c - 2(1 - \Pi)(\lambda_c - \lambda_d)}{\lambda_d + 2\lambda_c + (1 - \Pi)(\lambda_c - \lambda_d)} \right] \quad (5-6-3)$$

is applicable for high porosity systems, as in the derivation large spacings between particles compared to their sizes were assumed.

Later Burger [5-42] and Frick [5-43] extended Maxwell's theory to ellipsoidal particles, and Eucken [5-44] considered materials containing different shape particles.

Bruggeman [5-45] derived the relation applicable for calculation of conductivity of mixtures consisting of plane particles, ellipsoids and spheres. For the latter shape this relation takes the form:

$$1 - U_d = \frac{\lambda_d - \lambda_{eff}}{\lambda_d - \lambda_c} \left( \frac{\lambda_c}{\lambda_{eff}} \right)^{1/3} \quad (5-6-4)$$

By appraising the Maxwell-Eucken-Burger theory and by comparing the values, obtained on the basis of this theory, with numerous experimental data, de Vries [5-46] concluded that the theory gives good results for particle conductivity less than that of the medium. If the medium conductivity is more than that of particles, then the larger the ratio of conductivities the better the agreement between experimental data and the Bruggeman theory. Equations of the Maxwellian type show good agreement with experimental data for cellular materials and emulsions.

Many investigators neglect real structures of disperse materials, primarily, such important structural and mechanical properties as the type of structure as a whole, sizes and shapes of particles and pores, and the nature of particle contacts. It is natural that structural and mechanical properties define the picture of heat-transfer in disperse materials. Based on the analysis of structure of real structures of dispersed materials a number of calculation formulas have been obtained in [5-46]-[5-51] (Fig. 5-36).

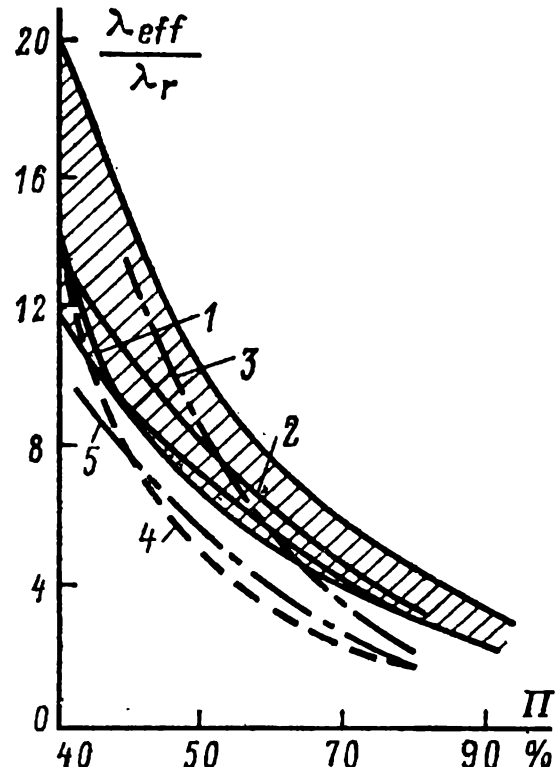


Fig. 5-36. Treatment of calculated relations reported by different authors as a function of  $\lambda_{eff}/\lambda_r = f(\Pi)$  for the case when  $\lambda_{TB}/\lambda_r = 100$

1 — reproduced from [5-52], [5-54]; 2 — reproduced from [5-57–5-59]; 3 — reproduced from [5-40]; 4 — reproduced from [5-84]; 5 — reproduced from [5-67]

It should however be mentioned that many calculation dependences show noticeable disagreement with experimental data, that may be attributed to many reasons including the following:

(1) the choice of the method for the calculation of effective thermal conductivities should strictly depend on mutual location and the number of constituent phases;

(2) no theory is capable of taking into account the effect of real distribution of particles and pores according to their sizes, their shapes and contacts between neighbouring particles on the contact conductivity;

(3) none of the theories available at present can correctly estimate temperature drop at the solid-gas interface;

(4) the theories either neglect the radiant component of thermal conductivity, or consider it inadequately.

In this connection it should be mentioned that in recent years the trend is to more detailed analysis of all mechanisms of heat transfer in materials under consideration, to derivation of predicted relations, which together with thermal conductivities of particles and gas would include contact and radiation thermal conductivities, and also the dependence of gas thermal conductivity upon pressure.

So, M. G. Kaganer [5-67] after considering the thermal resistance of a sphere with two contact spots, obtained for the determination of contact thermal conductivity of granular materials a formula in the form:

$$\lambda_k = 3.12(1 - \Pi)^{4/3} P^{1/3} / E \quad (5-6-5)$$

where  $E$  is the Young modulus.

In [5-52] the contact thermal conductivity of disperse systems is assumed to be the sum of contact thermal conductivity  $\lambda_l$  of loose material which depends on the geometry of particles, their mutual orientation, weight and porosity, and  $\lambda(P)$ , which is a function of the external mechanical load  $P$ :

$$\lambda_k = \lambda_l + \lambda(P) \quad (5-6-6)$$

$$\text{where } \lambda(P) = \frac{\lambda_M}{75} \frac{1}{Q} P^{2/3} K_c \text{ at } P < 3 \times 10^5 \text{ N/m}^2 \text{ and } \lambda(P) = \frac{\lambda_M}{E^{4/9}} \frac{1}{Q} P^{4/9} K_b$$

at  $P > 3 \times 10^5 \text{ N/m}^2$ ;  $Q = \sqrt{\frac{75}{100 - \Pi}}$ ,  $K_c$  and  $K_b$  are empirical coefficients. On varying the mechanical load from  $10^5$  to  $3 \times 10^5 \text{ N/m}^2$   $K_c$  varies between 0.35 and 0.45, and on varying the load from  $4 \times 10^5$  to  $12 \times 10^5 \text{ N/m}^2$   $K_b$  changes almost linearly from 2.2 to 2.9.

In [5-53] the values  $\lambda_l$  are presented for mineral particles  $\lambda_M < (1-2) \text{ W/(m} \cdot ^\circ\text{C)}$ ,  $\lambda_l = (2 \text{ to } 15) \times 10^{-3} \text{ W/(m} \cdot ^\circ\text{C)}$ , for metal particles  $\lambda_l = (3-10) \times 10^{-2} \text{ W/(m} \cdot ^\circ\text{C)}$  are confirmed by empirical values presented in [5-54] and [5-55]: for cast-iron shot  $\lambda_k = 0.06-0.07 \text{ W/(m} \cdot ^\circ\text{C)}$ ; for slag pellets  $\lambda_k = 0.015 \text{ W/(m} \cdot ^\circ\text{C)}$ .

The contact thermal conductivity may also be found from Rieman-Weber formula derived for cubic arrangement of spherical particles with diameter  $L$  and contact spot radius  $r_n$ :

$$\lambda_k = \lambda_M \left( \frac{L}{2r_n} + \frac{1}{\pi} \log_e \frac{L}{r_n} \right)^{-1} \quad (5-6-7)$$

The contact spot radius for two spherical particles may be found from Hertz formula [5-56].

L. L. Vasiliev [5-57] expressed contact resistance of granular systems as the sum of three components

$$R_k = R_L + R_{mr} + R_0 \quad (5-6-8)$$

where  $R_L$  is the thermal resistance caused by the decrease in effective cross-section for heat flux through two spherical particles in contact;  $R_{mr}$  is the thermal resistance of microroughnesses at the point of contact of two particles;  $R_0$  is the resistance of oxide film.

Contact thermal conductivity is defined by

$$\lambda_k = \frac{1}{R_k L} \quad (5-6-9)$$

A comparison of predicted and experimental values, which was carried out in [5-58] and [5-59], has demonstrated that Vasiliev's method is in good agreement with experiment.

The problem relating to the role of radiation in heat transfer processes in capillary-porous and disperse systems is a very complicated and poorly studied problem.

The difficulties in the calculation of heat transfer by radiation in disperse and capillary-porous materials arise because of the heterogeneous structure of these materials. Radiation in disperse systems depends on absorption, emission and scattering coefficients of particles, their shapes, sizes and mutual orientation. For calculations it is necessary to make certain assumptions.

The method, in which radiant heat transfer between two adjacent particles is identified with heat transfer between two infinite parallel plates, is widely used. It is evident that such a method gives very approximate results.

In most works, dedicated to the study of heat transfer by radiation in disperse and capillary-porous materials, a review of which is given in [5-39], the expression for the radiation component of the effective thermal conductivity is given in the form:

$$\lambda_p = 4f\sigma T^3 d \quad (5-6-10)$$

where  $d$  is the particle dimension.

Values of the parameter  $f$  are given in Table 5-8. V. S. Nikitin [5-55] obtained the following formula

$$f = 0.865 \frac{3\Pi\varepsilon_n + (1 - \Pi)\varepsilon_M}{1 + (1 - \Pi)(1 - \varepsilon_M)} \quad (5-6-11)$$

Calculations by formula (5-6-10) with the use of above coefficients are in some cases in good agreement with experimental data. But at present it is not possible to recommend any of the available formulas as the most suitable one.

Table 5-8

Parameter  $f$  after Different Authors

$f$	Authors
$\frac{1}{3}$	Bosworth [5-62]
$\varepsilon_M/(2 - \varepsilon_M)$	Argo and Smith [5-60]
$\varepsilon_M$	Schotte [5-61]
$2/(b + 2c) d$	Chen and Churchill [5-63]
$H\varepsilon_M^2/2d$	Chudnovsky [5-36]
$\left(1 + \frac{5 \times 10^{-5}}{r}\right) \frac{0.16}{(1 - \Pi)^{1.15}}$	Shorin, Zarudnyi, Serebryanyi [5-64]
$\varepsilon_M \Gamma$	Loeb [5-48]

Note.  $H$  is the pore dimension;  $\Gamma$  is the geometric factor;  $\Gamma = 2/3$  to 1.

Gas pressure in pores exerts appreciable effect on the effective thermal conductivity. From the kinetic theory of gases it is known that thermal conductivity of gases is independent of pressure under normal conditions, but this dependence becomes appreciable as pressure decreases when the molecular free path of the gas is of the same order as the distance  $\delta$  between heat exchanging surfaces or larger than this distance, i.e. when the Knudsen number ( $Kn = \lambda/\delta$ ) is close to unity or more.

Different methods are available to evaluate the Knudsen numbers at which thermal conductivity of gases starts depending on pressure.

For example, R. S. Prasolov [5-65] and several other authors consider three regions: the continuum region ( $Kn \ll 1$ ), the free-molecular region ( $Kn \gg 1$ ) where the thermal conductivity of a gas is independent of pressure and the so-called transient region ( $10^{-2} < Kn < 10^{+2}$ ), where the thermal conductivity of a gas is essentially dependent on pressure.

In capillary-porous and disperse materials the limit free molecular paths, at which effective thermal conductivity starts depending on pressure, are determined by pore sizes. In view of this, a remarkable peculiarity of fine-disperse porous materials with small pores is that for such materials Knudsen numbers are quite large even at atmospheric pressure. This implies

that at atmospheric pressure in such systems the effects take place which are peculiar of systems under vacuum. This is of great practical significance.

Mean pore dimension may be found from one of the formulas proposed in [5-53] and the mean free molecular path is related with pressure  $p$  and temperature  $T$  of gas by the expression [5-66]

$$\lambda = \frac{c^*}{p} \frac{1}{1 + \frac{T_y}{T}} \quad (5-6-12)$$

where  $c^*$  and  $T_y$  are constants depending on gas species. For air  $c^* = 8.42 \times 10^{-3}$  N/m and  $T_y = 113$  K.

A number of authors [5-67, 5-68] revealed that the effective thermal conductivity of disperse systems starts depending on pressure of the filler gas not at Knudsen numbers close to unity but at numbers much less than unity. This may be attributed to the fact that the choice of mean pore size as a characteristic dimension in the calculations of Kn is physically incorrect because a large portion of heat in disperse and capillary porous materials is transferred through the gas layer near the contact spot where the distance between the heat exchanging surfaces is considerably less than the mean pore size.

In literature a great number of theoretical dependences between thermal conductivity of a gas and pressure are available. The best known among them are the Smolukhovsky formulas [5-69] based on the assumption that there exists a temperature drop at the particle surface, and also on the Maxwell and Knudsen formulas. Recently, the generalized heat conduction equation for a gas, proposed in [5-65] has found wide acceptance

$$\lambda_g = \frac{\lambda'_g}{1 + \frac{2K}{K+1} \left( \frac{2-a_1}{a_1} + \frac{2-a_2}{a_2} \right) \frac{\text{Kn}}{\text{Pr}}} \quad (5-6-13)$$

where  $\lambda'_g$  is the thermal conductivity of the gas at normal pressure;  $K = c_p/c_v$ ;  $a_1$  and  $a_2$  are accommodation coefficients; Kn and Pr are the Knudsen and Prandtl numbers, respectively. Examples illustrating the use of this equation are given in [5-59], [5-65].

It should be pointed out that the available data on accommodation coefficients are scanty. This restricts the use of the above relation. In view of this the procedure described in [5-76] deserves attention. In this report the effect of incomplete energy exchange is accounted for by the coefficient determining the effective specific surface area of the material.

Despite numerous attempts to obtain a design formula for the determination of effective thermal conductivity of disperse and capillary-porous materials over a wide range of pressure and temperature variations of the filler gas, this problem has not been solved completely as yet. The processes of molecular heat transfer in disperse systems have been theoretically studied in greater detail both at normal and low pressures. Contact heat transfer may be estimated only tentatively and in some gases on the basis of experimental data. The processes of radiative heat transfer in disperse systems have been studied inadequately. In most papers no account has been taken of such important factors as the dependence of the degree of blackness of disperse material layer on material properties, granulometric composition, temperature, etc. Although general physical considerations lead in all calculations to correct relations between  $\lambda_p$ , on one hand, and the particle diameter and the temperature, on the other hand, the numerical factors still prove to be different, this is a drawback in the calculations.

Nevertheless, numerous reported data and comparisons of experimental and theoretical values show that it is quite realistic to solve the problem of calculating thermophysical properties of disperse materials and capillary-porous systems. For this it is necessary to have reliable data on porosity, thermal conductivity of the solid phase of material, particle size, temperature, pressure and other basic properties of the system.

Along with the above cited relations there exist a number of theoretical relations. The calculations made by these relations showed satisfactory and in many cases good agreement with experimental data over wide ranges of temperature, pressure, porosity and chemical composition.

Good results in the calculations of effective thermophysical properties of nonmetallic and metallic system are obtained by using the relation suggested by L. L. Vasiliev [5-57]—[5-59].

$$\frac{\lambda_{eff}}{\lambda_M} = \frac{1}{\frac{1}{(h/L)^2} + A} + \frac{\lambda_g}{\lambda_M} (1 - h/L) + \frac{2}{1 + h/l + \frac{1}{\lambda_g h / \lambda_M L}} \quad (5-6-14)$$

where  $h/l$  and  $h/L$  are the functions of system porosity;

$$A = \frac{1}{\frac{\lambda_k}{\lambda_M} + \frac{\lambda_g}{4\lambda_M K_k K_m} \left(\frac{h}{L}\right)^2 \times 10^3} \quad (5-6-15)$$

$K_k$  and  $K_m$  are empirical coefficients.



The relation was obtained on the basis of the generalized conductivity principle by isolating from the considered disperse system an elementary cell, which possesses all the properties of the system, by considering on the strength of the method of electrothermal analogy the thermal resistances of the cell and by calculating its thermal conductivity. With due regard to the correction for the dependence between the thermal conductivity of gas and pressure and temperature, this relation was widely used for calculation of effective thermal conductivity of nonmetallic and metallic granular and powdered materials. In many cases the results agreed satisfactory with experimental data.

The relation suggested in [5-67] is of the form

$$\lambda_{eff} = \lambda_g \left[ \frac{5.8(1 - \Pi)^2}{K} \left( \frac{1}{K} \log_e \frac{\lambda_M}{\lambda_g} - 1 - \frac{K}{2} \right) + 1 \right] \quad (5-6-16)$$

where  $K = 1 - \lambda_g/\lambda_M$  gives fairly good agreement with experimental data for powdered materials.

The relation presented in [5-71] shows satisfactory agreement between calculated and experimental data for granular materials having rather high thermal conductivity of solid phase as the authors when deriving this formula neglected thermal resistance of particles.

In [5-55] a semi-theoretical relation has been obtained for effective thermal conductivity of disperse systems in the form

$$\begin{aligned} \lambda_{eff} = \lambda_g \left[ 1 + 3.91(1 - \Pi) \lambda_g^{0.1} \log_e \frac{\lambda_M}{\lambda_g} \right] & \left[ \frac{1}{1 + \frac{7\gamma}{\gamma_T - \gamma} \left( \frac{\Lambda}{d} \right)^{0.55}} \right] \\ & + \frac{3.46\sigma t^3 d[3\Pi\epsilon_n + (1 - \Pi)\epsilon_M]}{1 + (1 - \Pi)(1 - \epsilon_M)} + \lambda_k \end{aligned} \quad (5-6-17)$$

where  $\gamma$  and  $\gamma_T$  are volume weight of the disperse material and specific weight of the solid phase material, respectively;  $\epsilon_M$  and  $\epsilon_n$  are emissivities of the material and pores.

This relation gave satisfactory results when calculating  $\lambda_{eff}$  of disperse materials with porosity of 30-60% at elevated temperatures.

It seems necessary to consider in more detail the procedure of calculating the effective thermal conductivity of capillary-porous solids. In this case the generalized conductivity method proved to be effective [5-59], [5-72]. For a two-phase system consisting of two or more species the following calculation procedure may be recommended.

The thermal conductivity of a two-component solid phase of the considered system may be determined from Odelevsky's relation [5-47] obtained

for calculating the conductivity of a matrix two-phase heterogeneous system:

$$\frac{\lambda_s}{\lambda_1} = 1 - \frac{V_2}{\frac{1}{1 - \lambda_2/\lambda_1} - \frac{1 - V_2}{3}} \quad (5-6-18)$$

where  $\lambda_1$  and  $\lambda_2$  are thermal conductivities of continuous and dispersed components of the solid phase, respectively;  $V_2$  is the volume fraction of the dispersed component. The component of a larger volume is assumed to be a continuous phase.

Along with formula (5-6-18) Maxwell-Eucken's relation may be used to calculate the thermal conductivity of the solid phase:

$$\frac{\lambda_s}{\lambda_1} = \frac{1 + 2V_2 \frac{1 - \lambda_1/\lambda_2}{2\lambda_1/\lambda_2 + 1}}{1 - V_2 \frac{1 - \lambda_1/\lambda_2}{2\lambda_1/\lambda_2 + 1}} \quad (5-6-19)$$

The second stage of the calculation procedure involves the determination of thermal conductivity of the capillary-porous system as a whole. If the test material has only communicating pores or the fraction of closed pores is small, then the formula presented in [5-73] may be used. This formula is obtained for a two-component disperse system with communicating pores:

$$\frac{\lambda_{eff}}{\lambda_M} = \left(\frac{h}{L}\right)^2 + \frac{\lambda_g}{\lambda_M} \left(1 - \frac{h}{L}\right)^2 + \frac{2 \frac{\lambda_g}{\lambda_M} \frac{h}{L} \left(1 - \frac{h}{L}\right)}{1 - \frac{h}{L} \left(1 - \frac{\lambda_g}{\lambda_M}\right)} \quad (5-6-20)$$

The solid phase in this case is assumed to be a homogeneous material.

The analysis of (5-6-20) reveals that if  $\lambda_{eff}$  of a material is determined experimentally, for example, in air, then knowing the porosity of this material the thermal conductivity of its solid phase may be found. With the known thermal conductivity of any other gas and using the same relation we may calculate  $\lambda_{eff}$  in atmosphere of that gas, that in several cases eliminates the necessity to perform complicated experiments.

In [5-59] are given the calculated and experimental data which verify the validity of relations discussed above.

In moist capillary-porous materials transfer of energy in the form of heat is accompanied by energy transfer due to mass transfer, the energy flux

$$\vec{j}_e = -\lambda \vec{\nabla} T + \sum_{i=1}^4 h_i \vec{j}_i \quad (5-6-21)$$

Therefore, to find actual thermal conductivity of a moist material, the mass transfer should be minimized. From this view-point the steady-state methods used for the calculation of thermal conductivity do not hold as they determine the conventional heat conduction coefficient  $\lambda_e$ :

$$\lambda_e = \frac{j_e}{\vec{\nabla} T} \quad (5-6-22)$$

Finally the relation between thermal conductivity  $\lambda_e$  and moisture content may be formulated as: with the increase in moisture content the heat conduction coefficient increases and then attains its maximum value. Further increase in moisture content has small effect on thermal conductivity, but for some moist materials such an increase lowers  $\lambda_e$  (the curve  $\lambda_e(W)$  has a maximum).

For heat-insulating materials V. I. Dubnitsky [5-74] experimentally obtained such a relation for the heat-conduction coefficient:

$$\lambda = \lambda_0 + A t u \exp(-Bu) \quad (5-6-23)$$

where  $\lambda_0$  is the heat conduction-coefficient of a perfectly dry body at  $t = 0^\circ\text{C}$ ;  $A$  and  $B$  are constants measured experimentally. For crushed diatomite  $A = 0.037$ ,  $B = 2$ .

Specific heat capacity of a moist material is classified as specific heat capacity per unit mass of a perfectly dry material ( $c$ ) and specific heat capacity per unit mass of a moist material ( $c_m$ ). If shrinkage is neglected, then the relation between them may be written as:

$$c_m \rho = c \rho_0 = (c_0 + c_1 u_1 + c_2 u_2) \rho = (c_0 + c_2 u_2) \rho_0 \quad (5-6-24)$$

where subscript 1 refers to vapour, 2 to water and 0 indicates the perfectly dry state.

Consequently

$$c_m = \frac{c}{1 + u} \quad (5-6-25)$$

With allowance made for shrinkage, we have

$$c_m \rho = c \rho_0 (1 + \beta_v u)^{-1} \quad (5-6-26)$$

where  $\beta_v$  is the volume shrinkage factor.

Specific heat capacities  $c_0$  of various perfectly dry capillary-porous materials differ slightly, and vary from 0.2 to 0.4 cal/(g·°C). The temperature has a slight effect on specific heat  $c_0$ . The specific heat capacity of the majority of moist materials is a linear function of the moisture content  $u$ . But for certain materials, such as wood,  $c$  varies with moisture content following a more complicated law.

The coefficient of temperature conductivity or thermal diffusivity ( $a = \lambda/c\rho$ ) increases with moisture content, then decreases, so that the curve  $a(u)$  has a maximum. This maximum corresponds to transition from one form of moisture bond to another.

The works by M. F. Kazansky [5-75] and his associates have revealed a number of singular points on the curves  $\lambda(u)$  and  $a(u)$ , which take account of the capillary moisture state in porous material. In particular, an analysis of  $\lambda(u)$  and  $a(u)$  shows the ranges of adsorption moisture, capillary moisture, pendular and funicular moistures and the maximum hygroscopic moisture content.

Thus, the study of relations between the transfer coefficients ( $\lambda, a$ ) of moist capillary-porous materials and the moisture content makes it possible to determine the forms of moisture bond with moist materials. This is a new method of physico-chemical analysis of moist materials.

## 5-7. MOISTURE TRANSFER IN POROUS MATERIALS

A capillary-porous body with the solid framework of lyophilic material, i.e. the capillary and pore walls adsorb gases, vapours and liquids, is taken as a test system. In this case the diffusion-osmotic bond of the adsorbed substance with the solid material may exist.

Such a system differs much from model disperse materials covered by the classical filtration theory.

The substance bound by the solid framework may be liquid, vapour, inert gas, solid or subcooled liquid depending on heat- and mass-transfer conditions. The freezing point of a liquid varies over wide ranges depending on the type of moisture bonds. Therefore, in capillary-porous materials at temperatures below 0°C some quantity of subcooled liquid (water) is always present.

It should also be mentioned that in most cases the pores and capillaries of the material are filled with liquid or ice, and partially by vapour-gas mixture. To simplify the analysis of the system under consideration water containing no soluble substances is taken as liquid. Presence of solutes in the liquid changes the mass-transfer process and gives rise to additional effects.

Subscript 1 will be used to denote vapourous moisture of vapour, liquid will be designated by subscript 2, subscript 3 will stand for solid moisture (ice), subscripts 4 and 0 denote inert gas and solid framework, respectively.

Volume concentration  $\omega$  of the bound material

$$\omega = \frac{m}{V} = \frac{1}{V} \sum m_i = \sum \omega_i \quad (5-7-1)$$

where  $m_i$  is the mass of bound material in the state  $i$  ( $i = 1, 2, 3, 4$ );  $\omega_i$  is the volume concentration of the  $i$ -th bound material, and is equal to

$$\omega_i = \frac{m_i}{V} \quad (5-7-2)$$

Mass of the whole bound material  $m$  equals the sum of masses of dry air, vapour, liquid and ice:

$$m = \sum_i m_i$$

The volume  $V$  of the material generally varies with concentration  $\omega$  following a linear law. The volumetric concentration  $\omega$  is, therefore, substituted by the relative concentration or moisture content  $u$  of the material

$$u = \frac{m}{m_0} \sum_i u_i = \frac{\omega}{\rho} \quad (5-7-3)$$

where  $\rho$  is the volumetric concentration of the solid framework defined by the relation  $\rho = m_0/V$ ;  $u_i$  is the relative concentration of the  $i$ -th bound material

$$u_i = \frac{m_i}{m_0} \quad (5-7-4)$$

If the volume of the material depends slightly on the moisture content, i.e. shrinkage may be neglected ( $V = V_0 = \text{const}$ ), then the volume concentration of the solid framework will be equal to the density of a perfectly dry material

$$\rho_0 = \frac{m_0}{V_0} = \frac{m_0}{V} = \text{const} \quad (5-7-5)$$

where  $V_0$  is the volume of a perfectly dry material.

Consequently, the relative concentration  $u_i$  and volume concentration  $\omega_i$  are directly proportional

$$\omega_i = u_i \rho_0; \quad \omega = u \rho_0 \quad (5-7-6)$$

If the walls of solid framework are slightly hydrophilic, and the variation in liquid density of a monomolecular adsorbed layer can be neglected, then the amount of moisture present in a physico-mechanical mode (capillary moisture, ice, vapour or air) may be found from

$$\omega^i = \Pi_v b_i \rho_i = \rho_0 u_i'' \quad (5-7-7)$$

In the case of a monocapillary-porous material the porosity  $\Pi_V$  is proportional to  $r^3$ , for a polycapillary-porous system

$$\Pi_V = \int_{r_0}^{r_{\max}} f_V(r) dr \quad (5-7-8)$$

where  $f_V(r)$  is the differential pore distribution curve;  $r_0$  and  $r_{\max}$  are extreme variations in pore radii. Saturation  $b_i$  of the pores is determined from the relation

$$\sum_i b_i = 1 \quad (5-7-9)$$

It should be pointed out that relation (5-7-7) will not hold for physico-chemically bound moisture, i.e.  $\omega'_i \neq \omega_i$  and  $u'_i \neq u_i$ . However, formula (5-7-7) may be taken as the basis for calculation of moisture concentrations for any bond form, but then  $b_i$  will not determine the degree of saturation of the pores bound by moisture, and relation (5-7-9) does not hold any more. Only for typical capillary-porous systems with small hygroscopic moisture content, equalities  $\omega'_i = \omega_i$ , and  $u'_i = u_i$ , may be taken, having retained in this case the validity of (5-7-9).

In case of molar (hydrodynamic) flow of moisture (liquid, gas, vapour) through material pores during filtration, the flow density  $j_{i \text{ mol}}$ , kg/(m<sup>2</sup>·sec) of the  $i$ -th species will be equal to

$$j_{i \text{ mol}} = \Pi_S \rho_i b_i v_i \quad (5-7-10)$$

where  $v_i$ , m/sec, is the mean linear molar flow velocity of the  $i$ -th species;  $\Pi_S$  is the surface porosity of the sample equal to the ratio of the total pore (hole) area to the cross-sectional area of the sample in the direction normal to the direction of vector  $\vec{j}_{i \text{ mol}}$ .

This quantity for a polycapillary-porous material is

$$\Pi_S = \int_{r_0}^{r_{\max}} f_S(r) dr \quad (5-7-11)$$

where  $f_S(r)$  is the differential distribution curve of surface porosity with respect to the pore radius. For homogeneous materials the surface porosity  $\Pi_S$  is approximately equal to the volume porosity.

Strictly speaking,  $b_i$  in relation (5-7-10) is different from  $b_i$  appearing in expression (5-7-7); however at  $\Pi_V = \Pi_S = \Pi$  it is the degree of filling of pores with the  $i$ -th bound material.

Thus it may be written

$$j_{i \text{ mol}} = \Pi \rho_i b_i v_i \quad (5-7-12)$$

Relation (5-7-12) may be used for the determination of capillary moisture flow density in a monocapillary porous material. In this case  $v_2 = v_{2\text{ cap}}$ , where  $v_{2\text{ cap}}$  is the linear fluid velocity in a monocapillary porous material due to capillary forces.

Besides, it should be pointed out that under normal conditions when the pressure of humid air in pores is slightly different from the barometric pressure, the mass of air and vapour in pores is negligible compared to that of liquid or ice. Certainly, in this case it is assumed that the body is at equilibrium with the surrounding humid air, i.e. its moisture content is not zero. After Posnov calculations [5-13] under normal conditions for materials with maximum porosity (ceramics, wood, etc.) the mass of humid air present in the pores amounts to about  $10^{-5}\%$  of the liquid mass corresponding to equilibrium moisture content of the material. The total moisture content of the system may therefore be assumed equal to the sum of the liquid moisture content  $u_2$  and of ice  $u_3$ :

$$u = \sum_i u_i = u_2 + u_3 \quad (5-7-13)$$

The above relation will be used in the calculation of bound material sources due to phase transitions.

#### (a) Transfer Equation

We shall primarily dwell upon the effect of porous structure on the transfer equation. Most of the transfer processes are governed by relations of the form (5-6-21): the mass flow density is proportional to the gradient of corresponding transfer potential. For a porous material this relation may be written as

$$j = -k\varepsilon \vec{\nabla} p = -\frac{k}{\mu} \vec{\nabla} p = -k_m \vec{\nabla} p \quad (5-7-14)$$

where  $\varepsilon$  is the coefficient of resistance to transfer. O. Krischer called the inverse of  $\varepsilon$  the resistance coefficient  $\mu$  ( $\varepsilon = \frac{1}{\mu}$ ).

If the surface porosity of a material is equal to the volume porosity ( $\Pi_v = \Pi_s = \Pi$ ), then  $\varepsilon = \frac{1}{\mu} = \Pi$ . However, the porosity  $\Pi$  of moist

materials depends on their moisture content; therefore,  $\varepsilon$  or  $1/\mu$  will also depend on moisture content; for example, when the moisture content of wood varies from 10 to 20%,  $\varepsilon$  increases by a factor of 10 [5-10], when the moisture content of potato changes from 6 to 16%  $\mu$  increases by 250 times [5-10]. Besides the coefficient  $\varepsilon$  depends on temperature. Therefore it won't be expeditious to determine  $\varepsilon$  or  $\mu$  as structural factors of moist materials. In our opinion the direct method of determining the transfer coefficient  $k_m$  for a test material ( $k_m = \varepsilon k$ ) is better.

We shall consider this method in more detail. Formula (5-7-14) is similar to the Poiseuille formula which is derived as follows. The Navier-Stokes equation for a flow through a cylindrical tube with constant velocity

$$\left( \frac{\partial v_x}{\partial \tau} = 0, \quad \frac{\partial v_x}{\partial x} = 0 \right) \text{ is of the form}$$

$$\eta \left( \frac{\partial^2 v_x}{\partial r^2} + \frac{1}{r} \frac{\partial v_x}{\partial r} \right) = - \frac{\partial p}{\partial x} = \text{const} \quad (5-7-15)$$

where  $x$  is the streamwise coordinate (along the tube).  $\partial p / \partial x$  is assumed constant; integration from  $r = 0$  (the tube axis) to  $r = R$  (the tube radius) gives

$$v_x = \frac{r^2}{4\eta} \frac{\partial p}{\partial x} + A \log_e r + B \quad (5-7-16)$$

Constant  $A$  is equal to zero ( $A = 0$ ), as at  $r = 0$ ,  $v_x \neq \infty$ . Constant  $B$  may be found from  $v_x = 0$  at  $r = R$  (the velocity at the tube wall is zero, the so-called stick condition).

Hence

$$v_x = - \frac{R^2 - r^2}{4\eta} \frac{\partial p}{\partial x} \quad (5-7-17)$$

The mean fluid flow velocity  $\bar{v}_x = v$

$$v = \bar{v}_x = - \frac{R^2}{8\eta} \frac{\partial p}{\partial x} \quad (5-7-18)$$

However when a liquid flows through a macrocapillary it is necessary to take into account all slips (hydrodynamic and thermal), but for a gas mixture flow the diffusional slip should also be considered. All forms of slip contribute to the velocity  $v$  of visible motion (molar transfer). All kinds of slip are to be taken into account if the Knudsen number lies within

$$0.01 < \text{Kn} < 0.1 \quad (5-7-19)$$

In the case of hydrodynamic slip the velocity at the wall is not zero (stick condition do not hold any more).

The hydrodynamic slip velocity  $v_{hs}$  is defined by the relation

$$v_{hs} = - K_{hs} \left( \frac{\partial v}{\partial y} \right)_w = - K_{hs} \left( \frac{\partial v_x}{\partial r} \right)_{r=R} \quad (5-7-20)$$

where  $\left( \frac{\partial v}{\partial y} \right)_w$  is the velocity gradient at the wall, in the present case it equals  $\left( \frac{\partial v_x}{\partial r} \right)_{r=R}$ .



From formula (5-7-17) it follows that

$$\left( \frac{\partial v_x}{\partial r} \right)_{r=R} = \frac{R}{2\eta} \frac{\partial p}{\partial x} \quad (5-7-21)$$

Then

$$v_x = \frac{R^2 - r^2}{4\eta} \frac{\partial p}{\partial x} - K_{hs} \frac{R}{2\eta} \frac{\partial p}{\partial x} = - \frac{(R^2 - r^2 + 2K_{hs}R)}{4\eta} \frac{\partial p}{\partial x} \quad (5-7-22)$$

The mean velocity will then be equal to

$$v = \bar{v}_x = - \left( \frac{R^2}{8\eta} + \frac{K_{hs}R}{2\eta} \right) \frac{\partial p}{\partial x} = - K \frac{\partial p}{\partial x} \quad (5-7-23)$$

where

$$K = \frac{R^2}{8\eta} + K_{hs} \frac{R}{2\eta} \quad (5-7-24)$$

Let us assume that temperature and vapour concentration gradients exist along the capillary tube wall (vapour-gas mixture motion is considered). If radial flows are neglected and pressure  $p$ , density  $\rho$  and concentration  $\rho_{k0}$  ( $k = 1, 2$ ) are assumed constant throughout the cross-section, the mean velocity  $\bar{v}$  then will equal to

$$\bar{v} = - K \frac{\partial p}{\partial x} + v_{ts} + v_{ds} \quad (5-7-25)$$

Thermal or diffusional slip is a molar (convective) transfer without a total pressure gradient.

Linear velocities of thermal  $v_{ts}$  and diffusional  $v_{ds}$  slips are

$$v_{ds} = K_{ds} \frac{\partial \rho_{k0}}{\partial x} \quad (5-7-26)$$

$$v_{ts} = K_{ts} \frac{\partial T}{\partial x} \quad (5-7-27)$$

respectively, where  $K_{ts}$  and  $K_{ds}$  are thermal and diffusional slip coefficients, respectively.

Let us now assume that the total pressure gradient  $\partial p / \partial x$  is small and it may be expressed as the sum of gradients of partial pressures of vapour ( $\partial p_1 / \partial x$ ) and noncondensable gas ( $\partial p_2 / \partial x$ )

$$\frac{\partial p}{\partial x} = \frac{\partial p_1}{\partial x} + \frac{\partial p_2}{\partial x}, \quad p = p_1 + p_2 \quad (5-7-28)$$

The gradient of partial vapour pressure is directly proportional to the relative concentration gradient  $\rho_{10}(\rho_{10} = \rho_1/\rho)$ :

$$\frac{\partial p_1}{\partial x} \frac{\partial \rho_{10} M_1 / M}{\partial x} \approx \frac{M_1}{M} \frac{\partial \rho_{10}}{\partial x} = \frac{M_1 \partial \rho_{10}}{M_2 \partial x} \quad (5-7-29)$$

provided that the molecular mixture mass is independent of concentration ( $\rho_{10} \ll 1$ ,  $M \approx M_2$ ).

When evaporation takes place from liquid meniscus through which noncondensable gas (air) does not flow, the gradient of partial air pressure is found from the condition that total (molecular and molar) air flow  $j_2$  through the capillary cross-section is zero. This is an indication of approximate constancy of total pressure or for individual flows  $p = \text{const}$  may be assumed

$$j_2 = -D_{21} \rho \partial \rho_{20} / \partial x \quad (5-7-30)$$

Hence

$$\frac{\partial p_2}{\partial x} = \frac{\bar{v} p_2}{D_{12}} \frac{M_2}{M_1} \quad (5-7-31)$$

as for a binary mixture  $D_{12} = D_{21}$ .

Formula (5-7-25) may then be written as\*

$$v = v_x = \frac{1}{1 + Kp \frac{(1 - \rho_{10})}{D_{12}} \frac{M_2}{M_1}} \times \left[ -Kp \frac{M_2}{M_1} \frac{\partial \rho_{10}}{\partial x} + K_{ds} \frac{\partial \rho_{10}}{\partial x} + K_{ts} \frac{\partial T}{\partial x} \right] \quad (5-7-32)$$

In the derivation of this formula the Stefan velocity has been accounted for. If slip is neglected ( $K_{ds} = K_{ts} = 0$ ) and  $p = \text{const}$  (absence of Poiseuille flow) is assumed that implies  $[Kp(1 - \rho_{10})/D_{12}] \frac{M_2}{M_1} \gg 1$ , then from (5-7-32) we obtain a formula for the Stefan flow

$$v_c = - \frac{D_{12}}{1 - \rho_{10}} \frac{\partial \rho_{10}}{\partial x} \quad (5-7-33)$$

Relation (5-7-33) for the Stefan flow may be obtained from formula (5-7-32) assuming that  $R \rightarrow \infty$  ( $K \rightarrow \infty$ ). This implies substitution of a capillary tube with the meniscus of free surface liquid, for which convective

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\* Strictly speaking, the barodiffusion effect should have been accounted for, but it is considerably smaller than the concentration diffusion.

mass transfer involves only the Stefan flow (absence of capillary walls, and hence the slip).

Besides, formula (5-7-32) makes it possible to introduce a correction factor for hydrodynamic, thermal and diffusional slips at gradientless convective transfer with a certain linear velocity. Such a correction is necessary for convective transfer in capillary-porous materials.

For this case we have

$$v = - \frac{D_{12}}{1 - \rho_{10}} \frac{\partial \rho_{10}}{\partial x} + \frac{M_1 D_{12}}{Kp(1 - \rho_{10})M_2} \times \left[ K_{ds} \frac{\partial \rho_{10}}{\partial x} + K_{ts} \frac{\partial T}{\partial x} \right] \quad (5-7-34)$$

The first term in the formula is the Stefan flow velocity, the second is the diffusional slip velocity with a correction for hydrodynamic slip and the third term denotes the thermal slip velocity with the same correction factor.

Although formula (5-7-32) has been derived with the assumption that  $\frac{\partial p}{\partial x} \rightarrow 0$ , but the Poiseuille formula may be obtained from it with the assumption

$$K_{ds} = K_{ts} = 0; [Kp(1 - \rho_{10})/D_{12}] M_2/M_1 \ll 1$$

### (b) Transfer Processes in Colloid Materials

In colloid materials moisture is mainly bound by adsorption and diffusion-osmotic forces. Liquid-like moisture is transferred by selective type diffusion due to the difference in osmotic pressure. Consequently, the flow density  $j_2$  of liquid-like moisture is proportional to the osmotic pressure gradient  $\vec{\nabla} \overline{p}_H$ . At ordinary moisture contents of finitely swelling materials osmotic pressure or swelling pressure is a function of the moisture content gradient; thus the swelling pressure gradient is proportional to the moisture content gradient. Consequently, we have

$$j_2 = - D_{p2} \vec{\nabla} \overline{p}_H = - D_{p2} \left( \frac{\partial \overline{p}_H}{\partial u} \right) \vec{\nabla} u = - a_{2k} \rho_0 \vec{\nabla} u \quad (5-7-35)$$

where  $D_{p2}$  is the diffusion coefficient of liquid-like moisture in colloid materials, based on the swelling pressure difference;  $a_{2k}$  is the liquid diffusion coefficient of liquid in a colloid material, based on the moisture content difference.

Moist colloid material consists of uniform micells located from each other at distances comparable with intermolecular ones. A material having numerous microcapillaries of molecular order can serve as a model of colloid material. Consequently, in such a model material typical liquid transfer occurs due to the wedging pressure gradient  $\mathcal{P}$  or the film thickness gradient  $H$ , since  $\mathcal{P} = f(H)^*$ . The liquid film thickness is proportional to the moisture content of the body ( $H \sim u$ ); therefore in a general case of nonisothermal film transfer we have

$$j_{2f} = -k_f \rho_0 \vec{\nabla} u + k_f^T \rho_0 \vec{\nabla} T \quad (5-7-36)$$

where  $k_f$  and  $k_f^T$  are the coefficients of isothermal and nonisothermal liquid film motion, respectively.

Thus, the total flow of liquid-like moisture equals

$$j_2 = -(a_{2k} + k_f) \rho_0 \vec{\nabla} u + k_f^T \rho_0 \vec{\nabla} T \quad (5-7-37)$$

Vapour transfer in the first approximation depends on molecular flow (effusion). The molecular flow density is proportional to the gradient of the quantity  $p_1/\sqrt{T}$

$$j_1 = 1.064 r_e \sqrt{\frac{M_1}{R}} \vec{\nabla} \left( \frac{p_1}{\sqrt{T}} \right) \quad (5-7-38)$$

where  $r_e$  is the equivalent radius of the molecular vapour flow

$$r_e = \frac{1}{r_{\max} - r_0} \int_{r_0}^{r_{e \max}} r f(r) dr$$

Where  $r_0$  is the minimal capillary radius;  $r_{e \max}$  is the maximal radius of capillaries involved in effusion. Radius  $r_e$  is measured in the units of length and depends on the pore-radius distribution curve  $f(r)$ .

If microcapillaries, through which vapour flows, are partially filled with liquid, then  $r_e$  increases with the moisture content. Because of the presence of factor  $r$  in the integrand the relation between  $r_e$  and  $u$  is not linear.

Partial vapour pressure  $p_1$  according to the sorption isotherm is a function of moisture content and temperature  $p_1(u, T)$ . Hence,  $p_1/\sqrt{T} = f(u, T)$ .

Then

$$j = -1.064 r_e \sqrt{\frac{M_1}{RT}} \left[ \left( \frac{\partial f}{\partial u} \right)_T \vec{\nabla} u + \left( \frac{\partial f}{\partial T} \right)_u \vec{\nabla} T \right] = a_{1e} \rho_0 \vec{\nabla} u - a_{1e}^T \rho_0 \vec{\nabla} T \quad (5-7-39)$$

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\* Liquid film motion takes place in capillary-porous materials also. The film motion in macrocapillaries is however insignificant compared to capillary motion.

where  $a_{1e}$  and  $a_{1e}^T$  are the coefficients of effusion and thermal effusion of vapour in a porous material, respectively.

$$\left. \begin{aligned} a_{1e} &= \frac{1}{\rho_0} 1.064 r_e \sqrt{\frac{M_1}{R}} \left[ \left( \frac{\partial f}{\partial u} \right)_T \right] \\ a_{1e}^T &= 1.064 r_e \sqrt{\frac{M_1}{R}} \left( \frac{\partial f}{\partial T} \right)_u \frac{1}{\rho_0} \end{aligned} \right\} \quad (5-7-40)$$

As the flow of vapourous and liquid-like moisture is defined by  $\overrightarrow{\nabla}u$  and  $\overrightarrow{\nabla}T$ , the total flow of moisture in colloid bodies equals

$$j = j_1 + j_2 = -a_{mk}\rho_0\overrightarrow{\nabla}u - a_{mk}^T\rho_0\overrightarrow{\nabla}T \quad (5-7-41)$$

where the coefficients of diffusion and thermal diffusion of moisture in a colloid material are

$$\left. \begin{aligned} a_{mk} &= a_{1e} + a_{2k} + k_f \\ a_{mk}^T &= a_{1e}^T + k_f^T \end{aligned} \right\} \quad (5-7-42)$$

### (c) Liquid Moisture Transfer in Capillary-Porous Materials

Let us assume that a capillary-porous material contains in macrocapillaries moisture bound in a physico-mechanical way (osmotic and adsorbed moistures are neglected). For a monocapillary structure material the mass flow of liquid-like moisture is proportional to liquid velocity in a single capillary  $v_{2cap}$ , i.e.

$$j_{2cap} = \Pi \rho_2 b_2 v_{2cap} \quad (5-7-43)$$

The capillary velocity is a linear function of  $1/x$ :

$$v_{cap} = B \frac{r^2 \rho_2}{\eta_2} \left( \frac{\psi_r}{x} - g_k \right) = \frac{B r^2 \rho_2}{\eta_2} \frac{\psi}{x} \quad (5-7-44)$$

where  $B$  is a constant dependent on the structure of material;  $\psi$  is the total potential ( $\psi_r - \psi_g$ ).

Unlike a polycapillary structure material, monocapillary structure material has constant capillary potential  $\psi_r$ , which is determined experimentally.

For a material having polycapillary structure the liquid flow is directly proportional to the capillary potential gradient  $\overrightarrow{\nabla}\psi$ :

$$j_{cap} = k_\psi \overrightarrow{\nabla}\psi \quad (5-7-45)$$

where  $k_\psi$  is the capillary conductance:

$$k_\psi = \frac{\rho_2^2}{8\eta_2} \int_{r_0}^r r^2 f_s(r) dr$$

The capillary potential gradient is

$$\vec{\nabla}\psi = \frac{2\sigma \cos \theta}{\rho_2 r^2} \vec{\nabla}r \quad (5-7-46)$$

For a monocapillary material  $f_s(r) = \infty$ , hence  $k_p = \infty$ ,  $\vec{\nabla}\psi = 0$ , since  $\vec{\nabla}r = 0$ , therefore formula (5-7-46) is inapplicable to this material.

Capillary potential  $\psi$  is a function of temperature and moisture content of the material. For isothermal conditions  $\vec{\nabla}\psi$  is directly proportional to  $\vec{\nabla}u$ :

$$\vec{\nabla}\psi = - \frac{2\sigma \cos \theta \rho_0}{r^2 \rho_2^2 f_s(r)} \vec{\nabla}u \quad (5-7-47)$$

Then

$$j_{2\text{ cap}} = - a_{2\text{ cap}} \rho_0 \vec{\nabla}u \quad (5-7-48)$$

where

$$a_{2\text{ cap}} = \frac{\sigma \cos \theta}{4\eta_2 r^2 f_s(r)} \int_{r_0}^r r^2 f_s(r) dr \quad (5-7-49)$$

Relation (5-7-48) is similar to the concentration diffusion law, therefore the capillary motion of liquid in a polycapillary material is called capillary diffusion and the coefficient  $a_{2\text{ cap}}$  — the coefficient of capillary diffusion of liquid.

For nonisothermal conditions the flow of capillary moisture equals

$$j_{2\text{ cap}} = k_\psi \vec{\nabla}\psi = - a_{2\text{ cap}} \rho_0 \vec{\nabla}u - a_{2\text{ cap}}^T \rho_0 \vec{\nabla}T \quad (5-7-50)$$

where  $a_{2\text{ cap}}^T$  is the coefficient of capillary thermodiffusion

$$a_{2\text{ cap}}^T = \frac{\rho_2 \cos \theta}{4\rho_0 \eta_2 r} \frac{d\sigma}{dT} \int_{r_0}^r r^2 f_s(r) dr \quad (5-7-51)$$

#### (d) Capillary Motion of Liquid with Evaporation and Condensation of Moisture

If the moisture content of a body exceeds the maximum hygroscopic one, then the macrocapillaries are partially filled with water. Under these conditions the capillary liquid motion occurs when there is a difference in the capillary potential. Unlike capillary liquid suction, which takes place when the solid and liquid are in close contact with each other, the capillary potential in this case is defined ambiguously. For example, if in a porous material with uniform capillaries (sand) a limited amount of liquid is added, the liquid does not fill the whole body but only a part of it. In this case the moist portion borders on the dry one. The behaviour of liquid in sand is very similar to its behaviour in a simple capillary with limited amount of liquid. In both the cases the capillary potential is zero, as the menisci curvature is uniform over the circumference of the moist spot. For a

simple capillary we have:

$$\psi = \frac{2\sigma}{\rho_2} \left( \frac{1}{r'_1} - \frac{1}{r'_2} \right) \quad (5-7-52)$$

Hence it follows that at  $r'_1 = r'_2$  the potential  $\psi$  is zero. Addition of liquid only increases the size of the moist spot, without affecting its potential.

This may be an explanation to the absence of moisture transfer between two moist samples of monodisperse structure at different moisture contents. Water accumulated in the form of finite assemblies neither moves within the sample, nor from the sample to the other. All the differences between the samples are only due to amount of accumulated liquid (macrocapillaries of these moist samples had almost similar diameters).

But if the moist spot is brought in contact with a dry material, having capillaries of a smaller radius, then the narrow capillaries will suck liquid from the wider ones. Thus, in this case, liquid moves from a zero potential region (a moist spot with uniformly filled capillaries) to a spot with lower negative potential (a dry body with narrow capillaries). This phenomenon is quite similar to heat transfer from a body at zero temperature to a body at a lower (negative) temperature.

Almost uniform distribution of liquid over the whole volume is possible only in bodies having pronounced polydisperse structure.

If liquid evaporates inside a partially filled porous material, then the liquid moves towards the evaporation zone through the capillary-porous system. This occurs due to various reasons. In the most simple case it may be thought that during liquid evaporation narrow capillaries suck liquid from wider ones in a manner similar to liquid motion from a wider capillary to a narrow one. Some researchers believe that during evaporation the meniscus curvature increases, and as a result of this liquid displaces towards the evaporation zone.

Other authors consider that liquid motion in a narrow section of a capillary is accompanied by evaporation and condensation in a wide section. According to these authors no common flow of liquid in capillaries exists. Liquid flows along the narrow section of a capillary and reaches its wide section, where evaporation occurs accompanied by vapour diffusion with subsequent condensation at the meniscus surface of the narrow capillary section. Numerous experiments on drying of moist materials have demonstrated that vapour formed during liquid evaporation diffuses through the capillary-porous system of materials without condensing into liquid. Vapour condensation at the meniscus surface in macrocapillaries requires some surplus vapour, this cannot happen even at complete saturation of the vapour-air mixture.

Thus, when liquid moves along capillaries due to evaporation or condensation the potential  $\psi$  for a capillary (or for connected capillaries) is

defined by relation (5-7-52). In this case the necessary condition of motion is the inequality  $r'_1 \geq r'_2$ .

If a conic capillary, the variation of whose radius along its length is determined in accordance with the surface porosity distribution versus radius curve  $f_s(r)$ , is taken as a model, then from relation (5-7-52) we get a formula similar to (5-7-44) for moisture flow. In this case the relation between the capillary potential and pore radius is defined by another expression.

### (e) Vapour Moisture Transfer in Capillary-Porous Materials

Generally, vapour transfer in a moist material if evaporation takes place is described by the diffusion law with a correction for convective Stefan flow [see formula (3-1-103)]. But this will be valid only for a macrocapillary tube ( $r > 10^{-5}$  cm). In microcapillaries the vaporous moisture flow is governed by the effusion law; besides, during moisture evaporation from porous bodies other transfer phenomena may also occur.

In capillaries at viscous molecular flow conditions, diffusion and effusion vapour transfers are accompanied by visible (convective) motion caused by hydrodynamic, thermal and diffusional slips. If the total pressure is assumed constant ( $p = \text{const}$ ), e.g.  $Kp(1 - \rho_{10}/D_{12}) \gg 1$ , then the convective transfer rate, as it follows from formula (5-7-34), equals

$$v_k = - \frac{D_{12}}{1 - \rho_{10}} \left[ 1 - \frac{K_{ds}M_1}{KpM} \right] \nabla \rho_{10} + \frac{D_{12}K_{ts}M_1}{Kp(1 - \rho_{10})} \nabla T \quad (5-7-53)$$

Then the total vapour flow consisting of molecular (diffusion) flow  $j_{1d}$ , convective flow  $\rho_1 v_k$  and effusion flow  $j_e$  is

$$j_1 = - \frac{\varepsilon_1 D_{12} M_1}{RT(1 - \rho_{10})} \left\{ \left( 1 - \frac{M_1}{M} \rho_{10} \frac{K_{ds}}{Kp} \right) \nabla \rho_1 - \frac{\rho_{10} K_{ts}}{K} \nabla T \right\} - \varepsilon_1 K_{eff} \nabla \frac{\rho_1}{\sqrt{T}} \quad (5-7-54)$$

Vapour pressure  $p_1$  is the function of moisture content and temperature  $p_1(u, T)$ , it may therefore be written

$$\nabla p_1 = \left( \frac{\partial p_1(u, T)}{\partial u} \right)_T \nabla u + \left( \frac{\partial p_1(u, T)}{\partial T} \right)_u \nabla T \quad (5-7-55)$$

Then formula (5-7-54) may be written in the form

$$j_1 = -a_{1cap} \rho_0 \nabla u - a_{1cap}^T \rho_0 \nabla T \quad (5-7-56)$$



where  $a_{1\text{ cap}}$  and  $a_{1\text{ cap}}^T$  are the coefficients of diffusion and thermal diffusion of vaporous moisture in a capillary-porous body, respectively,

$$a_{1\text{ cap}} \rho_0 = \left\{ \frac{\varepsilon_1 D_{12} M_1}{RT(1 - \rho_{10})} \left[ \left( 1 - \frac{M_1 \rho_{10} K_{ds}}{MKp} \right) \right] + \frac{\varepsilon_1 K_{eff}}{\sqrt{T}} \right\} \left( \frac{\partial p_1}{\partial u} \right)_T \quad (5-7-57)$$

$$a_{m\text{ cap}}^T \rho_0 = \frac{\varepsilon_1 D_{12} M_1}{RT(1 - \rho_{10})} \left( 1 - \frac{M_1 \rho_{10} K_{ds}}{MKp} \right) \left( \frac{\partial p_1}{\partial T} \right)_u + \varepsilon_1 K_{eff} \left( \frac{\partial p_1 / \sqrt{T}}{\partial T} \right)_u - \frac{A \rho_{10} K_{rs}}{K} \quad (5-7-58)$$

It should be noted that in formula (5-7-58) the third term has a minus sign. It implies that vaporous moisture transfer by diffusion slip has the reverse direction to the heat flux. Therefore, if the sum of first two terms in formula (5-7-58) is less than the third term, then the coefficient  $a_{1\text{ cap}}^T$  will be negative.

The total moisture flow is then

$$j = j_1 + j_2 = a_{m\text{ cap}} \rho_0 \vec{\nabla} u - a_{m\text{ cap}}^T \rho_0 \vec{\nabla} T \quad (5-7-59)$$

where  $a_{m\text{ cap}}$  and  $a_{m\text{ cap}}^T$  are the coefficients of diffusion and thermal diffusion of moisture

$$a_{m\text{ cap}} = a_{1\text{ cap}} + a_{2\text{ cap}}; \quad a_{m\text{ cap}}^T = a_{1\text{ cap}}^T + a_{2\text{ cap}}^T \quad (5-7-60)$$

Transfer of liquid and gas due to capillary forces is also discussed in the filtration theory. Capillary pressure  $p_{\text{cap}}$  is defined by the formula

$$p_{\text{cap}} = \frac{2\sigma \cos \theta}{(K/\Pi)^{1/2}} J(\omega) \quad (5-7-61)$$

where  $K$  is the permeability coefficient and  $J(\omega)$  is the Leverett function. The shape of the Leverett function for impregnation (displacement of a less wetting liquid by a more wetting one) and drainage (displacement of a more wetting liquid by a less wetting one) is shown in Fig. 5-37.

If it is assumed that each liquid flows independent of the other, then

$$j_i = \rho_i v_i = - \frac{K_i(\omega)}{v_i} \vec{\nabla} p_i, \quad i = 1, 2 \quad (5-7-62)$$

where  $K_i(\omega)$  are called phase permeabilities.

The ratio  $K_i(\omega)/K$  is called the relative phase permeability.

The shapes of phase permeability curves (Fig. 5-38) may be calculated from the pore distribution curves [5-78].

The main disadvantage of these theories is the assumption of independence of separate phase motions (each fluid moves in the occupied by it part of capillaries and pores as if the remaining space of the system were the solid framework).

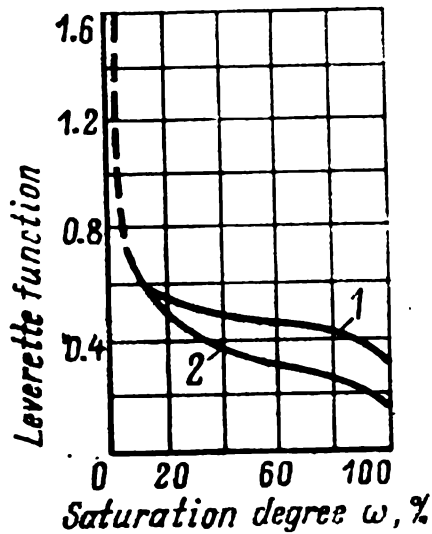


Fig. 5-37. Laverette function  
1 — depletion; 2 — suction

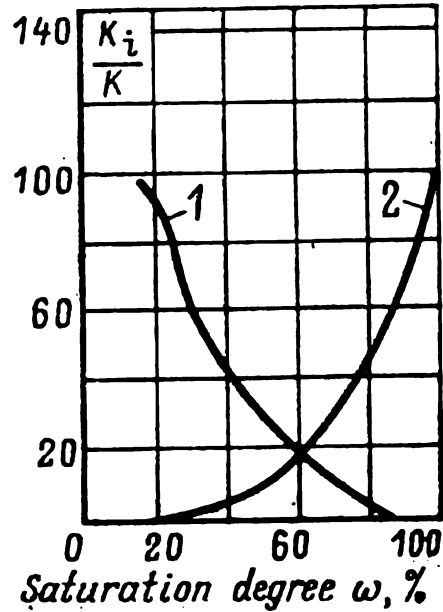


Fig. 5-38. Plot of relative phase permeability  $K_i/K$  versus saturation  $\omega$ , %  
1 — gas; 2 — liquid

The most correct approach to transfer processes in capillary-porous materials is the consideration of mutual effect of phases during multi-phase liquid transfer.

#### (f) Moisture Transfer in Colloid Capillary-Porous Materials

Moist materials are colloid capillary-porous bodies in which moisture transfer is controlled by various effects discussed above in the analysis of transfer processes in colloid and porous bodies.

If porous materials of monocapillary structure are excluded from the consideration, and the gravity forces are also neglected (typical capillary-porous colloid bodies), then moisture transfer is described by moisture content gradient and temperature. This happens because various thermodynamic moisture-transfer forces are the functions of  $u$  and  $T$ , so they may be expressed in terms of  $\vec{\nabla}u$  and  $\vec{\nabla}T$ .

Thus, moisture transfer in colloid capillary-porous materials is described by the formula:

$$j = -a_m \rho_0 \vec{\nabla}u - a_m^T \rho_0 \vec{\nabla}T = -a_m \rho_0 (\vec{\nabla}u + \delta \vec{\nabla}T) \quad (5-7-63)$$

where  $a_m$  and  $a_m^T$  are the coefficients of diffusion and thermal diffusion of moist materials, respectively;  $\delta$  is the relative coefficient of thermal diffusion ( $\delta = a_m^T/a_m$ ).

According to formulas (5-7-60), (5-7-51), and (5-7-49) coefficients  $a_m$ ,  $a_m^T$  and  $\delta$  are

$$\left. \begin{aligned} a_m &= a_{m \text{ cap}} + a_{mk}; & a_m^T &= a_{m \text{ cap}}^T + a_{mk}^T \\ \delta &= \frac{\delta_{\text{cap}} a_{m \text{ cap}} + \delta_k a_{mk}}{a_{m \text{ cap}} + a_{mk}} \end{aligned} \right\} \quad (5-7-64)$$

Coefficients  $a_m$  and  $\delta$  are the functions of moisture content and temperature.

The behaviour of  $a_m$  at different moisture contents depends on the type of moisture bond with material and mode of the moisture transfer (transfer of vapour or liquid). For typical capillary-porous materials the coefficient increases with moisture content and at large moisture contents it becomes constant.

Continuous increase of the coefficient  $a_m$  with moisture content of typical capillary-porous materials is characteristic of vapour transfer. If mass transfer takes place in the form of liquid, then the coefficient  $a_m$  may increase or remain constant as the moisture content increases depending on the shape of the differential pore-radius distribution curve.

Experimental data on diffusion coefficients for a number of materials correlate well with the values obtained by the empirical formula:

$$\frac{a_{m0}}{a_m} = 1 - A \rho_0 u \quad (5-7-65)$$

where  $a_{m0}$ ,  $A$  are constants found experimentally (see Table 5-9). For capillary-porous materials  $A = 0.035$  approximately, and for colloid capillary-porous materials  $A = 0.020$ . Coefficient  $a_{m0}$  depends on temperature

$$a_{m0} = a_{00} B \left( \frac{T}{1000} \right)^n \quad (5-7-66)$$

where  $a_{00}$  is a constant whose values for different materials are given in Table 5-9; approximately for capillary-porous materials  $B = 10^5$ ,  $n = 20$ ; for colloid capillary-porous materials  $B = 1$ ,  $n = 10$ .

After [5-79] the approximate value of coefficient  $A$

for capillary-porous materials. . . . .	0.0032
for colloid materials . . . . .	0.0008
for colloid capillary-porous materials . . . . .	0.0019

Coefficients  $a_{m0}$  are found from the formula [5-79]

$$a_{m0} = \frac{1}{A_0} \left( \frac{T}{273} \right)^n \quad (5-7-67)$$

Table 5-9

Constants in the Formula Relating Moisture Diffusion Coefficient with Moisture Content and Temperature for Some Materials

Material	$\rho_0$ , kg/m <sup>3</sup>	T, K	$\bar{u}$ , kg/kg	$A\rho_0$	$a_{60}$	Authors
Foamed diatomite brick	270	321	0.7-1.25	0.83	2.62	E. I. Sizyakova
Asbestos cement blocks	390	293	0.1-0.6	1.52	10.83	V. I. Dubnitsky
Autoclave concrete	400	293	0.1-0.4	1.67	6.11	V. I. Dubnitsky
Fixed diatomite blocks	500	293	0.1-0.5	2.44	12.08	V. I. Dubnitsky
Diatomite crumb	510	298-338	0.1-0.7	1.6	0.50	V. I. Dubnitsky
Gypsum	980	286	0.16-0.30	3.27	1.87	A. V. Luikov
Quartz sand	1 410	318	0.02-0.213	4.86	50.58	N. N. Babiev
Porous ceramics	1 684	292-321	0.04-0.16	5.1	0.323	V. P. Zhuravleva
Porous ceramics	1 816	292-321	0.04-0.11	5.65	0.645	V. P. Zhuravleva
Gelatine	65	293-297	45-75	0.110	0.190	A. V. Luikov F. M. Polonsky
Lowland peat (decomposition 20%)	100	318	0.5-455	0.20	24.6	N. N. Babiev
Foxum peat (decomposition 20%)	170	293	0.3-8.5	0.122	6.66	S. S. Korchunov and S. M. Dulkan
Lump peat	185	323-353	0.8-2.4	0.46	0.602	B. I. Pyatachkov
Wood (pine)	495	313-373	0.1-0.35	2.60	1.955	P. D. Lebedev
Wheat dough	950	318	0.3-0.5	2.04	0.002	A. V. Liukov
Macaroni dough	1 150	303-338	0.22-0.44	2.23	0.0004	I. S. Melnikova
Clay	1 460	318	0.035-0.516	3.22	6.10	N. N. Babiev
Caoline	1 700	298	0.05-0.3	3.37	13.95	A. V. Luikov

Constants  $A_0$  and  $n$  are listed in Tables 5-10 and 5-11.

Table 5-10

Constants  $A_0 \times 10^{-5}$  and  $n$  for Particular Moist Materials

Material	$\rho_0$	$T, K$	$\bar{u}, \%$	$A_0 \times 10^{-5}$	$n$
Fireclay brick	1900	307-320	8-16	3.30	16.6
Foam concrete	450	295-318	10-10	3.86	1.0
Gas concrete with mixed bond	712	293-313	10-30	27.9	10.0
Foam concrete with mixed bond	885	297-307	5-35	31.2	20.0
Diatomite crush	510	323-338	10-60	40.3	20.0
Ceramics	1684	293-321	4-16	57.4	20.0

Table 5-11

Constants  $A_0$ ,  $n$  and  $a_{m0}$  for Clays from Different Deposits

Deposit	$\rho_0$	$T, K$	$\bar{u}, \%$	$\bar{u}_0, \text{kg/kg}$	$A_0 \times 10^{-5}$	$a_{m0} \times 10^5$	$n$
Prinevskoe	1 650	294-305	6-15	0.214	0.61	1.637	11.76
Osinovskoe	1 820	289-308	6-15	0.264	1.20	0.833	8.35
Chasov-Yarskoe	1 850	308-338	10-25	0.320	1.50	0.666	6.85
Revdinskoe	1 650	296-308	10-25	0.279	1.65	0.606	9.00
Polynkovskoe	1 865	296-315	10-27	0.348	2.90	0.345	6.40
Vasyutinskoe	1 750	293-315	10-25	0.300	4.40	0.227	11.10

For a number of materials the curve  $a_m = f(u)$  has a complicated appearance. The coefficient  $a_m$  is then calculated by the formula

$$a_{m0}/a_m = 1 + B(u - u_M) \quad (5-7-68)$$

where  $u_M$  is the moisture content corresponding to the maximum of the curve  $a_m = f(u)$ ; constant  $a_{m0}$  is calculated from formula (5-7-67).

For example, for wool felt ( $0.05 < u < 0.15$ )  $B = 45$ ;  $A_0 = 5.95 \times 10^6$ ;  $n = 15$ ;  $u_M = 0.05$ .

In most cases the coefficient of thermal diffusion  $\delta$  increases with moisture content, attains the maximum, and then remains constant or decreases.

According to the experimental studies made by V. I. Dubnitsky, the thermogradient coefficients of heat insulating materials are practically independent of temperature, they may be calculated from the empirical formula:

$$\delta = Au^2 \exp(-Bu) \quad (5-7-69)$$

where  $A$  and  $B$  are constant, the latter being an integer. For example, for diatomite crush  $A = 9.12$ ,  $B = 9$ .

*(g) Convective Filtration Moisture Transfer in Moist Materials*

When a moist material is heated intensively its temperature rapidly rises. At temperatures above 100°C the partial pressure of saturated vapour  $p_1$  exceeds the barometric pressure of the surrounding air ( $p_1 > p$ ). As a result the diffusion transfer of vapour in a porous material is replaced by the molar one. Such a mechanism of vapourous moisture transfer, which proceeds due to total pressure gradient, is called the convective filtration or filtration transfer of vapour.

It should be emphasized that the total pressure gradient ( $\overline{\nabla p}$ ) appears not only at temperatures above 100°C. If the material is heated from within (for example, drying in a high-frequency field), then the total pressure gradient appears also at temperatures below 100°C. This may be explained by molecular air flow along microcapillaries and by slip diffusion in microcapillaries.

Relaxation of total pressure gradient occurs in an open-capillary system, but because of intense vaporization within the body and of the resistance of the moist material framework a certain gradient  $\overline{\nabla p}$  establishes, which is different from zero. This phenomenon was first experimentally corroborated by Maksimov upon drying moist materials with internal sources of heat.

The presence of total pressure gradient  $\overline{\nabla p}$  inside a capillary-porous material induces molar transfer of vapour-gaseous mixture (vapour and air) similar to gas filtration through porous materials. This molar transfer of vapour is not taken into account by the mass-transfer equation (5-7-59); for this reason an additional term, which accounts for convective vapour transfer, should be introduced.

From the Darcy law the filtration flux of humid air through a porous body is

$$j_a = \rho v = -k \overline{\nabla p} \quad (5-7-70)$$

where  $k$  is the penetration factor for air, kg/(m·h·mm Hg);  $\rho$  is the density of humid air. The amount of vapour transferred by this filtration air flux is

$$j_{1F} = \rho_1 v = -k \rho_{10} \overline{\nabla p} = -k_p \overline{\nabla p} \quad (5-7-71)$$

where  $\rho_{10} = \rho_1/\rho$  is the relative concentration of vapour equal to the ratio of vapour concentration to humid air density;  $k_p$  is the molar transfer coefficient

$$k_p = k \rho_{10} = k \frac{d}{1+d} \quad (5-7-72)$$

where  $d$  is the humidity of air.

The total moisture flow with gradient  $\overline{\nabla p}$  is

$$j = -a_m \rho_0 \overline{\nabla u} - a_m^T \rho_0 \overline{\nabla T} - k_p \overline{\nabla p} \quad (5-7-73)$$

Relation (5-7-73) does not account for liquid transfer due to gravity field or hydrostatic pressure gradient (liquid filtration through porous materials). The intensity of such molar transfer of liquid is by many times larger than moisture transfer due to diffusion and capillary forces. Therefore, liquid filtration in disperse materials is generally considered independent of moisture transfer.

The penetration factor  $k$  for air is a proportionality factor between the humid air flux and the total pressure gradient, i.e. it characterizes the vapour-gaseous transfer (vapour and dry air).

The ratio of the relative amount of vapour transferred by such a humid air filtration flow to the total flux of vapour-gaseous mixture equals the ratio of coefficients

$$\frac{k_p}{k} = \frac{d}{1 + d} \quad (5-7-74)$$

#### *h) Moisture Conductivity and Thermal Moisture Conductivity of Moist Materials*

Moisture content is an analogue of heat content and therefore it cannot serve as the moisture transfer potential. Therefore, when applying formula (5-7-73) to systems consisting of several moist bodies in contact, we may get incorrect results if the moisture content drop at their contact boundary is not accounted for.

At isothermal conditions ( $\overline{\nabla}T = 0$ ) in the absence of total gradient of pressure ( $\overline{\nabla}p = 0$ ) according to formula (5-7-73) moisture is transferred from high moisture content places to the low moisture content spots. This holds only if the mass capacity remains unchanged. At different mass capacities (a system of bodies) moisture may be transferred in the reverse direction, as shown in Fig. 5-19. Quartz sand at a moisture content of 0.1 kg/kg ( $W_1 = 10\%$ ) has a mass-transfer potential  $\theta = 600^\circ\text{M}$ , but peat at moisture content of 3 kg/kg ( $W_2 = 300\%$ ) has a potential  $\theta = 350^\circ\text{M}$ . When these materials are brought into contact moisture is transferred from the material having high potential but low moisture content (sand) to the material with low mass-transfer potential but high moisture content (peat). A similar situation may be observed in the case of heat transfer from a lead plate, whose specific enthalpy at  $t_1 = 200^\circ\text{C}$  equals 6 kcal/kg, to an aluminum plate having specific enthalpy equal to 20 kcal/kg at  $t_2 = 100^\circ\text{C}$ .

Only for a homogeneous moist material the moisture transfer formula holds at isothermal conditions.

$$j = -a_m \rho_0 (\overline{\nabla}u)_T \quad (5-7-75)$$

Similarly, the Fourier heat-conduction law ( $q = -\lambda \overline{\nabla}T$ ) may be written as

$$q = -\lambda \overline{\nabla}T = -a_p \rho \overline{\nabla}h \quad (5-7-76)$$

since

$$\vec{\nabla} T = \left( \frac{dT}{dh} \right) \vec{\nabla} h \quad (5-7-77)$$

where  $dh/dT = c_p$  is the specific heat at constant pressure;  $h$  is the specific enthalpy of the material;  $a_p$  is the thermal diffusivity or the coefficient of heat diffusion (for a solid  $a_p = a_v = a$ ).

Hence, the Fourier heat-conduction law may be formulated as follows: the heat flux density is directly proportional to the specific enthalpy gradient. The heat diffusion coefficient characterizes enthalpy transfer in a similar way the mass diffusion coefficient  $a_m$  describes mass transfer. The moisture-transfer potential  $\theta$  is a function of moisture content and temperature:  $\theta(u, T)$ . The moisture content gradient  $(\vec{\nabla} u)_T$  may be expressed in terms of the gradient of potential  $\theta$ :

$$(\vec{\nabla} u)_T = \left( \frac{\partial u}{\partial \theta} \right)_T \vec{\nabla} \theta = c_m \vec{\nabla} \theta \quad (5-7-78)$$

where  $c_m$  is the specific isothermal mass capacity or moisture capacity of the material.

Relation (5-7-76) is valid for a one-component system or a solid. Consequently, the isothermal moisture transfer law is of the form:

$$j = -a_m \rho_0 c_m \vec{\nabla} \theta = -\lambda_m \vec{\nabla} \theta \quad (5-7-79)$$

where  $\lambda_m = a_m \rho_0 c_m$  will be called a mass conductivity or moisture conductivity coefficient similar to thermal conductivity  $\lambda = a c \rho_0$ .

Moisture transfer due to potential  $\vec{\nabla} \theta$  is called moisture conduction, and relation (5-7-79) — the moisture conduction law.

At nonisothermal conditions the moisture transfer law is of the form:

$$j = -\lambda_m \vec{\nabla} \theta - \frac{\lambda_m}{c_m} \delta_\theta \vec{\nabla} T - k_p \vec{\nabla} p \quad (5-7-80)$$

since

$$a_m^T \rho_0 = a_m \rho_0 \delta = \frac{\lambda_m}{c_m} \delta_\theta + \lambda_m \theta_T$$

In formula (5-7-80) all forms of moisture transfer are expressed in terms of moisture transfer potential gradients  $(\vec{\nabla} \theta, \vec{\nabla} T, \vec{\nabla} p)$ . Consequently, the moisture thermal diffusion law

$$j_t = -a_m \rho_0 \delta \vec{\nabla} T = -\frac{\lambda_m}{c_m} \delta_\theta \vec{\nabla} T = \lambda_m^T \vec{\nabla} T \quad (5-7-81)$$

may be called the thermal moisture conduction law. Then the coefficient  $\lambda_m^T \left( \lambda_m^T = \frac{\lambda_m}{c_m} \delta_\theta \right)$  is the thermal moisture conductivity and  $\delta' = \delta_\theta / c_m$  is the dimensionless thermal moisture conductivity.



Different explanations of some physical quantities are often encountered in physics. For example, the surface tension coefficient is defined as a surface tension force acting on unit length of an arbitrary contour, on the section line of the liquid surface, dyne/cm, (property of a force). On the other hand, the surface tension coefficient is numerically equal to the free potential energy when a unit area of surface liquid is formed, erg/cm<sup>2</sup> (a power characteristic).

Exactly in the same manner the thermal diffusivity  $a$  ( $a = \lambda/c\rho_0$ ), is, on one hand, the heat diffusion coefficient (see above), and on the other hand, characterizes the isotherm propagation velocity. Therefore, the coefficient  $a$  measured in cm<sup>2</sup>/s, may also be called the thermal diffusivity. The reverse of thermal diffusivity  $1/a$  characterizes the internal properties of material with respect to isotherm displacement.

As temperature is a heat-transfer potential, the coefficient  $a$  may be called the potential conductivity of heat transfer. Similar relations may be found for moisture transfer. The moisture diffusion coefficient  $a_m$  ( $a_m = \lambda_m/c_m\rho_0$ ) may be called the potential diffusivity of moisture transfer as it characterizes the rate of propagation of the isopotential surface of material at isothermal conditions. Consequently, the quantity  $1/a_m$  characterizes inertial properties of material with respect to isopotential surface displacement in the moist material  $\theta = \text{const}$  ( $a_m = AW_\theta$ ). The coefficients  $a$  and  $a_m$  have the same dimensions (cm<sup>2</sup>/s), they are defined as

$$a = \frac{\lambda}{c\rho_0}; \quad a_m = \frac{\lambda_m}{c_m\rho_0} \quad (5-7-82)$$

A complete analogy exists in the measuring units of the heat conduction coefficients  $\lambda$ , cal/(m·h·°C) and moisture conductivity  $\lambda_m$ , kg (m·h·°M), and also of heat capacity  $c$ , kcal/(kg·°C) and mass capacity  $c_m$ , kg/(kg·°C).

### (i) Moisture Transfer in an Electromagnetic Field

Moist capillary-porous bodies are polarized systems in which the applied strengths of electric and magnetic fields relax. Consequently such systems are relaxation and polarized ones.

Let us denote electric and magnetic polarizations by  $\pi_e$  and  $\pi_M$ , respectively, and the strengths of electric and magnetic fields by  $E$  and  $B$ , then the chemical potential gradient of the  $k$ -th species is

$$\vec{\nabla}\mu_k = -S_k\vec{\nabla}T + V_k\vec{\nabla}p - (\vec{\nabla}E)\pi_e - (\vec{\nabla}B)\pi_m + \sum_{i=1}^{n-1} \frac{\partial\mu_k}{\partial\rho_{i0}} \vec{\nabla}\rho_{i0} \quad (5-7-83)$$

where  $S$  is the entropy and  $V$  is the specific volume.

Thus, aside from the thermodynamic forces  $\vec{\nabla}u$  and  $\vec{\nabla}T$  in a nonuniform electromagnetic field will act the thermodynamic forces

$$X_e = (\vec{\nabla}E)\pi_e; \quad X_M = (\vec{\nabla}B)\pi_M \quad (5-7-84)$$

Thus, in a uniform electromagnetic field moisture transfer is caused not only by diffusion ( $\bar{\nabla}u$ ) and thermodiffusion ( $\bar{\nabla}T$ ) forces, but also by the forces  $X_e$  and  $X_M$ . Hence, in a general case, in accordance with the Onsager equation

$$j_2 = L_{21}X_1 + L_{22}X_2 + L_{23}X_3 + \dots \quad (5-7-85)$$

the moisture flow is defined by relation

$$j = -a_m \rho_0 \bar{\nabla}u - a_m^T \rho_0 \bar{\nabla}T - a_m^e \rho_0 (\bar{\nabla}E) \pi_e - a_m^M \rho_0 (\bar{\nabla}B) \pi_M \quad (5-7-86)$$

where the first term determines the value of moisture diffusion contribution, the second term is the thermodiffusion of moisture. The third term corresponds to moisture transfer due to nonuniform electric field. This mode of transfer may be called electrodiffusion of moisture\*. A moisture flow caused by the force  $[(\bar{\nabla}B)\pi_M]$  will be called magnetodiffusion of moisture in a nonuniform magnetic field.

The coefficients  $a_m^e$  and  $a_m^M$  will be the coefficients of electro- and magneto-diffusion in moist materials.

If the electromagnetic field is uniform, but variable, then the moisture flow will be defined not only by the effect of thermodynamic forces  $\bar{\nabla}u$  and  $\bar{\nabla}T$ , but also the motive force  $\tilde{E}$ , which is the difference between the field strengths at equilibrium  $E_e$  and at a certain moment  $E$  ( $\tilde{E} = E_e - E$ ).

As to the thermodynamic force  $\tilde{B}$  ( $\tilde{B} = B_e - B$ ) the following is to be said: in a general case  $\tilde{B}$  is a tensor and, in accordance with the Curie principle, cannot be consistent with the vectors  $\bar{\nabla}u$ ,  $\bar{\nabla}T$  and  $\tilde{E}$ . But at certain conditions  $\tilde{B}$  affects moisture transfer. Thus for a variable, electric field we have:

$$j = -a_m \rho_0 \bar{\nabla}u - a_m^T \rho_0 \bar{\nabla}T - \tilde{a}_m^e \rho_0 \tilde{E} \quad (5-7-87)$$

where  $\tilde{a}_m^e$  is the electrodiffusion coefficient in a variable electric field.

The moisture flow due to electrodiffusion is

$$j_e = -a_m^e \rho_0 (\text{grad } E) \pi_e - \tilde{a}_m^e \rho_0 \tilde{E} \quad (5-7-88)$$

Similar relations are obtained for a variable magnetic field

$$j_M = -\bar{a}_m^M (\text{grad } B) \pi_M - \tilde{a}_m^M \rho_0 \tilde{B} \quad (5-7-89)$$

where the bar over  $B$  indicates averaging and the decrease in the order of tensor.

Relations (5-7-88) and (5-7-89) show the effect of an electromagnetic field on moisture transfer. Obviously, the thermodynamic forces affect heat transfer also.

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\* It is necessary to identify moisture diffusion due to electro-osmotic forces.

Researches on drying of moist materials in a variable magnetic field provide support to the fact that a variable magnetic field affects moisture transfer. But, because of the lack of experimental data it is not possible to calculate the coefficients of electric and magnetic diffusion.

#### 5-8. APPLICATION OF CAPILLARY-POROUS MATERIALS IN SPACE ENGINEERING

A vast and extremely important field of application of capillary-porous materials is afforded by recent developments in space engineering. In this field the most important problems are associated with the maintenance of optimal temperatures necessary for proper functioning of various devices and units of space vehicles. The solution of the problems essentially amounts to the removal of heat liberated in vehicles and its transfer to the surroundings. If under ordinary terrestrial conditions the methods of cooling by gas injection and liquid evaporation are, to a known extent, equivalent, then under space conditions (high vacuum, weightlessness, stringent requirements to temperature control systems) transpiration cooling is not only the optimal means but the only possible. In space all the advantages of transpiration cooling are fully manifested, such as high efficiency of evaporation due to the high evaporation rate in a vacuum; high performance economy due to strong endothermic effect of phase transition, no exacting requirements to temperature control of the cooling agent; no need of special systems for the supply of coolant since under weightlessness the capillary potential of liquid coolant supply to the surface to be cooled is theoretically infinite. Transpiration cooling is applicable both for external thermal protection and the transfer of internal heat separately and to complex cooling of all the units. Besides transpiration cooling lends itself to easily automatic control by controlling the coolant supply.

The task of ensuring life activities of cosmonauts during space voyages, especially when the astronaut leaves his spaceship presents exclusive difficulties. As the design of a space suit conforms primarily to the requirements of air-tightness, radiation and meteorite protection, but on the whole is far from the temperature control requirements, it is necessary not only to safeguard the astronaut from external heat fluxes (particularly from solar radiation) but also to ensure the removal of heat liberated by the astronaut. The requirements to the temperature control system are very rigid. The temperature of the inner surface of the suit should be lower than that of the human body, the system should be independent, small in size and light in weight. An alternative to the system may be the set of contact heat exchangers located close to the body. The heat exchangers represent a capillary-porous material comprising space suit and communicating with a liquid coolant tank.

Properties of the coolant and capillary-porous materials should be so selected that the heat exchanger would operate at the conditions of a

double phase transition (the supplied liquid coolant freezes due to intense phase transition and then it sublimates) ensuring low temperature at economical consumption of coolant. The phase conversion zone in this case should lie within the capillary-porous material.

Fulfillment of the last requirement leads to a curious situation: the thermal conductivity of a porous heat exchanger in the direction towards the astronaut body proves to be very low because of high heat resistance of the dry region acting as an insulation, while the thermal conductivity within the material in the moist zone is high enough.

Such diode properties of capillary-porous systems with a recessed phase conversion front prove to be very useful and valuable in most complicated cases when protection from external and internal heat fluxes is necessary.

Metering and control equipment, astronavigation equipment and many other high precision instruments installed in the spaceship are also very sensitive to high temperatures. Utilization of different modifications of transpiration cooling for the absorption of heat liberated by metering instruments and operation loops, allows elimination of thermal interference, suppression of thermal noise and improvement of the quality of microradio-electronic devices. The role of miniature radioelectronic devices cannot be overestimated not only for space vehicles but also for preset control systems, computers and cybernetic units so important in the national economy.

Transpiration cooling offers reliable means for temperature control of thermal system elements, for protection against overheat of fuel tanks in order to decrease the losses of cryogenic rocket propellant and for adhering to explosion danger rules. In this case both special liquid and cryogenic fuel may serve as a coolant. In the latter case it is easy to achieve self-freezing of the fuel.

The problem of nullifying the effect of internal heat release in a spacecraft is closely related to the need of heat removal. An ideal solution for the heat transfer problem may be obtained by using a device similar to a heat pipe. A heat pipe is a sealed capillary-porous wick saturated with a volatile liquid. With the use of evaporation-condensation mechanism of heat transfer a heat pipe allows an increase in thermal conductivity by tens of thousands of times compared to best known heat conductors (metals). A heat pipe is essentially an unusual automatic superconductor. It is only in the space, i.e. under weightlessness, heat pipes are independent of all geometrical and space limitations, and they (heat pipes) attain special significance from the design point of view. Particularly, the use of heat pipes allows not only elimination of inadmissible temperature deformations of the spacecraft and removal of temperature stresses from the structure, which appear due to strong overheat of the spacecraft from the illuminated side and sharp cooling from the shaded side, but also utilization of these on the whole adverse effects.

One of the processes the working cycle of heat pipes mounted in semi-circles over the spacecraft diameter between the hot and cold sides of the spacecraft is capillary pumping of the working fluid through the porous

wick from the cold condensation section of the pipes to the hot evaporation section. With correctly chosen characteristics of the working fluid and porous wick this will result in a potential flow of polar liquid through the porous material, and the heat pipe starts functioning as thermoelectric converter supplying "cheap" energy to the spacecraft.

Capillary-porous materials are however promising not only for the solution of space heat problems enumerated here in brief. The materials under consideration allow solution of a serious problem of liquid transport at zero-g.

As mentioned above the capillary potential of liquid transport in the absence of body forces is infinite and can ensure normal functioning of fuel systems of spacecrafts.

Using capillary-porous materials of cryogenic pumping it is possible to develop pumping devices which function for a long period due to capillary suction of condensed gases under steady-state conditions.

Capillary-porous materials in weak gravity fields have a number of special properties. Among them are:

1. Capillary suction potential, which characterizes interaction of liquid with capillary walls.
2. Permeability for gases and liquids.
3. Variation of effective transfer coefficients in wide ranges (for example, variation of effective thermal conductivity depending on the fact whether the temperature and concentration fields in the material coincide). Effective thermal conductivity of capillary-porous materials can vary from  $10^{-3} \text{ W/(m} \cdot ^\circ\text{C)}$  to  $10^3 \text{ W/(m} \cdot ^\circ\text{C)}$ , i.e. by 6 orders.
4. Large specific area.
5. Low density (compared to the solid material of which the considered porous material is made).
6. Electrokinetic effects when liquid flows through a capillary-porous material (a flow potential).
7. Selective properties, i.e. materials are permeable for some liquids and impermeable for others; permeable for heat fluxes in one direction and impermeable in the opposite direction.
8. Possibility of phase transitions in pores, which are accompanied by energy generation or absorption, and a number of other interesting properties.

These properties allow capillary-porous materials to be used as materials for various kinds of heat exchangers in space (condensers, evaporators, sublimers): to protect from external heat fluxes, for heat disposal in a vacuum (thermostatting of liquid tanks, under cooling of liquids, thermostatting of different devices exposed to solar heating), for liquid pumping, for converting heat energy into mechanical or electric energy with high efficiency, for making heat-transport devices of the heat pipe type with thermal conductivity as large as 100 times that of copper. Finally, the use of porous materials as engine nozzles makes it possible to obtain in a vacuum

gaseous jets of very uniform structure with a well developed isentropic kernel. Porous insulation is the most effective insulation in a vacuum.

Some practical application of capillary-porous materials in space will be considered next.

#### (a) *Liquid Transport by Capillary Wicks*

Owing to capillary suction forces porous materials can function as pumps for pumping a liquid from one vessel to another. In the absence of the gravitational field the forces counteracting the capillary suction forces are inertial and friction forces.

The pressure drop  $\Delta p$  at the wick section  $\Delta x$  is equal to the Laplace capillary pressure:

$$\Delta p_\sigma = 2\sigma \left( \frac{1}{r'_1} - \frac{1}{r'_2} \right) \quad (5-8-1)$$

where  $r'_1$  and  $r'_2$  are the meniscus radii at the inlet and outlet of the section  $\Delta x$ .

We shall assume that this pressure drop is balanced by friction forces  $\Delta p = \Delta p_\sigma$ .

If the Darcy filtration law is used, then

$$\frac{\Delta p}{\Delta x} = \frac{\eta \bar{v}}{K} = \frac{2\sigma}{\Delta x} \left( \frac{1}{r'_1} - \frac{1}{r'_2} \right) \quad (5-8-2)$$

Assuming that  $\frac{1}{r'_2} \rightarrow 0$  ( $r'_2 \rightarrow \infty$ ) for a bulk liquid flow ( $\text{cm}^3/\text{s}$ ) we get the following expression

$$\dot{V} = A\bar{v} = \frac{2\sigma KA}{\eta r'_1 \Delta x} \quad (5-8-3)$$

where  $A$  is the surface area of capillary-porous material.

Thus, the quantity  $K/r_1$  is a parameter of capillary pump. The meniscus radius  $r'_1$ , as will be shown in section 5-9, is equal to

$$r'_1 = \frac{2\sigma}{\rho g l_{\max}} \quad (5-8-4)$$

where  $g$  is the acceleration due to gravity, and  $l_{\max}$  is the capillary lift level.

Hence

$$\dot{V} = \left( \frac{\rho g}{\eta} \right) \left( \frac{A}{\Delta x} \right) l_m K \quad (5-8-5)$$

Quantity  $l_m K$  may be considered as a parameter characterizing liquid transferability of a capillary pump.

At present, porous materials with conical capillaries are widely used. For example, an original system is used for the supply of liquid in ion

engines which have no moving parts, and is applicable for operation under condition of weightlessness (Electrooptical Company, USA). In this system the supply of liquid from the tank to the heated evaporator

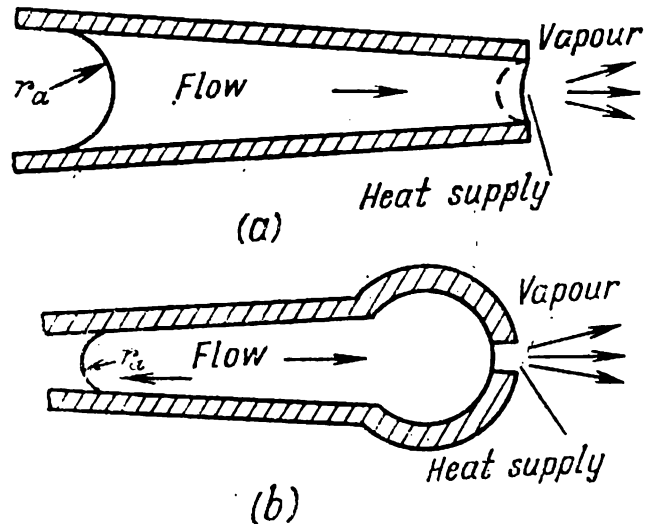


Fig. 5-39. Schematic of a capillary supply system based on liquid surface tension  
(a) wetting liquid; (b) nonwetting liquid.

surface is furnished by surface tension forces. The schematic of a capillary supply system based on liquid surface tension is shown in Fig. 5-39. Evaporation occurs at that part of the heated tube where the liquid meets a membrane. For nonwetting liquids (say, mercury) a very fine metal net may be used as the membrane.

For the supply of cesium Electrooptical developed two systems based on surface tension forces. In one of them the tank is filled with thin porous metallic discs arranged closely together. To obtain the required taper, different pore size discs are taken and arranged in a manner so that the pore size gradually decreases in the direction towards the evaporator (Fig. 5-40).

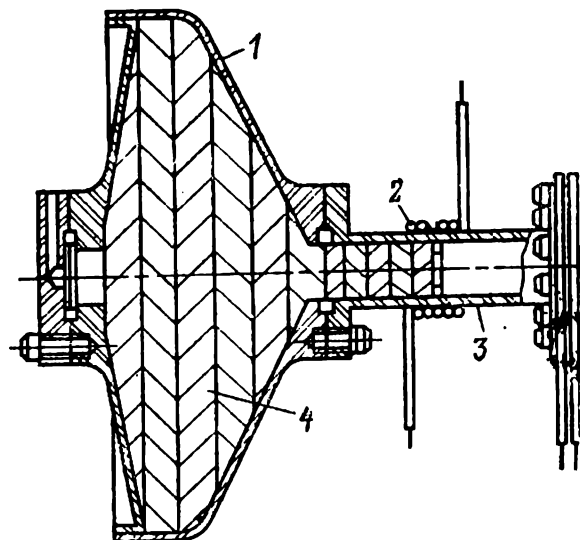


Fig. 5-40. Schematic section of a supply system with porous discs  
1 — jacket; 2 — heater; 3 — evaporator; 4 — porous discs

In the second design converging plates are used (Fig. 5-41). At the centre of the device a rod is mounted which acts as a wick and supplies liquid (cesium) directly to the surface where it evaporates.

The application of a conical porous structure for increasing the heat transfer coefficient both in space and in the conditions of weightlessness is of interest.

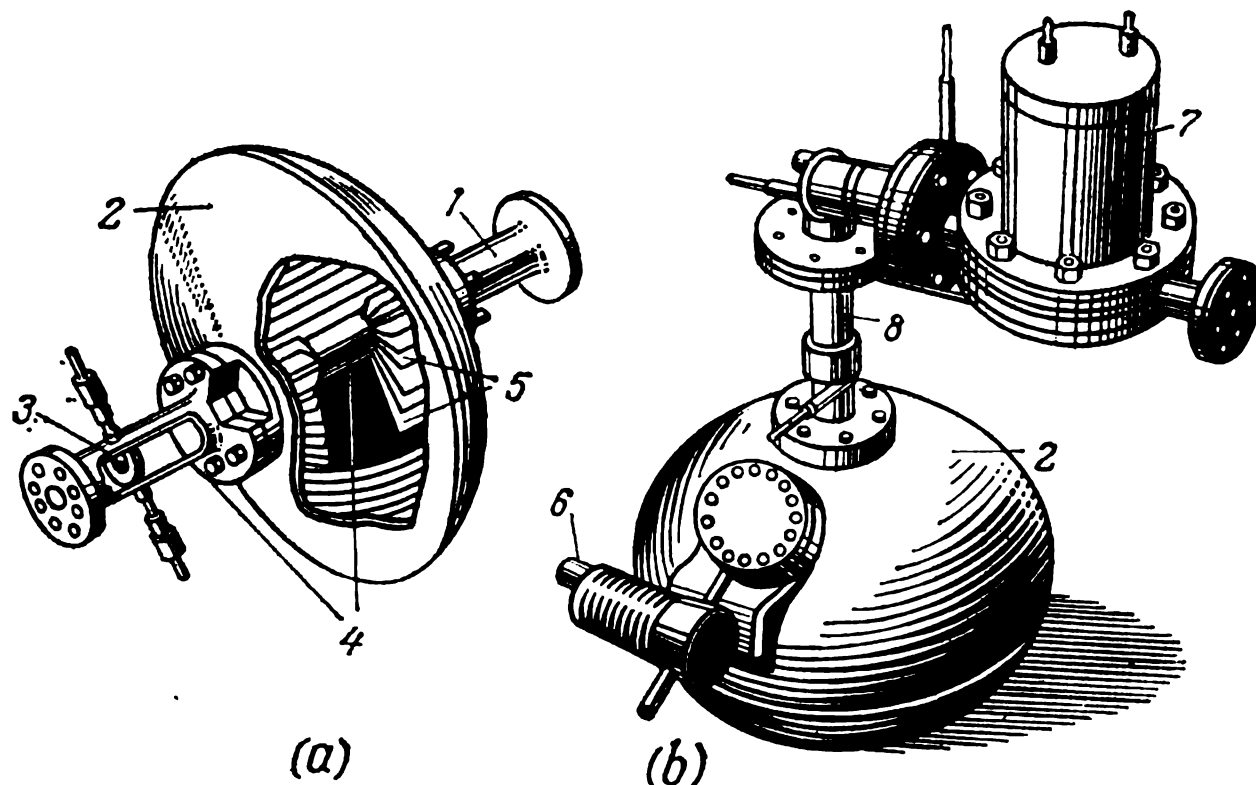


Fig. 5-41. Supply devices with porous plates

(a) section of the tank containing working fluid; (b) general view; 1 — supply pipe; 2 — working fluid tank; 3 — internal heater; 4 — porous metallic shaft; 5 — radial plates; 6 — evacuation valve; 7 — vapour valve; 8 — evaporator

### (b) Heat Pipes

The main points of heat pipe operation may be described as follows: liquid evaporates out of a capillary-porous wick. Vapour diffuses into a cold zone where it condenses. The condensate is transferred back by the same wick because during evaporation the meniscus curvature increases, whereas during condensation the curvature decreases. As a result liquid

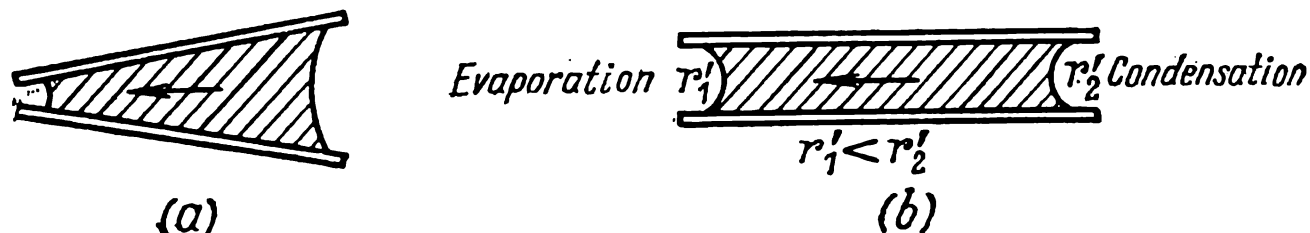


Fig. 5-42. Analogy between liquid motion through a conical (a) and cylindrical (b) capillary with evaporation and condensation

moves due to a capillary potential similar to liquid displacement in conical capillaries (Fig. 5-42). Therefore, to achieve better efficiency of the device conical capillaries are used in which liquid flow gets accelerated due to



meniscus curvature during evaporation and condensation. Such a device is shown in Fig. 5-43.

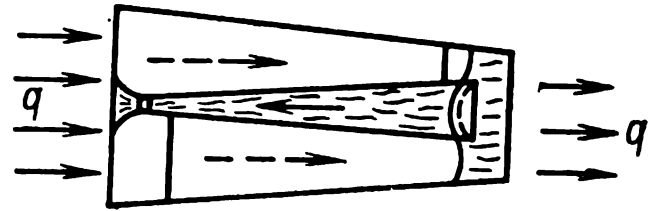


Fig. 5-43. Schematic of a heat pipe

### (c) Heat Exchangers Used in Space for Heat Disposal

A schematic diagram of a heat exchanger used in space for heat disposal is shown in Fig. 5-44. Layer 1 is made of water-repellent material with high porosity, having insignificant hydraulic resistance and low thermal conductivity, say of fluoroplastic with pores of 0.1-0.5 mm (thermal conductivity of such a material is less than that of the liquid).

Layer 2 is made of water-retaining material with high thermal conductivity of the solid framework (higher than that of the liquid). The heat to be discharged into vacuum in order to cool the liquid is supplied to the inner surface of the heat exchanger.

At the interface of two porous materials and somewhat above this interface, evaporation or sublimation of the coolant occurs accompanied by heat absorption. Because of large evaporation surface area large heat fluxes may be taken off from a unit area of the porous material.

The evaporation rate of liquid or ice in vacuum is high enough therefore it depends not only on the mass diffusion velocity (in the case of molecular viscous flow) or effusion velocity (molecular evaporation) but also on the kinetic rate of phase transition.

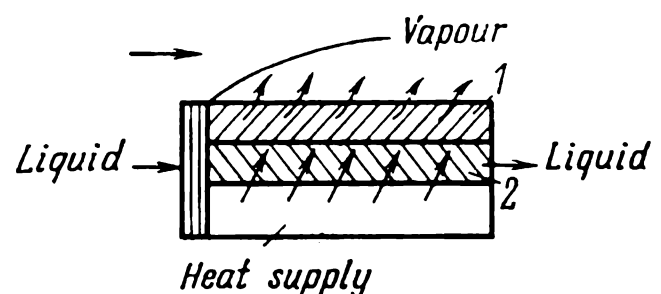


Fig. 5-44. Schematic of a heat exchanger used for the disposal of heat energy in space

If evaporation takes place at the surface, the evaporation rate (the mass flow density) may be determined by formula [5-80]:

$$j_m = K^* \bar{u} (C_w - C_\infty) \quad (5-8-6)$$

where  $C_w$  and  $C_\infty$  are the vapour concentrations at the surface and at a distance from it;  $\bar{u}$  is the mean thermal molecular velocity of vapour;  $K^*$  is the mass transfer coefficient

$$\frac{1}{K^*} = \frac{3}{\text{Nu}_D K_n} + \frac{4}{B}, \quad (5-8-7)$$

where  $B$  is the evaporation coefficient determined from the relations of molecular kinetic evaporation theory;  $Nu_D$  and  $Kn$  are the Nusselt diffusion and Knudsen numbers.

If evaporation takes place within a capillary-porous material, then the resistance to mass transfer  $1/K^*$  consists of the resistance defined by phase transition kinetics  $1/B\bar{u}$ , the resistance  $1/\beta$  of the diffusion boundary layer and the resistance  $\xi/K_c$  to the diffusion-filtration vapour motion through the capillaries of the material, i.e.

$$\frac{1}{K^*} = \frac{1}{\beta} + \frac{\xi}{K_c} + \frac{1}{B\bar{u}}, \quad (5-8-8)$$

where  $\beta$  is the mass-transfer coefficient;  $\xi$  is the distance of the evaporation surface from the material surface;  $K_c$  is the coefficient of vapour diffusion in the porous material based on the concentration difference.

Relations (5-8-7) and (5-8-8) hold for adiabatic evaporation. At non-adiabatic evaporation (internal heat supply) conditions are created for the boundary layer to get oversaturated, which are accompanied by the vapour condensation effect. Besides, at evaporation in a vacuum the material volume sharply increases (at a pressure of about 1 mm Hg column the volume at the vapour-ice transition increases by about  $10^6$  times). This effect of sharp volume increase gives rise to a rarefaction wave (similar to a shock wave). In this case the vapour pressure at the material surface is not equal to the saturation pressure at the given temperature [5-81].

The earlier considered heat exchanger may be used in space for heat disposal by evaporation or sublimation from a space suit or any other part with internal heat liberation.

This principle of cooling is also applicable for thermostating cryogenic liquids in space, say, for liquid oxygen subcooling.

### 5-9. TRANSFER EFFECTS UNDER CONDITIONS OF WEIGHTLESSNESS

To analyse transfer processes taking places in apparatuses and porous materials in space it is necessary, first of all, to know the behaviour of liquid under conditions of weightlessness. Heat transfer processes in liquids in the absence of gravity are of great importance.

The molecular interaction forces acting in liquids dominate under conditions of weightlessness.

#### *(a) Hydrostatics of Bulk Liquid at Reduced and Zero Gravity*

At normal gravity liquid poured into a vessel fills it and acquires the shape of the vessel. Gravity forces prevail over the inter-molecular

forces at the liquid surface (surface tension force), and the free liquid surface sets on a level (Fig. 5-45).

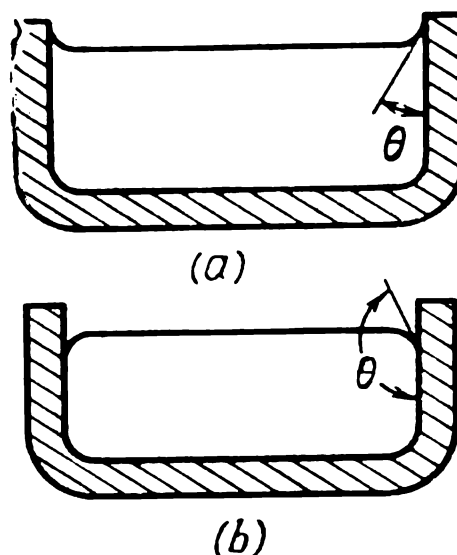


Fig. 5-45. Liquid equilibrium in a rectangular vessel under gravity and surface tension

(a) wetting liquid; (b) nonwetting liquid

In the state of weightlessness and at states close to it the general picture of liquid behaviour is quite different because of the variation in the ratio between surface tension and inertia forces. Estimation of this proportion is therefore necessary [5-82].

Benedickt [5-83] estimated the order of capillary forces (surface tension forces), acting on the liquid in a vessel of linear dimensions  $L$ , in terms of  $\sigma L$ , and the inertial forces acting on the liquid in terms of  $ng\rho L^3$ , where  $n$  is the load factor,  $g$  is the acceleration due to gravity. Then the dimensionless ratio of these two quantities

$$Bo = \frac{\sigma}{\rho ng L^2} \quad (5-9-1)$$

is a measure of relative contribution of capillary forces in the accelerated liquid and is called the Bond number. From this ratio it follows that surface tension forces start playing significant role at  $Bo \gg 1$ , i.e. in the case of liquids, subjected to acceleration corresponding to  $n < n_*$ , where

$$n_* = \frac{\sigma}{\rho g L^2} \quad (5-9-2)$$

To verify the above said, Benedickt theoretically considered the statistical behaviour of liquid contained in a vessel with a rectangular cross-section (see Fig. 5-45), with vertical side walls infinite in the direction normal to the cross-section.

The calculated shapes of menisci for water ( $\rho = 1 \text{ g/cm}^3$ ,  $\sigma = 72.75 \text{ dyn/cm}$ ,  $\theta = 14^\circ\text{C}$ ) and mercury ( $\rho = 13.546 \text{ g/cm}^3$ ,  $\sigma = 513.0 \text{ dyn/cm}$ ,  $\theta = 40^\circ\text{C}$ ) are shown in Fig. 5-46.

As is seen from Fig. 5-46 with the decrease of load factor the contribution of surface tension forces increases and at  $n = 0$  the surface becomes

spherical. But the author has not reported any data on the effect of load factor on the contact angle.

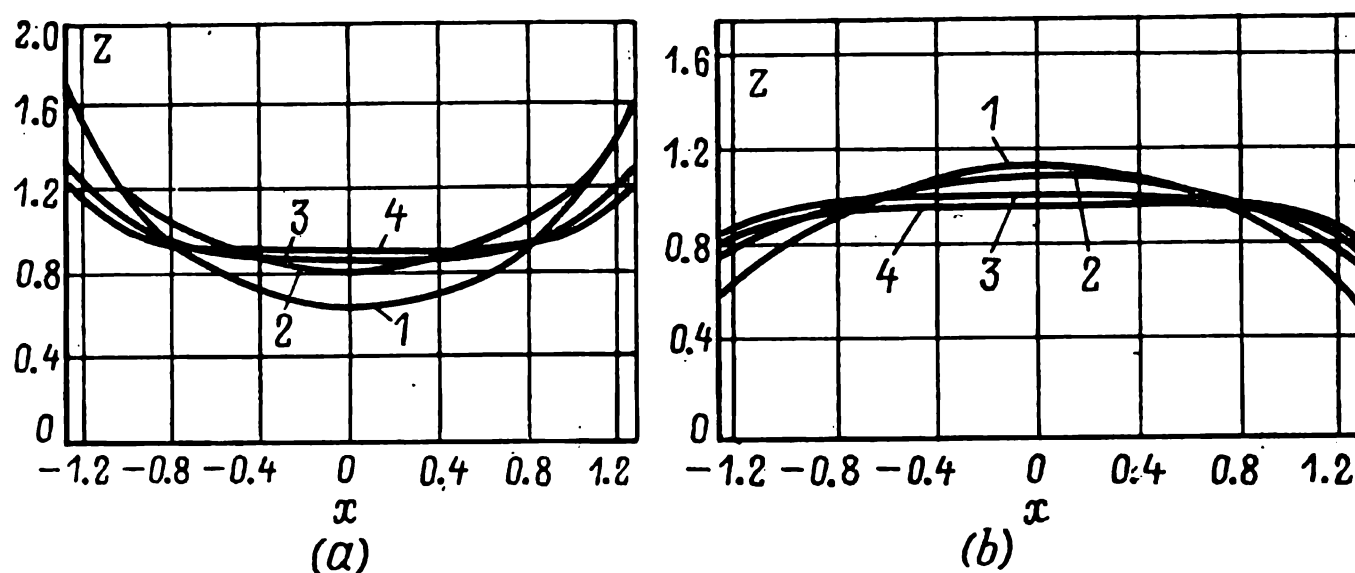


Fig. 5-46. Water (a) and mercury (b) menisci at different load factors  
 $1 - n = 0$ ;  $2 - n = 0.05$ ;  $3 - n = 0.5$ ;  $4 - n = 10$ .

The early experiments, described in literature, on the observation of liquid surface shape at reduced gravity were made by Reynolds [5-82], but they were quite qualitative and provisional. Based on these experiments a conclusion was drawn that liquids wetting the vessel would spread over the walls in a way to form a gas pocket in the centre. If the liquids do not wet the walls, then they quickly retract from the walls, and the gas continues to be in contact with the walls. But wetting liquids cannot spread infinitely: the liquid surface area is restricted by the amount of energy which may be spent on liquid stretching, and for this reason the wetting liquid cannot cover completely the vessel walls. In a similar way non-wetting liquid may tend to form more than one gas globules. The amount of energy required to separate from the bottom may prove to be insufficient to lift the centroid to the level necessary for the formation of a sphere. In this case, in the final shape of the liquid two or more spheres participate [5-82].

Statical shapes of the meniscus for the case of normal gravity and weightlessness in a spherical vessel are reported in [5-82] where it is shown that in the state of weightlessness the liquid surface should necessarily be spherical, and the radius of this sphere is completely defined by the contact angle at the interface, the vessel shape and the amount of liquid present in the vessel. The angle between the spherical liquid surface and the wall should be equal to the contact angle  $\theta$  which is always independent of the gravity field. Taking into consideration these boundary conditions a unique solution may only be obtained for liquid configuration in a spherical vessel, which is shown in Fig. 5-47.

Independence of the contact angle of the gravity conditions follows from V. V. Shuleikin's works [5-84]. The author considers changes in the

shape of water meniscus in a glass (of radius  $R$ ) at different values of gravity and at weightlessness. The general equation of equilibrium of water surface

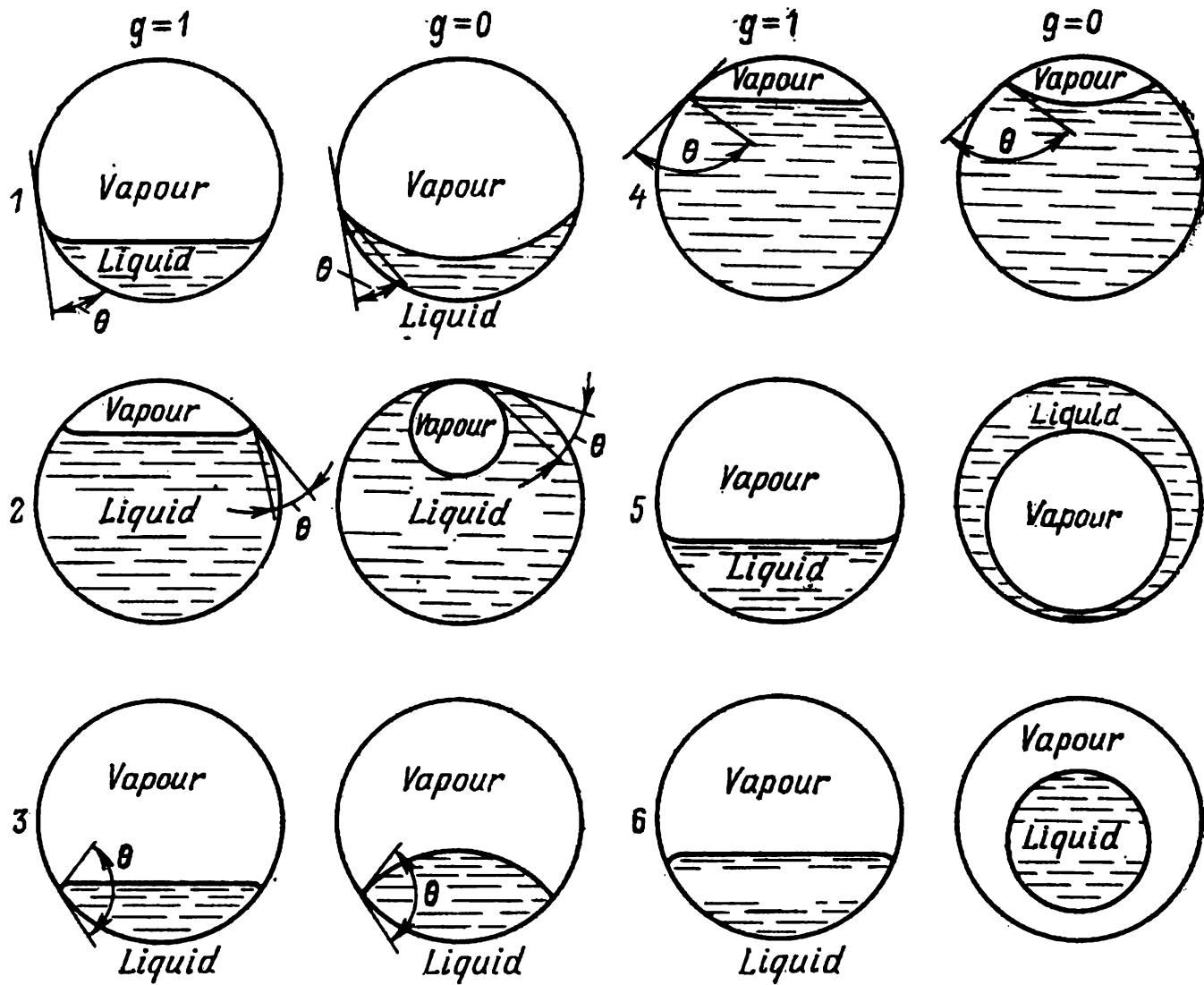


Fig. 5-47. Liquid-vapour system at normal gravity  $g = 1$  and at weightlessness  $g = 0$   
1, 2, 4, 5 — wetting liquid; 3, 6 — nonwetting liquid

with the major curvature radii  $r_1$  and  $r_2$ , given the surface tension  $\sigma$ , is taken as:

$$\rho g_1 z = \sigma \left( \frac{1}{r_1} + \frac{1}{r_2} \right) \quad (5-9-3)$$

where  $g_1 < g$ .

Assuming that

$$\frac{z}{h} = s; \quad \frac{r_1}{h} = \rho_1; \quad \frac{r_2}{h} = \rho_2$$

where  $h^2 = \frac{2\sigma}{\rho g_1}$ , equation (5-9-3) may be written as

$$s = \frac{1}{2} \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right) \quad (5-9-4)$$

Equation (5-9-4) is valid both for terrestrial conditions (when  $g_1 = g$ ) and at a small but finite positive  $g_1$ . From equation (5-9-4) differential equations of the surface of revolution, bounded between air and water, may be obtained. If we denote by  $x$  the distance from a certain point of this surface to the coordinate axis which serves as the axis of revolution of the contour and by  $\varphi$  the angle between the axis and the normal at this point of the surface, after certain transformations we obtain from (5-9-4) a set of equations

$$\frac{ds}{d\varphi} = \rho_1 \sin \varphi; \quad \frac{d\xi}{d\varphi} = \rho_1 \cos \varphi; \quad \frac{1}{\rho_1} = 2s - \frac{\sin \varphi}{\xi} \quad (5-9-5)$$

where  $\xi = \frac{x}{h}$ .

The initial profile  $a$  (Fig. 5-48) was obtained by integrating the system of equations (5-9-5) at the following boundary conditions: glass is perfectly wetted by water and  $R/h \gg 1$  at  $g_1 = g$ . The rest of the curves were obtained by approximate integration methods. The limit profile at complete weightlessness was plotted with the assumption of constant pressure at all parts of the free surface in the absence of external forces

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} = \text{const}$$

This equation is satisfied by a single surface which simultaneously satisfies the condition at the glass-liquid interface; spherical surface of radius  $R$  touching the cylinder (curve  $e$ , Fig. 5-48).

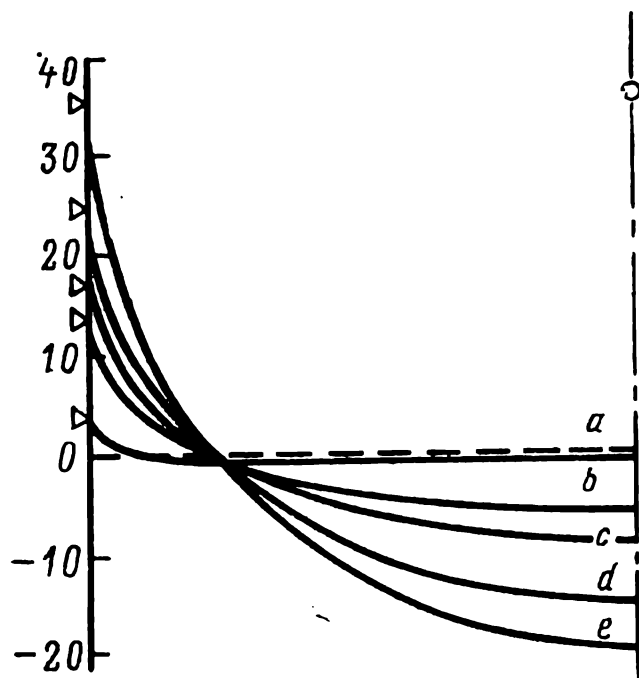
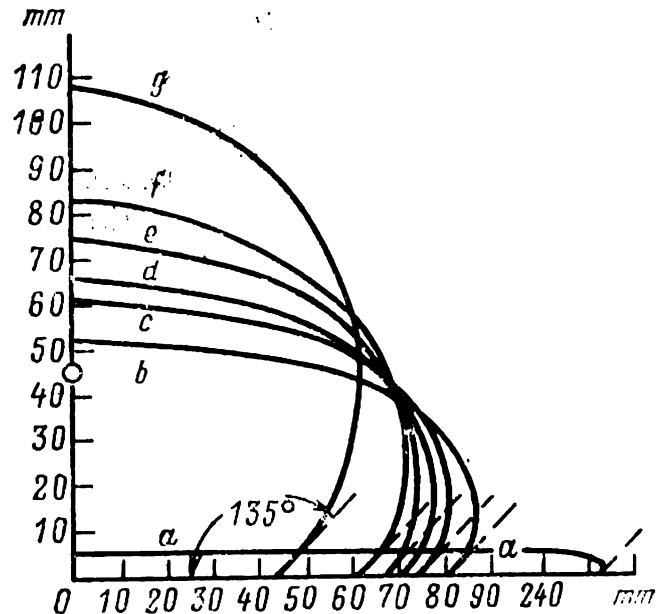


Fig. 5-48. Static profiles of water surface, appearing in a glass of 110 mm diameter, at vanishing gravity

In [5-84] statical profile of a water droplet meniscus in contact with a water-repellent surface of zinc stearite ( $\theta = 135^\circ$ ) have been calculated as

gravity decreases to zero. The profiles are shown in Fig. 5-49. Curve  $g$  corresponds to the case when  $g = 0$ . The results were experimentally verified. A very interesting fact should be mentioned: if a certain amount

Fig. 5-49. Right-hand sides of meridional sections of water surface over zinc stearite (curves for weightlessness) (a) initial position; (b) through (f) at different gravities; (g) complete weightlessness [5-84]



of liquid (wetting liquid) is present in a closed cylindrical vessel (Fig. 5-50), then the vessel surface not in contact with macroscopic liquid layers gets covered by a microscopic film of the liquid. Development of the film is fairly facilitated by weightlessness, i.e. at  $ng = 0$ .

In [5-85] the following expression for the thickness of this film is given

$$\Delta = \left( \frac{-Ar_0}{\sigma} \right)^{1/4} \quad (5-9-6)$$

where  $r_0$  is the radius of curvature of vessel walls and  $A$  is a constant.

To find the order of magnitude of  $\Delta$ , in this work the thickness of the film was calculated for liquid oxygen ( $A = -9.8 \times 10^{-20}$  dyn/cm<sup>2</sup>) at  $r_0 = 20$  cm:

$$\Delta \approx 2 \times 10^{-5} \text{ cm}$$

At appreciable gravity the film thickness is less by an order.

The estimation of thickness  $\Delta$  order, presented in [5-85], agrees with that obtained by a different method in [5-86]. The authors calculated for



Fig. 5-50. Wetting liquid in a closed cylindrical vessel under weightlessness

liquid oxygen (90 K) and liquid hydrogen (20 K) the thickness of microscopic film, which was of the order of  $7.5 \times 10^{-5}$  cm. No experimental data on the measurement of film thickness are available, since for conducting

such an experiment it is necessary to “maintain at constant temperature” the tank containing cryogenic liquid with an accuracy of  $\pm 10^{-8}$  K for oxygen and  $\pm 10^{-9}$  K for hydrogen [5-86].

*(b) Hydrodynamics of the Bulk Liquid Behaviour at Vanishing Gravity*

Earlier we considered the static shapes of liquid menisci. But the shape of liquid surface, which is present at the spacecraft board or in a fuel tank, at any moment is a function of many quantities, including the interface energies, tank shapes, filling coefficient, mechanical disturbances, etc. Thus, for example, if the liquid present in a certain volume is disturbed from the state of equilibrium, then surface tension forces will tend to bring the liquid to an equilibrium space configuration, but the configuration will not attain equilibrium until the excess free surface energy, compared to the equilibrium state energy, is spent [5-86].

In [5-82] are described the liquid surface oscillations during transition to the state of weightlessness. The oscillation period  $T$  of the liquid surface is defined by

$$T = \sqrt{\frac{3}{4} \pi \frac{m}{\sigma}}$$

where  $m$  is the liquid droplet mass.

Existence of the oscillations is verified by experiments, although hydrodynamics of the effect has not been studied adequately.

In a general case, for hydrodynamic systems three critical dimensionless parameters exist, which indicate whether the flow will be controlled mainly by gravitational, inertia or capillary forces (surface tension). These three quantities, which characterize different flow patterns, are the Reynolds number  $Re$ , the Bond number  $Bo$  and the Weber number  $We$ . One more important parameter, which may be obtained by combining  $Bo$  and  $We$ , is the Froude number  $Fr$ .

In accordance with the definitions of  $Re$ ,  $Bo$  and  $We$  different flow patterns are possible when the (1) inertial forces; (2) capillary forces and (3) gravity forces dominate (see Fig. 5-13).

Using these dimensionless parameters we shall review some works dealing with fluid dynamics under weightlessness and reduced gravity. Thus, for example, in [5-87] a theoretical analysis is made of an axisymmetric fluid flow in a conical tank, which is formed by a circular cone (Fig. 5-51) with a half expansion angle  $\alpha$  and a sphere of unit radius. The flow is induced by axial acceleration applied to the tank, which is characterized by the overload  $n(\tau)$ .

The liquid volume was prescribed by the ordinate  $h$  of the level plane corresponding to the liquid surface in the absence of surface tension. The liquid was assumed to be ideal and the flow was assumed to be vortex-free. The obtained profiles are depicted in Fig. 5-52.



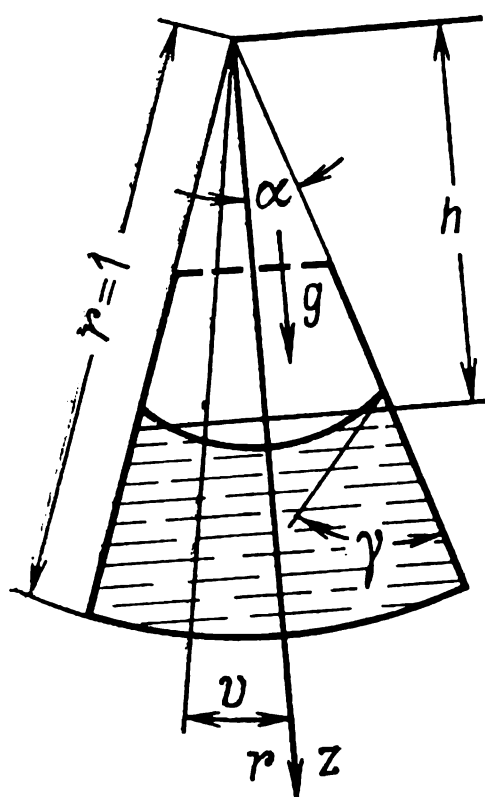


Fig. 5-51. Calculation of liquid motion in a circular cone [5-87]

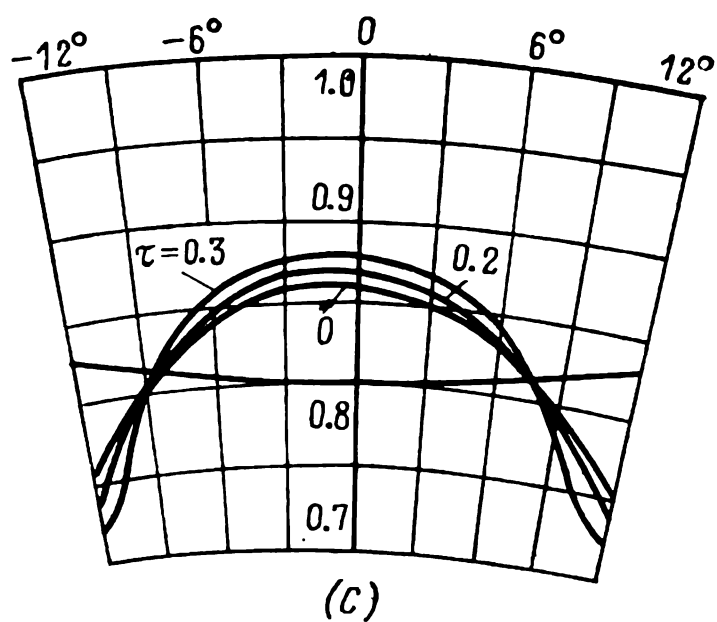
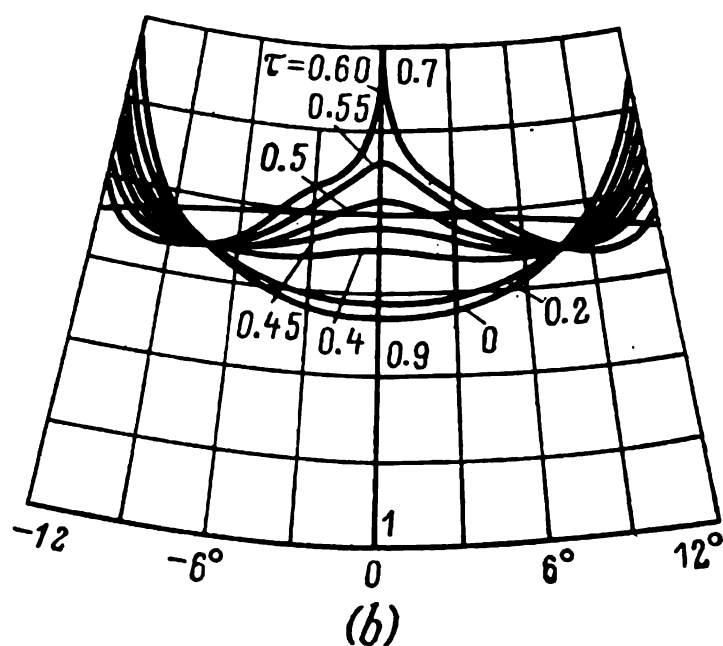
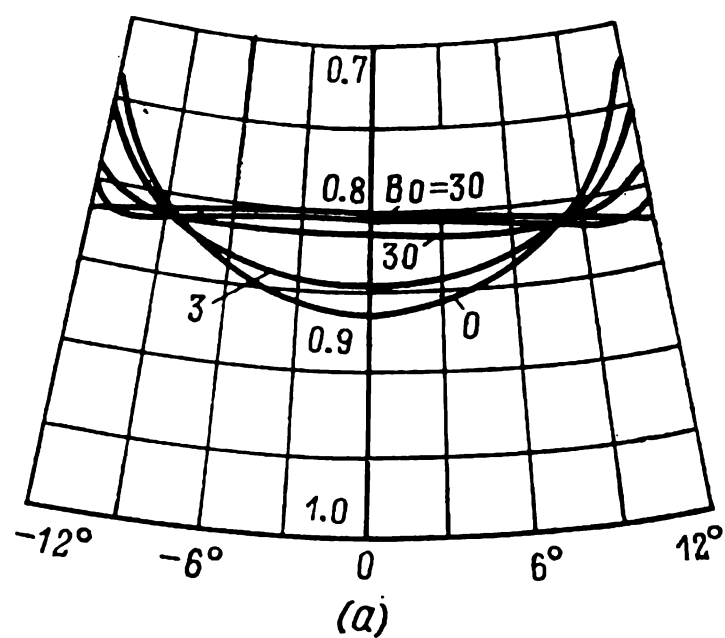


Fig. 5-52. Free surface shapes of liquid in a conical vessel ( $\alpha = 12^\circ$ )

(a) equilibrium state,  $\gamma = 5^\circ$ , for varying  $Bo$  from 0 to 300; (b) successive shapes,  $\gamma = 5^\circ$ :  $Bo = 1$ ,  $n = -0.5$  (negative overload) for times  $\tau = 0$  to 0.6; (c) positive overload ( $n = 1$ ) at  $Bo = 0$  for times  $\tau = 0$  to 0.3

The author concludes that the liquid excursion height and velocity increase as the Bond number decreases, that is with large curvature of the initial surface. A fair agreement between experimental and calculated values is found.

A more general problem has been solved in [5-88] where the shape of a free liquid surface was investigated with the assumption that it was an axisymmetric velocity distribution. The results of solutions in dimensionless quantities are plotted in Fig. 5-53.

The authors have come to the conclusions which to some extent are extended to the results of [5-87]:

(1) the surface velocity has a great effect on the free liquid surface shape and the meniscus height;

(2) the relative effect of  $We$  on the meniscus height decreases with  $Bo$ , i.e. the relative effect of velocity on the meniscus shape decreases as the gravity increases;

(3) for the selected distribution of velocity the meniscus height increases with  $We$  in most cases.

### c) Capillary Liquid Behaviour under Weightlessness

The behaviour of capillary liquid under weightlessness is of great interest, since, as it follows from relation (5-9-1), the surface force effect increases due to simultaneous decrease of  $ng$  and  $h^2$ . Therefore, the phenomena observed in capillaries at normal gravity should increase in the state of weightlessness. The transfer processes in capillaries at  $g = 0$  may, apparently, be described by eliminating the gravity term from equations governing transfer processes in capillaries at normal gravity.

In [5-89], on the assumption that the liquid flow in a capillary is laminar, the following equation for capillary suction of liquid has been obtained

$$\frac{d^2 l}{d\tau^2} + \frac{1}{l} \left( \frac{dl}{d\tau} \right)^2 + \frac{8\eta}{r^2 \rho} \frac{dl}{d\tau} + g \sin \alpha - \frac{2\sigma \sin \theta}{r \rho l} = 0 \quad (5-9-7)$$

where  $l$  is the liquid column length;  $\alpha$  is the angle of inclination of the capillary axis to the horizontal plane;  $r$  is the capillary radius;  $\rho$  is the liquid density;  $\tau$  is the time. It is difficult to solve such an equation. Therefore, if we assume that the first and second terms are infinitesimal, the following expression for the velocity  $v$  of capillary lift is obtained:

$$v = \frac{dl}{d\tau} = \frac{r^2 \rho}{8\eta} \left( \frac{2\sigma \cos \theta}{r \rho l} - g \sin \alpha \right) \quad (5-9-8)$$

and for the maximum lift

$$h_{\max} = l_{\max} \sin \alpha = \frac{2\sigma \cos \theta}{r \rho g} \quad (5-9-9)$$

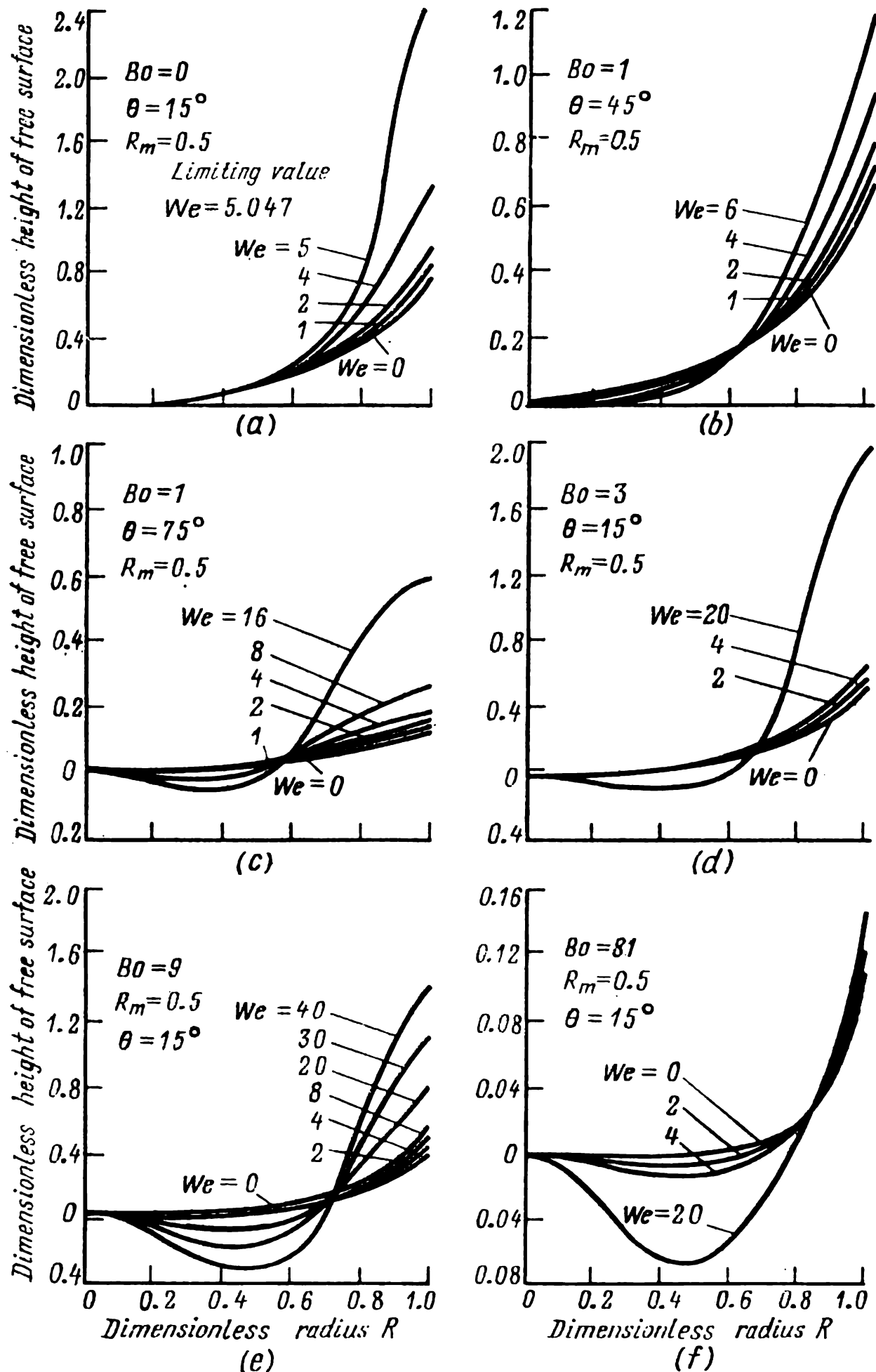


Fig. 5-53. Distribution of dimensionless height of a free surface at various  $Bo$ ,  $\theta$ ,  $R_m$  and Weber numbers  $We$

Thus, for the case  $g = 0$

$$v = \frac{dl}{d\tau} = \frac{r\sigma \cos \theta}{4\eta l} \quad (5-9-10)$$

$$h_{\max} = \infty \quad (5-9-11)$$

The solution of equation (5-9-7) for zero- $g$  is given in [5-90] at the initial conditions

$$\tau = 0; \quad l = h_0 + \frac{2\sigma \cos \theta}{r\rho g} = l_0; \quad \frac{dl}{d\tau} = 0 \quad (5-9-12)$$

in the form:

$$l^2 = \frac{1}{16} \frac{\sigma r^3}{\nu v^2} \cos \theta (e^{-\frac{8\nu\tau}{r^2}} - 1) + \left( \frac{1}{2} \frac{\sigma R}{\rho \nu} \cos \theta \right) \tau + l_0^2 \quad (5-9-13)$$

where  $h_0$  is the depth to which the tube is submerged in the liquid (Fig. 5-54);  $\nu$  is the kinematic viscosity,  $\nu = \eta/\rho$ .

For large times we may write

$$l = \left[ \left( \frac{1}{2} \frac{\sigma r}{\rho \nu} \cos \theta \right) \tau \right]^{1/2} \quad (5-9-14)$$

In [5-91] the capillary suction curves for conditions of weightlessness were calculated by formula (5-9-13). It was found that formula (5-9-13) gives exaggerated results which differ still further from experimental data for larger diameters of tubes ( $d > 2$  cm). The experiments were carried out with capillaries in the conditions of zero- $g$ . In Figure 5-55 photographs of one of the experiments are presented.

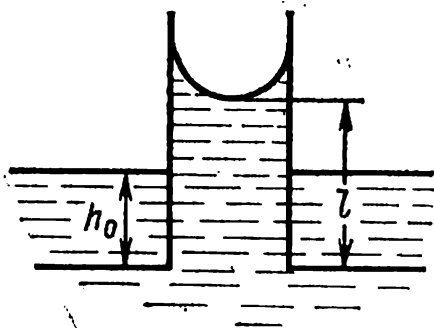


Fig. 5-54. Derivation of capillary liquid equations

To achieve better agreement with experimental data the inlet losses due to the variation in momentum, required for the development of velocity profiles, are accounted for.

The equation of motion

$$\frac{d^2 l}{d\tau^2} + \frac{1.14}{l} \left( \frac{dl}{d\tau} \right)^2 + \frac{8\eta}{r^2 \rho} \frac{dl}{d\tau} + \frac{g_0}{l} \left( \frac{2\sigma \cos \theta}{r\rho g} + hn \right) - ng_0 = 0 \quad (5-9-15)$$

differs from Porkhaev's equation [5-89] only by a factor at  $(dl/d\tau)^2$ . Hence, it may be considered that the solution for the zero- $g$  case in the form of (5-9-13) holds also for equation (5-9-15).

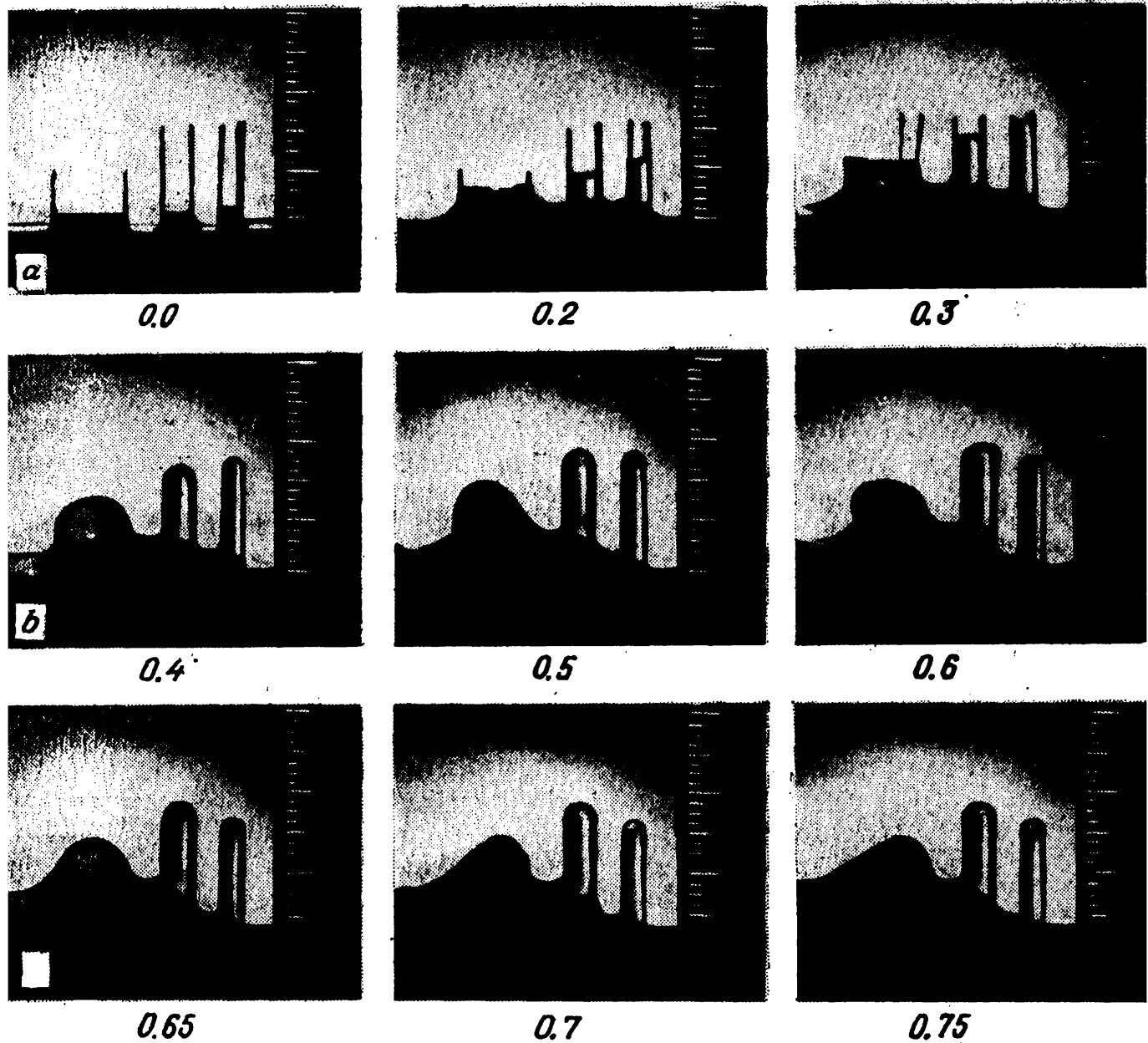


Fig. 5-55. Capillary lifts at normal gravity and under weightlessness at different times (sec)

Tube diameters: 0.196, 0.317 and 0.750 inches

Thus, the calculations made by equation (5-9-15) will also not agree with the experimental data. For this reason in [5-91] an attempt was made to account for turbulence effect, weakening of surface tension at unsteady state and the inertia of liquid in the tank. As a result was obtained the following equation

$$\begin{aligned} & \left(1 - 0.25 \frac{\pi r}{l}\right) \frac{d^2 l}{d\tau^2} - \frac{5}{l} \left(\frac{dl}{d\tau}\right)^2 - \frac{8\nu}{r^2} \frac{dl}{d\tau} \\ & + \frac{1}{l} \frac{2\sigma}{r\rho} \cos \theta \left(1 - 0.1 \frac{dl}{d\tau}\right) + \frac{g_0 h n}{l} - ng = 0 \end{aligned} \quad (5-9-16)$$

which gives values of capillary lift at zero- $g$  that coincide with experimental ones. It is quite difficult to solve this equation, but if the terms with  $d^2l/d\tau^2$  and  $(dl/d\tau)^2$  are neglected, we get the following formula for the capillary lift rate at zero- $g$

$$v = \frac{dl}{d\tau} = \frac{2\sigma r^2 \cos \theta}{0.2\sigma r^2 \cos \theta + 8\eta l r} \quad (5-9-17)$$

In formula (5-9-17) for the capillary suction rate the attenuation effect of the surface tension is accounted for. The account of this effect has not been taken in expression (5-9-10). From formula (5-9-13) we calculate the velocity

$$v = \frac{dl}{d\tau} = \frac{\sigma r}{4\eta l} \cos \theta (1 - e^{-\frac{8\eta r}{r^2}}) \quad (5-9-18)$$

These three formulas for the capillary suction rate differ very slightly, and at large times formulas (5-9-10) and (5-9-18) give similar results. From the analysis of formulas (5-9-10), (5-9-17) and (5-9-18) it follows that

$$(l_{\max})_{g=0} = \infty$$

It is difficult to judge about the real value of  $(l_{\max})_{g=0}$ , since very few experimental data on capillary suction at zero- $g$  are available. The theory gives such a result, apparently, for the reason that in the equation of fluid flow through capillaries all the dissipative processes have not been accounted for, which at normal gravity have no significance compared to the intensity of gravitational forces. It should be mentioned that capillary suction processes for very narrow capillaries (up to  $0.1 \mu\text{m}$ ) at normal gravity are described by [5-92]

$$\frac{dl}{d\tau} = \frac{\sigma r \cos \theta}{4\eta l} \quad (5-9-19)$$

This equation was verified many times. Equation (5-9-19) is similar to equation (5-9-10). Hence, the processes occurring in narrow capillaries at zero- $g$  and normal gravity are similar.

The behaviour of liquid in capillary-porous materials at zero- $g$  and reduced gravity will, probably, be similar to their behaviour in a single capillary, i.e. the capillary lift velocity is more significant than at normal gravity, and the capillary height tends to  $\infty$ . Therefore capillary-porous materials may serve as good carriers of liquid at zero- $g$ .

#### (d) Application of Liquid Properties at Zero- $g$

Since liquid behaviour at zero- $g$  is governed by intermolecular forces at the surface, the required behaviour of liquid at zero- $g$  may be ensured. One simple solution of this problem involves the use of conical vessels.

Let us consider a conical vessel with liquid (Fig. 5-56). In the case of a nonwetting liquid surface forces make the liquid mass to move towards the vessel end of larger cross-section, in the case of a wetting liquid — in the opposite direction.

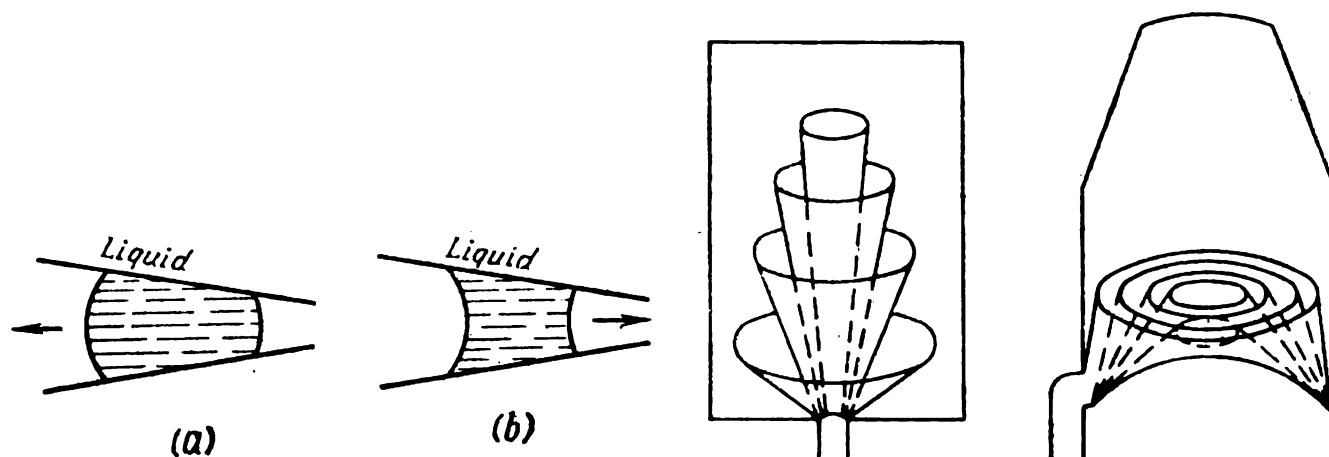


Fig. 5-56. Liquid under weightlessness in a capillary vessel

(a) nonwetting liquid; (b) wetting liquid

Fig. 5-57. Liquid collectors under weightlessness

Thus, any liquid may be moved or pumped as required. This phenomenon has already found application for the supply of working medium to ion rocket engines; developed by Electrooptical, and without rotating parts. These engines are quite useful for operation at zero-g.

This effect may probably be used for designing tanks and pipelines for spaceships. For example, in [5-86] the design of a nozzle, consisting of cones with a common apex at the tank outlet, has been proposed for collecting a wetting liquid. This nozzle will function as "a wick" for liquid. The liquid from the storage tank walls will move towards the centre. Two such nozzles are shown in Fig. 5-57.

#### (e) *Heat-Transfer Processes in Liquids at Zero-g and Reduced Gravity*

Absence of gravitational forces should certainly affect heat transfer at zero-g.

In the available literature such consideration has been given to heat transfer processes during building at zero-g and at reduced gravity, and also to the boiling process. The results of the studies of boiling processes, which are available in [5-82, 5-93 and 5-97], lead to the following: at zero-g the nucleate boiling period is very small and film boiling predominates. The burnout heat flux is proportional to  $(g_1/g)^{1/4}$ . The most exhaustive data are presented in [5-95] where heat transfer mechanism has been studied in a hydrogen tank at zero-g.

The following results are obtained at moderate but constant heat fluxes the heat-transfer process in a boiling liquid not subjected to disturbances at zero-g may be divided into 5 stages:

1. Due to heat conduction a thin thermal layer is developed near the walls, which grows and as a result a superheated liquid film is formed.

2. Due to heat fluxes from the walls and thermal diffusion in superheated liquid vaporization nuclei appear and grow, they are distributed more or less uniformly.

3. As the vapour bubbles grow the specific heat of evaporation starts playing ever increasing role.

4. In the process of growing bubbles start protruding the thermal layer and come in contact with the cold bulk liquid.

5. Vapour bubbles get deformed due to their expansion in the lateral direction and acquire the shape of spherical segments. Then they start uniting with each other and liquid particles lying between them take part in the process of evaporation. As a result of this the thermal resistance of the interlayer between the wall (the heat source) and the cold bulk liquid grows with time.

Thus, at zero-g heat is transferred by two modes, namely, molecular heat conduction and evaporation.

It has also been found that during nucleate boiling at zero-g or reduced gravity the bubble rise velocity decreases and the bubble diameter increases [5-82, 5-93, 5-95].

The heat transfer coefficient does not vary at developed nucleate boiling [5-82] and decreases with time at zero-g (at film boiling) [5-95].

At forced liquid motion the effect of weightlessness on the possibility of appearance of boiling crisis is different [5-94].

Interesting results are obtained on vapour condensation at zero-g but they are qualitative. For example, during the condensation of mercury vapours at zero-g condensate may be accumulated in a tubular condenser owing to liquid viscosity and vapour pressure [5-82]. A stable liquid-vapour interface may be maintained.

It is quite possible that evaporation processes from a free surface are not affected by the weightlessness in the same manner as zero-g does not affect the evaporation processes taking place within the bulk liquid at boiling [5-86].

No free convection at zero-g will be observed, but random convection may occur [5-82].

It should also be pointed out that as gravity decreases the thermal stability of liquid with respect to small disturbances increases and vice versa [5-96].

## 5-10. HEAT PIPES

The Grover heat pipe [5-98] is a self-contained engineering device having thermal conductivity much greater than that of a device made of a uniform piece of any available metal. This property is achieved by liquid evaporation within a closed jacket, by vapour transfer to the other part



of the vessel, by vapour condensation and by returning the condensate to the evaporator through a filtering device.

The advantages of heat pipes are best utilized in long and thin pipes, i.e. a long cylinder or elongated thin plane structure is taken. For illustration we consider a circular cylinder with a large length-to-diameter ratio. The procedure analysis may evidently be applied to different geometries, although it is not always possible. As may be seen from Fig. 5-58 such a heat pipe consists of a closed pipe of length  $l$  and external radius  $r_p$  and circular capillary structure impregnated with a wetting liquid, of external radius  $r_w$  and vapour space of radius  $r_v$ .

As heat is supplied to and removed from the heat pipe through the wall by ordinary heat conduction, the wall should be as thin as possible in order to minimize the radial difference in temperature, but it should naturally withstand the difference of external and internal pressures. Heat pipes become effective at internal pressures of the order of about a hundredth of one atmosphere, and their efficiency improves with pressure.

The wick designs may be different. The wick should not necessarily be placed at the inner surface of the vessel, although this is the best location. Since evaporation and condensation take place at the liquid-vapour interface, this location of the wick makes it possible to achieve necessary radial heat transfer through a medium of higher thermal conductivity, and thereby minimizes the radial difference in temperature. Therefore, the hydraulic diameter of the vapour space should be as large as possible to minimize the axial pressure gradient in the vapour flow. The wick may be made of twisted cloth, felt, slag and other similar materials or may even be made in the form of grooves on the jacket. The capillary structure is characterized

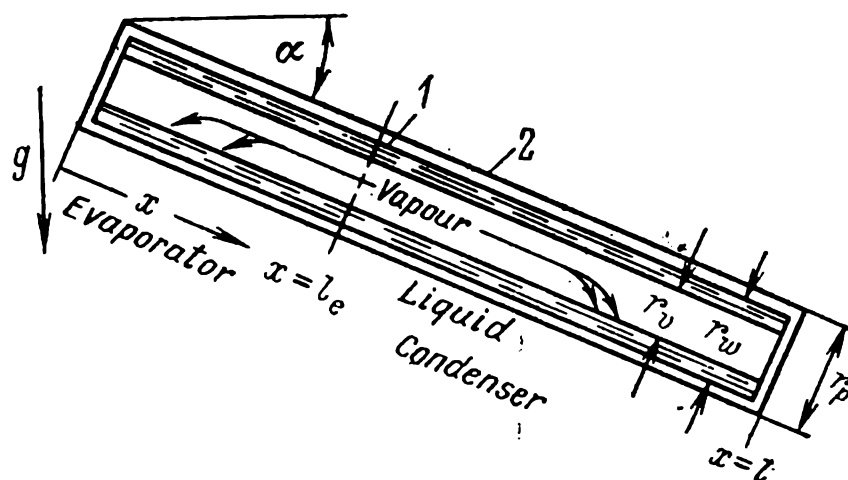


Fig. 5-58. Cylindrical heat pipe  
1 — capillary wick; 2 — outer jacket

by a mean pore radius, permeability and the liquid fraction volume. The wick material should be wetted by the working liquid; it is desirable that the jacket walls should also be wetted as it improves heat transfer. It is permitted to have a small excess of the liquid required for wick impregnation. Lack of liquid may lower the heat transfer maximum due to the

decrease in the effective volume of the wick in the heat pipe evaporation zone.

The following factors characterize the performance of heat pipes:

1. Mass (liquid) transfer in a capillary-porous wick, i.e. the work of capillary pump.

2. Heat removal by liquid evaporation from a capillary-porous material, i.e. actual heat flux in the evaporator.

3. Hydrodynamics of the process of mass transfer in a vapour phase from the evaporator to the condenser.

4. Heat transfer during vapour condensation at the porous surface and removal of heat by conduction through the wick and pipe wall.

Any of these factors may prove to be a limiting one, but in most practical cases the first two are bottle neck in the effective use of heat pipes.

In most studies on hydrodynamics and heat transfer in heat pipes the problems are solved approximately, without the account of interrelation between heat and mass transfer, although heat pipes represent typical heat transfer devices where heat and mass transfer processes are closely related. For this reason the problem of convective heat transfer is a conjugate one.

In a laminar flow of incompressible vapour within a heat pipe the conjugate problem is mathematically formulated as follows (see Fig. 5-59) [5-99]:

(a) equations of heat transfer within a heat pipe

$$\operatorname{div} \vec{v} = 0; \quad \rho_v \vec{v} (\vec{\nabla} \cdot \vec{v}) = - \vec{\nabla} p_v + \eta_v \nabla^2 \vec{v} \quad (5-10-1)$$

$$\rho_v c_v \vec{v} \vec{\nabla} T_v = \lambda_v \nabla^2 T_v \quad (5-10-2)$$

where subscript  $v$  refers to vapour;

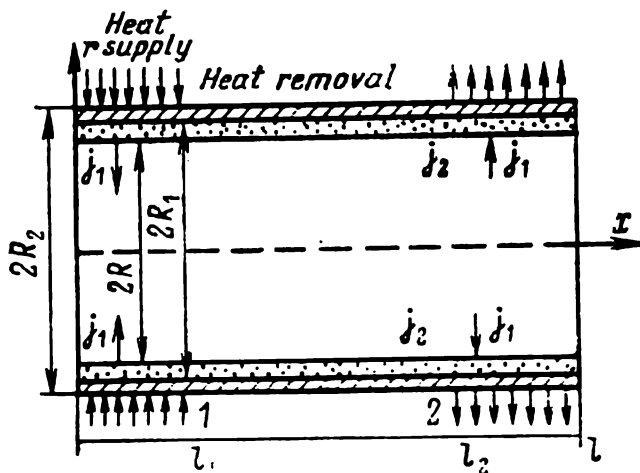


Fig. 5-59. Calculation of a cylindrical heat pipe

(b) equation of heat transfer in a capillary-porous jacket, when the pores are completely filled with condensate, may be written as:

$$c_f \rho_f w_f \vec{\nabla} T_f = \lambda' \nabla^2 T_f \quad (5-10-3)$$

where  $w_f$  is the fluid velocity; subscript  $f$  refers to fluid.

The system of equations (5-10-1) through (5-10-3) should be supplemented by an equation of heat conduction in the pipe envelope and by an equation of heat carrier filtration through capillary structure. The thickness of pipe casing ( $R_2 - R_1$ ) is considerably less than the capillary wick thickness ( $R_1 - R$ ), therefore redistribution of heat along the casing may be neglected. The relation between the heat fluxes  $q_{s2}$  and  $q_{s1}$  at the external and internal surfaces, respectively, is (see Fig. 5-59)

$$q_{s1} = q_{s2} R_2 / R_1 \quad (5-10-4)$$

As an approximation we shall assume that vapour or liquid (condensate) filtration takes place in different directions. For example, in a cylindrical pipe the liquid flows in the direction of  $x$  and vapour moves only along  $\vec{r}$ , i.e.

$$(j_v)_x = (j_f)_r = 0; \quad j_x = j_f; \quad j_r = (j_v)_r \quad (5-10-5)$$

Then the heat-transfer equations become

$$\frac{\partial}{\partial x} \left[ r \left( j_f h_2 - \lambda'_x \frac{\partial T}{\partial x} \right) \right] + \frac{\partial}{\partial r} \left[ r j_f - \lambda'_r \left( \frac{\partial T}{\partial r} \right) \right] = 0 \quad (5-10-6)$$

Here, the heat-conduction coefficients of the heat-transfer agent and the wick material are assumed to be close to each other and quite high because of the assumption of equal temperatures of the jacket and the fluid.

It is necessary to distinguish between evaporation and condensation zones, as both the regions have different conjugation conditions. The point is that the temperature of the vapour formed during evaporation differs from the liquid surface temperature. Therefore, for boundary conditions two relations are required, one for the evaporation rate, the other should relate liquid and vapour temperatures at the evaporation surface. Besides, it is necessary to consider that liquid in the porous wick evaporates only from the menisci surface. A correction factor  $\varepsilon$  is therefore introduced, which is the ratio of the evaporation surface area to the whole area of the wick through which vapour passes.

With due regard for these remarks, the boundary conditions take the form:

$$(v_x)_{x=0} = (v_r)_{x=0} = (v_x)_{x=l} = (v_r)_{x=l} = (v_x)_{r=R} = 0 \quad (5-10-7)$$

$$\left( \frac{\partial T_v}{\partial x} \right)_{x=0} = \left( \frac{\partial T_v}{\partial x} \right)_{x=l} = \left( \frac{\partial T_f}{\partial x} \right)_{x=0} = \left( \frac{\partial T_f}{\partial x} \right)_{x=l} = 0 \quad (5-10-8)$$

$$j_R L = -\lambda'_r \left( \frac{\partial T_f}{\partial r} \right)_{r=R}; \quad -\lambda'_r \left( \frac{\partial T_f}{\partial r} \right)_{r=R_1} = \frac{R_2 q_{s2}}{R_1} \quad (5-10-9)$$

$$j_R = \frac{2\alpha\varepsilon}{2 - \alpha} \left( \frac{\mu}{2\pi R^*} \right)^{1/2} \left[ \frac{p_v}{\sqrt{T_v}} - \frac{p(T_0)}{\sqrt{T_0}} \right]_{r=R} \quad (5-10-10)$$

at  $j_R < 0$

$$(T_v)_{r=R} = T_0 \left\{ 1 + \left( \frac{2\pi R^* T_0}{\mu} \right)^{1/2} \frac{j}{8\epsilon p(T_0) [\alpha + \beta(1 - \alpha)]} \right\}_{r=R} \quad (5-10-11)$$

at  $j_R > 0$

$$\left( \frac{\partial T_v}{\partial r} \right)_{r=R} = 0 \quad (5-10-12)$$

at  $j_R = 0$

$$\lambda'_r \left( \frac{\partial T_2}{\partial r} \right)_{r=R} = \lambda_v \left( \frac{\partial T_1}{\partial r} \right)_{r=R} \quad (5-10-13)$$

The necessary conditions of heat pipe performance are:

(1) the limiting capillary pressure  $p_{c \max}$  of the wick with cylindrical pores of radius  $r_c$  should be

$$p_{c \max} = \frac{2\sigma \cos \theta}{r_c} - \frac{\sigma}{R} \quad (5-10-14)$$

(2) conservation of the momentum component normal to the evaporation or condensation surface

$$p_v + \rho_v v_r^2 = p_f + \rho_f w_r^2 + \sigma K \quad (5-10-15)$$

where  $K$  is the surface curvature.

Using relation (5-10-15) for any two points 1 and 2 of the evaporation and condensation zones, we may write (Fig. 5-59)

$$\begin{aligned} p_{v1} - p_{v2} - (p_{f1} - p_{f2}) + \rho_{v1} v_{r1}^2 - \rho_{f1} w_{r1}^2 \\ - (\rho_{v2} v_{r2}^2 - \rho_{f2} w_{r2}^2) = \sigma_1 K_1 - \sigma_2 K_2 \end{aligned} \quad (5-10-16)$$

This relation defines the capillary pressure necessary to achieve the prescribed conditions of vapour and liquid flows. On the other hand, at steady-state conditions the capillary pressure defined by formula (5-10-16) should not exceed  $p_{c \max}$ . Since relation (5-10-14) holds for any two points in the evaporation and condensation zones, it should also hold at those points where the left-hand side of formula (5-10-16) has a maximum. Thus the operating conditions for a heat pipe take the form:

$$\begin{aligned} \max \{ (p_{v1} - p_{v2}) - (p_{f1} - p_{f2}) + \rho_{v1} v_{r1}^2 - \rho_{f1} w_{r1}^2 \\ - (\rho_{v2} v_{r2}^2 - \rho_{f2} w_{r2}^2) \} \leq p_{c \max} \end{aligned} \quad (5-10-17)$$

This relation is the most general condition, which makes it possible to calculate a heat pipe and find its maximum heat-transferability. The calculation procedure is as follows:

(1) calculation of fluid flow through a capillary structure;

- (2) calculation of vapour flow in the heat pipe cavity;
- (3) determination of the maximum of the left-hand side of equation (5-10-16) as a function of two variables — coordinates of the first and second points and verification of conditions (5-10-17).

Calculation of the vapour flow is complicated. Vapour may be incompressible or compressible depending on the heat load of vapour, and the flow may be laminar or turbulent. The compressible vapour flow is accompanied by considerable pressure drops. Such working conditions are therefore avoided. No data on  $Re_{cr}$  (the critical Reynolds number in a pipe with injection and suction) are available. As a first approximation  $Re_{cr} = 1250$  may be taken. Then we calculate the Reynolds and Mach numbers based on the mean vapour velocity in the heat-protected zone from the formulas

$$Re = \frac{QR}{\pi R^2 L \eta_v}; \quad M = \frac{Q}{\pi R^2 L p \gamma} \sqrt{\gamma R^* T_v / \mu_v}; \quad \gamma = \frac{c_{pv}}{c_{vv}} \quad (5-10-18)$$

where  $R^*$  is the universal gas constant.

Then the laminar and turbulent flow regions may be distinguished (Fig. 5-59). Unlike the vapour flow, the calculation of filtration should include the gravity force. For this aim the quantity  $\rho_f g x \sin \alpha$  should be subtracted from  $p_f$ , liquid pressure, where  $\alpha$  is the pipe inclination angle to the horizontal plane.

For alkali metals  $Re_{rf} < 1$ , therefore the convective terms in the equation of motion may be neglected. For the calculation of liquid motion through a capillary structure the Darcy formula may be used.

Liquid pressure along the capillary tube may be expressed as

$$p_f(x) = p_f(0) - \frac{v_f}{K} \Gamma \int_0^x dx \int_0^x (j_r)_{r=R} dx \quad (5-10-19)$$

where  $K$  is the permeability,  $\Gamma$  is a factor which depends on the porous wick structure.

Capillary pressure depends on the pore radius at the point of location of meniscus. If a porous wick has different-size capillaries, then it corresponds to the case when coarse capillaries are partially drained and fine capillaries transfer liquid to the wick surface. Consequently, when a heat pipe functions under conditions of partially drained capillaries, the degree of wick filling with liquid varies along the pipe, hence the penetration factor also changes. In case of partial drainage heat from the wick is transferred under conditions of the evaporation surface recession (see Chapter 4).

Generally, the heat-conduction coefficients of the fluid and the frame of the wick are taken close to one another, that is consistent with the assumption of their equal temperatures.

Then

$$j_{fx} \frac{\partial h_f}{\partial x} - \frac{\partial}{\partial x} \left( \lambda'_x \frac{\partial T}{\partial x} \right) = - (h_v^0 - h_v) \\ \times \frac{1}{r} \frac{\partial r j_{vr}}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \lambda'_r \frac{\partial T}{\partial r} \right) - j_{vr} \frac{\partial h_v}{\partial r} \quad (5-10-20)$$

where  $h_v^0$  is the vaporization enthalpy at  $T_v^0$ :

$$T_v^0 = T + \frac{T^{3/2} \alpha}{4p(T) (2 - \alpha) [\alpha + \beta(1 - \alpha)]} \left[ \frac{p_v}{\sqrt{T_v}} - \frac{p(T)}{\sqrt{T}} \right] \quad (5-10-21)$$

The variation in the vapour flow along the axis  $r$  ( $\partial r j_{vr} / \partial r$ ) is caused by evaporation within the wick. If  $\varepsilon_1$  stands for the total area of the free surface of liquid in a unit volume of the wick, then we may write

$$\frac{\partial r j_{vr}}{\partial r} = - \frac{2\alpha}{2 - \alpha} \sqrt{\frac{\mu}{2\pi R^*}} \left[ \frac{p_v}{\sqrt{T_v}} - \frac{p(T)}{\sqrt{T}} \right] r \varepsilon_1$$

If  $\varepsilon_r^0$  denotes the porosity of drained pores for the wick cross-section normal to the axis  $r$ , then

$$j_{vr} \frac{\partial h_v}{\partial r} = \frac{2\lambda_v \text{Nu} f_1}{\alpha^2} (T - T_v) \varepsilon_r^0 \quad (5-10-22)$$

where  $f_1$  is a factor defined by the geometry of capillaries, i.e. by their tortuosity and porosity.

Equations (5-10-20)–(5-10-22) describe heat propagation through a capillary structure in partially drained regions. Filtration of the heat-transfer agent is described by the Darcy equation

$$\frac{dp_f}{dx} = - \frac{\nu_f}{K} j_{fx} \quad (5-10-23)$$

For the determination of liquid pressure along the wick it is necessary to prescribe its value at some point. During wick drainage surplus heat-transfer agent appears which is carried away by the vapour flow to the condensation zone. Therefore, at the minimum curvature  $x_{\min}$  point of the free surface holds the relation

$$(p_f)_{x_{\min}} = (p_v)_{x_{\min}} - \frac{\sigma}{R} \quad (5-10-24)$$

and  $x_{\min}$  is found from the condition  $\frac{d}{dx} [p_v - p_f]_{x_{\min}} = 0$ .

If in the condensation zone the equation has no root, then  $x_{\min} = l$ . Based on the above listed equations in [5-98] a sodium heat pipe was calcu-

lated for the following parameters: the wick hole radius was 0.1 mm, porosity 0.5, the condensation and accommodation coefficients  $\alpha = 0.1$ ;  $\beta = 0.1$ . The results are plotted in Fig. 5-60 for three temperatures at proportional change of each zone ( $l_1/l = 0.36$ ;  $l_2/l = 0.5$ ;  $R_1 = 1$  cm).

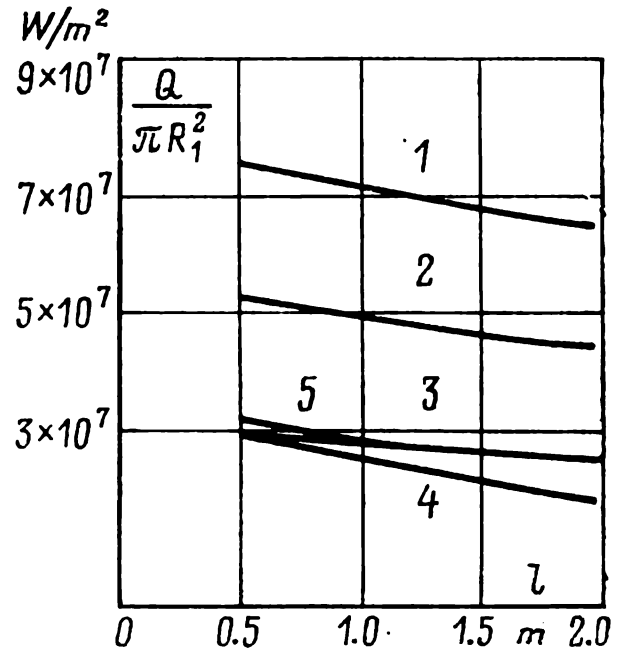


Fig. 5-60. Plot of limiting heat load ( $W/m^2$ ) versus pipe length  $l$  (heat-transfer agent — sodium,  $k_1 = 1$  cm);  
 $\alpha = 0$   
 1 — 117 K; 2 — 1075 K; 3 — 973 K; 4 — 973 K;  
 $\alpha = \frac{1}{2} \pi$ ; 5 — 973 K

When the heat pipe operates in the vertical position (curve 4)  $Q_{\max}$  increases slightly compared to the horizontal position of the pipe.

Concurrent with the considered method of calculation let us make a simplified calculation of a heat pipe. The theory of this calculation is available in the first edition of this handbook. Let us now consider the work of a heat pipe in the steady state by making the following assumptions:

1. The condenser area is much in excess of the evaporator area.
2. Heat flux, liquid and vapour temperatures are uniform over the length  $x_m$  of the condenser, in addition to this the vapour pressure  $p_v$  is constant.
3. Vapour condenses at the surface of the condenser and has a constant velocity  $v_v$  normal to the surface.
4. The porous wick is isotropic and incompressible. Then, we obtain a general integral of energy momentum equation in the form:

$$-\int_{R'(x=0)}^{R''(x=x_m)} 2\sigma \frac{dR}{R^2} - \int_0^{x_m} k \frac{q}{h_0 \delta} \frac{\eta_f}{\rho_f} x dx = \int_0^{x_m} \frac{2q^2 x dx}{g_0 h_0^2 \gamma_f \Pi^2 \delta^2} \quad (5-10-25)$$

where  $x_m$  is the condenser length;  $h_0 = h_v - h_f$ ;  $\gamma_f$  is the specific weight  $k = K^{-1}$ .

To integrate equation (5-10-25) it is necessary to know the integration limits.

In the heat-pipe condensation zone the vapour-liquid interface radius is large enough, and it may be assumed to be approaching infinity

( $R'' \rightarrow \infty$ ). In the liquid evaporation zone  $R'$  should be minimum. It may be estimated as  $R_{\min} = r_c \sec \theta$ , where  $r_c$  is the equivalent capillary radius. In practice  $R'$  may be found from the Juraine formula.

In the calculation of a heat pipe its maximum length is found at the prescribed heat flux, or, inversely, heat flux is determined at the prescribed length.

For the calculation of the maximum condenser length  $x_m$ , we assume that

at  $x = 0$ ,  $R'' \rightarrow \infty$

at  $x = x_m$ ,  $R' \rightarrow R_{\min}$

With these limits it is possible to integrate equation (5-10-25) and to solve it for  $x_m$ :

$$x_m = \left\{ \frac{\rho_f h_0 \sigma}{\eta_f} \left[ \frac{4\delta}{qkR_{\min} \left( 1 + \frac{2q}{\eta_f h_0 \delta \Pi^2 k} \right)} \right] \right\}^{1/2} \quad (5-10-26)$$

From formula (5-10-26), in its somewhat simplified form, we get for the maximum heat flux

$$Q_{\max} = 2N \left( \frac{b\delta}{x_m} \right) \left( \frac{g}{g_0} \frac{\gamma_f}{\sigma} \right) \left( \frac{l_{\max}}{k} \right) \quad (5-10-27)$$

It should be pointed out that the gravitation effect has not been taken into account in this formula. If this effect is accounted for, then equation (5-10-27) is written as:

$$Q_{\max} = \left[ 2N \left( \frac{b\delta}{x_m} \right) \left( \frac{g}{g_0} \frac{\gamma_f}{\sigma} \right) - \frac{2\rho_f h_0 \gamma_f}{\eta_f} \frac{g}{g_0} \left( \frac{b\delta}{l_{\max}} \right) \sin \theta \right] \frac{l_{\max}}{k}. \quad (5-10-28)$$

Fig. 5-61 shows the relation between the maximum pipe length and the diffused heat flux for different ratios  $l_{\max}/k$ , which is calculated by formula (5-10-27).

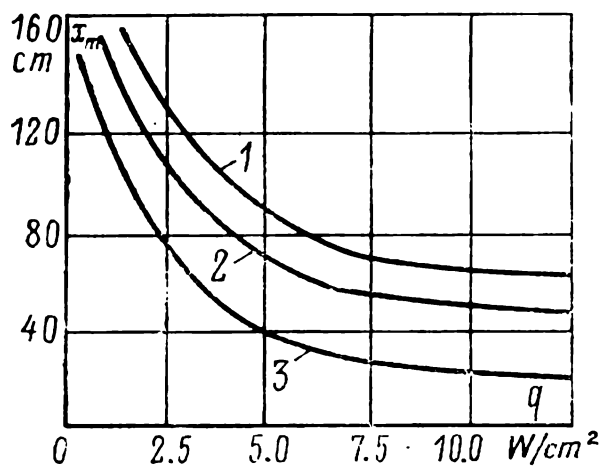


Fig. 5-61. Plot of condenser length  $x_m$  versus heat flux  $q$  for various  $l_{\max}/k$  (cm<sup>3</sup>) for water at  $\delta = 0.25$  cm and  $T = 350$  K  
 1 -  $l_{\max}/k = 1.57 \times 10^{-4}$ ; 2 -  $l_{\max}/k = 0.68 \times 10^{-4}$ ; 3 -  $l_{\max}/k = 0.13 \times 10^{-4}$

In the majority of heat pipes the areas of evaporation zones are smaller than those of the condensation zones, they do not affect the work of



the condenser until heat flux becomes critical. However in the devices designed to transfer small heat fluxes evaporator may be much longer than the condenser.

To conclude the chapter, it may be mentioned that for effective use of heat pipes in space the following two factors, affecting condenser operation, must be accounted for:

1. Heat flux at which apparent film boiling starts in the evaporator.
2. Maximum amount of fluid which can be pumped by the capillary-porous material into the evaporator.

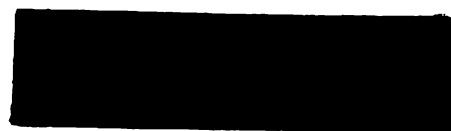
To estimate the efficiency of a capillary pump a simplified schematic was analysed. According to this schematic the work of a capillary pump depends on parameters  $N$  and  $l_{\max}/k$ . Both the parameters should be as large as possible for satisfactory operation of the pump.

The ratio of the capillary force maximum to the friction force when the liquid moves throughout the wick is a parameter of the capillary pump.

Constant  $l_{\max}$  is a quantity characterizing the function of capillary forces in a porous wick. From the physical point of view  $l_{\max}$  is the highest level up to which a liquid can rise in a vertical capillary-porous material.

Constant  $k$  characterizes the effect of friction forces which appear when liquid flows through a capillary-porous material, when the Darcy law holds. This constant may be obtained by measuring pressure drop in the porous material through which liquid flows.

Thus, for the analysis of wick efficiency it is necessary to carry out separate experimental studies for the evaluation of  $l_{\max}$  and  $k$ .



## 6-1. DIFFERENTIAL HEAT- AND MASS-TRANSFER EQUATIONS

In the present chapter the results of recent works in the field of unsteady-state heat and mass transfer in capillary-porous and porous materials are presented. The obtained analytical solutions may be employed to calculate simultaneous heat and mass transfer in other systems. A large contribution to the development of methods for solving differential heat-and mass-transfer equations has been made by Yu. A. Mikhailov [6-1] and M. D. Mikhailov [6-2], who successfully continue their work in this field.

The following notations will be used in the foregoing: subscript 1 denotes vapour, 2 — liquid (water), 3 — ice, 4 — dry air with reduced molecular mass  $M = 29$ ; 0 — porous body skeleton.

As follows from the mass conservation law, the local derivative of volume concentration of the  $i$ -th bound material with respect to time is equal to the sum of divergence of density of flows of mass and the  $i$ -th material source due to phase transitions, i.e.

$$\frac{\partial(u_i \rho)}{\partial \tau} = - \operatorname{div} (\vec{j}_{i \text{ dif}} + \vec{j}_{i \text{ mol}}) + J_i \quad (6-1-1)$$

where  $\vec{j}_{i \text{ dif}}$  and  $\vec{j}_{i \text{ mol}}$  are the densities of diffusional and molar flows of the  $i$ -th bound material, respectively.

In accordance with relation (5-7-10), equation (6-1-1) may be rewritten as

$$\frac{\partial(u_i \rho)}{\partial \tau} = - \operatorname{div} (\vec{j}_{i \text{ dif}} + \Pi \rho_i b_i \vec{v}_i) + J_i \quad (6-1-2)$$

Here, it should be remembered that  $v_i$  is the rate of convective (molar) transfer; its physical nature is quite different from that of linear diffusional transfer rate  $\vec{v}_{i \text{ dif}} = \vec{j}_{i \text{ dif}} / \rho_i$ .

As sources of the  $i$ -th material are caused by phase transitions, their sum equals zero

$$\sum_i J_i = 0 \quad (6-1-3)$$

Neglecting the shrinkage of body ( $\rho = \rho_0 = \text{const}$ ) equation (6-1-2) may be rewritten as

$$\rho_0 \frac{\partial u_i}{\partial \tau} = - \text{div} (\vec{j}_{i \text{ dif}} + \Pi \rho_i b_i \vec{v}_i) + J_i \quad (6-1-4)$$

Differential equations of heat transfer may be obtained from the energy transfer equation: the local derivative of the volume energy concentration is equal to the divergence of density of the energy flow. Generally, for a solid the isochoric heat capacity  $c_v$  is assumed to be equal to isobaric heat capacity  $c_p$  ( $c_p = c_v = c$ ). Thus, for a capillary-porous body the local derivative of a volume enthalpy concentration with respect to time equals the divergence of density of enthalpy flux including enthalpy transfer due to convective and diffusional (molecular) motion

$$\frac{\partial}{\partial \tau} (h_0 \rho_0 + \sum h_i \rho_0 u_i) = - \text{div} \left[ \vec{j}_q + \sum_i h_i (\vec{j}_{i \text{ dif}} + \Pi \rho_i b_i \vec{v}_i) \right] \quad (6-1-5)$$

where  $\vec{j}_q$  is the density of heat conduction flux

$$\vec{j}_q = - \lambda \text{grad } T = - \lambda \vec{\nabla} T \quad (6-1-6)$$

Here,  $\lambda$  is the total heat-conduction coefficient which describes molecular heat transfer through the porous skeleton and the bound material.

If the specific heat capacity is denoted by

$$c_i = \frac{dh_i}{dT} \quad (6-1-7)$$

then

$$\begin{aligned} (c_0 \rho_0 + \sum_i c_i \rho_0 u_i) \frac{\partial T}{\partial \tau} + \rho_0 \sum_i h_i \frac{\partial u_i}{\partial \tau} = \text{div} (\lambda \vec{\nabla} T) - \sum_i c_i (\vec{j}_{i \text{ dif}} \\ + \Pi \rho_i b_i \vec{v}_i) \cdot \vec{\nabla} T - \sum_i h_i \text{div} (\vec{j}_{i \text{ dif}} + \Pi \rho_i b_i \vec{v}_i) \end{aligned} \quad (6-1-8)$$

If all the terms of equation (6-1-4) are multiplied by  $h_i$ , and summarized over all  $i$  ( $i = 0, 1, 2, 3, 4$ ) and the obtained relations are substituted for

$\sum_i h_i \rho_0 \frac{\partial u_i}{\partial \tau}$ , then

$$c \rho_0 \frac{\partial T}{\partial \tau} = \text{div} (\lambda \vec{\nabla} T) + \sum_i h_i J_i - \sum_i (\vec{j}_{i \text{ dif}} + \Pi \rho_i b_i \vec{v}_i) c_i \vec{\nabla} T \quad (6-1-9)$$

where  $c$  is the reduced specific heat capacity of the solid, equal to

$$c = c_0 + \sum_i c_i u_i \quad (6-1-10)$$

Equation (6-1-9) together with (6-1-4) represents the most general system of differential equations for mass and heat transfer in capillary-porous bodies.

### (a) Differential Equations of Filtration

Equations of liquid motion in porous materials may be obtained as a particular case of equation (6-1-4).

If it is assumed that  $\rho_0 u_i = \Pi b_i \rho_i$  (i.e. there is no physico-chemical bond with the porous body skeleton),  $J_i = 0$  (no mass source) and  $\vec{j}_{i\, dif} = 0$  (no diffusional transfer), then from equation (6-1-4) the equation for filtration of immiscible liquids ( $i = 1, 2, 3$ ) in a porous material is obtained

$$\Pi \frac{\partial(\rho_i b_i)}{\partial \tau} = - \operatorname{div} \vec{j}_{i\, mol} = - \operatorname{div} \left[ k \frac{k_i}{\eta_i} (\vec{\nabla} p - \rho \vec{\nabla} h) \rho_i \right] \quad (6-1-11)$$

The filtration flow density  $\vec{j}_{i\, mol}$  is found from the relation

$$\vec{j}_{i\, mol} = \Pi \rho_i b_i \vec{v}_i = \rho_i k \frac{k_i}{\eta_i} (\vec{\nabla} p - \rho \vec{\nabla} h) \quad (6-1-12)$$

where  $k$  is the total permeability and  $k_i$  is the relative permeability of the  $i$ -th phase.

### (b) Monocapillary Structure

In a monocapillary-porous body (with monocapillary structure) the liquid is transferred at a certain finite rate  $\vec{v}_{2\, cap}$  governed by capillary forces ( $\vec{v}_2 = \vec{v}_{2\, cap}$ ). In most cases the vapour and air within the pores are transferred by diffusion (interdiffusion of vapour and air).

Filtration transfer of vapour and air within the porous body may occur only at total pressure gradient  $[\operatorname{grad} p]$ . In case of liquid evaporation from the meniscus surfaces of capillaries, interdiffusion of vapour and air is known to give rise to convective transfer of humid air (the Stefan flow); this mode of transfer may, however, be accounted for by introducing a factor in the formula for vapour diffusion in a vapour-air mixture, as the linear velocity of the Stefan convective transfer  $v_k$  is directly proportional to vapour gradient concentration or the partial pressure gradient. A correction for diffusional slip is introduced in a similar way. Besides, under the effect of thermal slip humid air moves at a certain velocity, which is proportional to the temperature gradient. This mode of transfer is, therefore, accounted for by a thermodiffusional transfer term.

Thus, we assume that  $\vec{j}_{2dif} = 0$  (i.e. there is no liquid diffusion), then the system of differential mass-transfer equations takes the form

$$\rho_0 \frac{\partial u_1}{\partial \tau} = -\text{div} (\vec{j}_{1dif}) + J_1 \quad (6-1-13)$$

$$\rho_0 \frac{\partial u_2}{\partial \tau} = -\text{div} (\Pi \rho_2 b_2 \vec{v}_{2cap}) + J_2 \quad (6-1-14)$$

$$\rho_0 \frac{\partial u_4}{\partial \tau} = -\text{div} (\vec{j}_{4dif}) + J_4 \quad (6-1-15)$$

As in an inert gas (say, dry air) no chemical conversions occur,  $J_4 = 0$ ; phase transformations at temperatures above  $0^\circ\text{C}$  correspond to liquid-vapour transition.

Thus  $J_2 = -J_1$ . As the content of vapour and air in the pores ( $u_1 + u_4$ ) is very small compared to the moisture content present in liquid  $u_2$  ( $u = u_1 + u_2 + u_4 \approx u_2$ ), then the variation in the concentration of humid air in capillaries may be neglected ( $\rho_0 \partial u_1 / \partial \tau = 0$ ;  $\rho_0 \partial u_4 / \partial \tau = 0$ ).

Therefore

$$\Pi \rho_2 \frac{\partial b_2}{\partial \tau} - \text{div} (\Pi \rho_2 b_2 \vec{v}_{2cap}) - \text{div} \vec{j}_{1dif} \quad (6-1-16)$$

$$J_2 = -J_1 = -\text{div} \vec{j}_{1dif} \quad (6-1-17)$$

$$\text{div} \vec{j}_{4dif} = 0 \quad (6-1-18)$$

Then from equations (6-1-16) through (6-1-18) it follows that

$$\Pi \rho_2 \left( \frac{\partial b_2}{\partial \tau} + \vec{v}_{2cap} \vec{\nabla} b_2 \right) = -\Pi \rho_2 b_2 \text{div} \vec{v}_{2cap} - \text{div} \vec{j}_{1dif} \quad (6-1-19)$$

The capillary velocity  $\vec{v}_{2cap}$  depends on the mean capillary radius  $\bar{r}$ , surface tension  $\sigma$  and viscosity  $\eta_2$ .

The capillary velocity is defined by formula (5-7-23). For a one-dimensional problem differential equation (6-1-19) may, therefore, be written as

$$\left( \frac{\partial b_2}{\partial \tau} - \frac{B \bar{r}^2 \rho_2}{\eta_2} \frac{\psi}{x} \frac{\partial b_2}{\partial x} \right) = b_2 \frac{B \bar{r}^2 \rho_2}{\eta_2} \frac{\psi}{x^2} \quad (6-1-20)$$

where  $B$  is a constant.

Thus for a macrocapillary-porous body having monocapillary structure  $\text{div} \vec{v}_{2cap}$  is not zero.

*(c) Colloidal Capillary-Porous Materials*

Earlier it has been established that moisture transfer of vapourous ( $i = 1$ ) and liquid ( $i = 2$ ) moisture is described by the following relations

$$\vec{j}_i = a_{mi}\rho_0\vec{\nabla}u - a_{mi}^T\rho_0\vec{\nabla}T = -a_{mi}\rho_0(\vec{\nabla}u + \delta_i\vec{\nabla}T); \quad i = 1, 2 \quad (6-1-21)$$

and the total transfer of vapour and moisture is equal to

$$\vec{j} = \vec{j}_1 + \vec{j}_2 = -a_m\rho_0\vec{\nabla}u - a_m^T\rho_0\vec{\nabla}T = -a_m\rho_0(\vec{\nabla}u + \delta\vec{\nabla}T) \quad (6-1-22)$$

The system of differential mass-transfer equations will thus take the form\*

$$\rho_0 \frac{\partial u_1}{\partial \tau} = -\operatorname{div} \vec{j}_1 + J_1 \quad (6-1-23)$$

$$\rho_0 \frac{\partial u_2}{\partial \tau} = -\operatorname{div} \vec{j}_2 + J_2 \quad (6-1-24)$$

Summation of (6-1-23) and (6-1-24) yields

$$\rho_0 \frac{\partial u}{\partial \tau} = -\operatorname{div} \vec{j}_1 - \operatorname{div} \vec{j}_2 \quad (6-1-25)$$

Substitution of expressions for  $\vec{j}_1$  and  $\vec{j}_2$  gives

$$\rho_0 \frac{\partial u}{\partial \tau} = \operatorname{div} [a_{m1}\rho_0\vec{\nabla}u + a_{m1}^T\rho_0\vec{\nabla}T] + \operatorname{div} [a_{m2}\rho_0\vec{\nabla}u + a_{m2}^T\rho_0\vec{\nabla}T] \quad (6-1-26)$$

The differential heat transfer equation then becomes

$$c\rho_0 \frac{\partial T}{\partial \tau} = \operatorname{div} (\lambda\vec{\nabla}T) + r_{12}J_{12} - \sum_i c_i \vec{j}_i \vec{\nabla}T \quad (6-1-27)$$

The liquid source  $J_2 = J_{12}$  is found from equation (6-1-23); for this purpose  $\partial u_1/\partial \tau = 0$  is assumed

$$J_2 = J_{12} = -J_1 = -\operatorname{div} \vec{j}_1 \quad (6-1-28)$$

Hence

$$c\rho_0 \frac{\partial T}{\partial \tau} = \operatorname{div} (\lambda\vec{\nabla}T) + r_{12} \operatorname{div} (a_{m1}\rho_0\vec{\nabla}u + a_{m1}^T\rho_0\vec{\nabla}T) - \sum_i c_i \vec{j}_i \vec{\nabla}T \quad (6-1-29)$$

---

\* Further the subscript *dif* at the mass flux *j* will be omitted.

For the zonal system of calculation, when for each interval (zone)  $u$  and  $T$  the transfer coefficients  $\lambda$ ,  $a_{m1}$ ,  $a_{m2}$ ,  $a_{m1}^T$ ,  $a_{m2}^T$  are assumed constant, the system of differential heat- and mass-transfer equations is of the form

$$\frac{\partial u}{\partial \tau} = a_m \nabla^2 u + a_m^T \nabla^2 T = a_m [\nabla^2 u + \delta \nabla^2 T] \quad (6-1-30)$$

$$\frac{\partial T}{\partial \tau} = \left( a + a_{m1} \frac{r_{12}}{c} \right) \nabla^2 T + a_{m1} \frac{r_{12}}{c} \nabla^2 u \quad (6-1-31)$$

$$- [(c_1 a_{m1} + c_2 a_{m2}) \vec{\nabla} u + (c_1 a_{m1}^T + c_2 a_{m2}^T) \vec{\nabla} T] \vec{\nabla} T / c$$

Some particular cases will be considered next.

*Bound material liquid-ice* ( $i = 2, 3$ ). There is a porous material-liquid-ice system; the processes of ice evaporation (sublimation) and direct condensation of vapour into ice (ablimation) are neglected ( $J_{31} = -J_{13} = 0$ ).

The differential mass-transfer equation for a solid phase ( $i = 3$ ) of bound material is considerably simplified, because ice transfer ( $j_3 = 0$ ) does not take place

$$\rho_0 \frac{\partial u_2}{\partial \tau} = -\operatorname{div} \vec{j}_2 + J_2 \quad (6-1-32)$$

$$\rho_0 \frac{\partial u_3}{\partial \tau} = J_3 \quad (6-1-33)$$

where  $J_2 = J_{32} = -J_{23} = -J_3$  is the liquid source caused by ice melting. Equation (6-1-32) may thus be written as

$$\rho_0 \frac{\partial u_2}{\partial \tau} = -\operatorname{div} \vec{j}_2 + \rho_0 \frac{\partial u_3}{\partial \tau} \quad (6-1-34)$$

Here it is assumed that the porous material has a polycapillary structure wherein liquid is transferred by capillary or molecular diffusion.

When considering ice-containing systems the ice-content coefficient  $\varepsilon_3$  is generally introduced, which is equal to the ratio of ice mass  $m_3$  to the total liquid mass  $m$

$$\varepsilon_3 = \frac{m_3}{m} = \frac{u_3}{u_1 + u_2 + u_3 + u_4} = \frac{u_3}{u_2 + u_3} \quad (6-1-35)$$

since vapour and air mass contents are negligible compared to those of liquid and ice.

For bodies containing no ice ( $u_3 = 0$ )  $\varepsilon_3 = 0$ . If all the water becomes ice ( $u_2 = 0$ ), then  $\varepsilon_3 = 1$ . In most cases  $\varepsilon_3 < 1$  ( $0 < \varepsilon_3 < 1$ ).

From equation (6-1-35) it may be found that

$$du = \varepsilon_3 du_3 \quad (6-1-36)$$

Hence

$$J_3 = -J_2 = \frac{\varepsilon_3}{1 - \varepsilon_3} \rho_0 \frac{\partial u}{\partial \tau} = \varepsilon_3 \rho_0 \frac{\partial u_2}{\partial \tau} \quad (6-1-37)$$

The differential heat- and mass-transfer equations then become

$$\frac{\partial u_2}{\partial \tau} = a_{m2} \nabla^2 u + a_{m2}^T \nabla^2 T - \frac{\varepsilon_3}{1 - \varepsilon_3} \frac{\partial u_2}{\partial \tau} \quad (6-1-38)$$

$$\frac{\partial T}{\partial \tau} = a \nabla^2 T + \frac{r_{23} \varepsilon_3}{c(1 - \varepsilon_3)} \frac{\partial u_2}{\partial \tau} + c_2 (a_{m2} \vec{\nabla} u + a_{m2}^T \vec{\nabla} T) \vec{\nabla} T / c \quad (6-1-39)$$

where  $r_{23}$  is the specific heat of liquid freezing (ice melting) and  $c$  is the specific heat

$$c = c_0 + c_2 u_2 + c_3 u_3 \quad (6-1-40)$$

This system of differential equations may be written as

$$\frac{\partial u_2}{\partial \tau} = (1 - \varepsilon_3) (a_{m2} \nabla^2 u + a_{m2}^T \nabla^2 T) \quad (6-1-41)$$

$$\frac{\partial T}{\partial \tau} = \left[ a + \varepsilon_3 a_{m2}^T \frac{r_{23}}{c} \right] \nabla^2 T + \varepsilon_3 a_{m2} \frac{r_{23}}{c} \nabla^2 u + c_2 (a_{m2} \vec{\nabla} u + a_{m2}^T \vec{\nabla} T) \vec{\nabla} T / c \quad (6-1-42)$$

*Bound material vapour-ice* ( $i = 1, 3$ ). Using the above procedure of finding the source  $J_{31}$  and assuming  $\varepsilon_3 = 1$ , we find

$$J = \rho_0 \frac{\partial u_3}{\partial \tau} = \rho_0 \frac{\partial u}{\partial \tau} \quad (u = u_3)$$

and obtain a system of differential mass-transfer equations

$$\frac{\partial u}{\partial \tau} = [a_{m1} \nabla^2 u + a_{m1}^T \nabla^2 T] \quad (6-1-43)$$

$$\frac{\partial T}{\partial \tau} = \left( a + \frac{r_{13}}{c} a_{m1}^T \right) \nabla^2 T + \frac{r_{13}}{c} a_{m1} \nabla^2 u + \frac{c_1}{c} (a_{m1} \vec{\nabla} u + a_{m1}^T \vec{\nabla} T) \vec{\nabla} T \quad (6-1-44)$$

where  $r_{13}$  is the specific heat of ice sublimation and  $c$  is the specific heat

$$c = c_0 + c_1 u_1 + c_3 u_3 \quad (6-1-45)$$



As no filtrational mass transfer takes place in the porous body, the terms  $\sum_i c_{ij} \vec{\nabla} T$  appearing in the differential heat-transfer equation may be neglected. The system of differential heat and mass transfer equations then takes the form

$$\frac{\partial u}{\partial \tau} = K_{11} \nabla^2 u + K_{12} \nabla^2 T \quad (6-1-46)$$

$$\frac{\partial T}{\partial \tau} = K_{22} \nabla^2 T + K_{21} \nabla^2 u \quad (6-1-47)$$

Here the coefficients  $K_{12}$  and  $K_{21}$  are not equal.

For a system  $i = 1, 2$

$$K_{11} = a_m; \quad K_{12} = a_m^T = a_m \delta \quad (6-1-48)$$

$$K_{22} = a + a_{m1}^T \frac{r_{12}}{c}; \quad K_{21} = a_{m1} \frac{r_{12}}{c} \quad (6-1-49)$$

For a system  $i = 2, 3$

$$K_{11} = a_{m2}(1 - \varepsilon_3); \quad K_{12} = (1 - \varepsilon_3)a_{m2}^T = (1 - \varepsilon_3)a_{m2}\delta_2 \quad (6-1-50)$$

$$K_{22} = a + (1 - \varepsilon_3)a_{m2}^T \frac{r_{23}}{c}; \quad K_{21} = (1 - \varepsilon_3)a_{m2} \frac{r_{23}}{c} \quad (6-1-51)$$

For a system  $i = 1, 3$

$$K_{11} = a_{m1}; \quad K_{21} = a_{m1}^T \quad (6-1-52)$$

$$K_{22} = a + a_{m1}^T \frac{r_{23}}{c}; \quad K_{21} = a_{m1} \frac{r_{13}}{c} \quad (6-1-53)$$

Equations (6-1-46) and (6-1-47) are valid not only for an unsteady state but also for a steady state ( $\partial u / \partial \tau = 0$  or  $\partial T / \partial \tau = 0$ ).

## 6.2. DIFFERENTIAL MOISTURE-TRANSFER EQUATION IN DRYING PROCESSES

In moist capillary-porous bodies during drying the convective heat-transfer component  $c_{ij} \vec{\nabla} T$  appearing in heat-transfer equation (6-1-29) is small compared to the conductive component  $\text{div}(\lambda \vec{\nabla} T)$ .

This result may be obtained on the basis of the following analysis. For  $(\text{GrPr}) < 1 \cdot 10^3$  the total heat-conduction coefficient in disperse materials is known to be equal to the molecular heat-conduction coefficient, i.e. transfer of heat takes place by conduction.  $\text{GrPr} = 1 \cdot 10^3$  corresponds to the equivalent number  $\text{Re}_e = 22$ . Let us now estimate this number.

The highest rate of moisture transfer in convective-type driers is about  $40 \text{ kg}/(\text{m}^2 \cdot \text{h})$ . In the most unfavourable case the equivalent capillary diameter  $d_e$  is about 3 mm (void fraction is 70%). The viscosity of water  $\eta_2$  at  $30^\circ\text{C}$  amounts to  $2.88 \text{ kg}/(\text{m} \cdot \text{h})$ . Then

$$\text{Re}_e = \frac{j_2 d_e}{\eta_2 \Pi} = \frac{40 \cdot 3 \cdot 10^{-3}}{0.7 \cdot 2.88} \approx 5.2 \cdot 10^{-2}$$

i.e. it is considerably less than 22. Therefore, the term  $\sum_i c_i \vec{j}_i \vec{\nabla} T$  compared to  $\text{div}(\lambda \vec{\nabla} T)$  may be neglected. The system of differential moisture transfer equations, when there is no gradient of total pressure, will take the form

$$\frac{\partial u}{\partial \tau} = K_{11} \nabla^2 u + K_{12} \nabla^2 T \quad (6-2-1)$$

$$\frac{\partial T}{\partial \tau} = K_{22} \nabla^2 T + K_{21} \nabla^2 u \quad (6-2-2)$$

where the coefficients  $K_{11}$ ,  $K_{12}$ ,  $K_{22}$ ,  $K_{21}$  are determined from relations (6-1-48) and (6-1-49).

The system of equations (6-2-1) and (6-2-2) is the most general system. It is valid not only for drying processes but also for any type of moisture transfer.

The drying process is a typical unsteady-state heat- and mass-transfer process. For this process the moisture source  $J_2 = -J_1$  may be expressed in terms of the local derivative (with respect to time) of the moisture content  $(\partial u / \partial \tau)$ . The calculation procedure, originally reported in [6-3], will be used.

The total change in the moisture content  $du$  is the sum  $d_e u$  of the change in  $d_i u$  due to moisture transfer and the change of  $d_i u$  due to liquid vaporization

$$du = d_e u + d_i u \quad (6-2-3)$$

Here it is assumed that the total moisture content of the body is equal to the specific liquid content ( $u = u_1 + u_2 = u_2$ ) i.e. in the system under consideration  $i = 1, 2$ ,  $u_1 = 0$ .

In case of unsteady-state moisture transfer ( $du \neq 0$ ), the ratio  $d_i u / du$  is finite and describes the change in moisture content due to evaporation relative to the total change in moisture content at the given point of the body. This quantity is called the criterion or coefficient of phase transformation of liquid into vapour and is denoted by

$$\varepsilon = \frac{d_i u}{du} \quad (6-2-4)$$

If  $d_i u = 0$ , then the coefficient  $\varepsilon = 0$ , i.e. the variation in moisture content takes place only due to liquid transfer; in the absence of liquid transfer ( $d_e u = 0$ ), when the moisture content at any point of the body varies only due to evaporation,  $\varepsilon = 1$ . Thus, in a general case,  $\varepsilon$  varies between 0 and 1 ( $0 \leq \varepsilon \leq 1$ ) and the system of differential moisture-transfer equations takes the form

$$\frac{\partial u}{\partial \tau} = a_{m2} \nabla^2 u + a_{m2} \delta \nabla^2 T + \varepsilon \frac{\partial u}{\partial \tau} \quad (6-2-5)$$

$$\frac{\partial T}{\partial \tau} = a \nabla^2 T + \varepsilon \frac{r_{21}}{c} \frac{\partial u}{\partial \tau} \quad (6-2-6)$$

Comparison of (6-2-5), which may be written as

$$\frac{\partial u}{\partial \tau} = \frac{a_{m2}}{(1 - \varepsilon)} [\nabla^2 u + \delta_2 \nabla^2 T] \quad (6-2-7)$$

with the equation

$$\frac{\partial u}{\partial \tau} = a_m (\nabla^2 u + \delta \nabla^2 T) \quad (6-2-8)$$

which is valid for the most general case, leads to a conclusion that the equations will be identical provided the equalities hold

$$a_m = a_{m2}/(1 - \varepsilon) \text{ and } \delta = \delta_2 \quad (6-2-9)$$

Hence

$$\varepsilon = \frac{a_{m1}}{a_{m1} + a_{m2}} = \frac{a_{m1}}{a_m} \quad (6-2-10)$$

The systems of moisture-transfer equations (6-2-5) and (6-2-6) or (6-2-6) and (6-2-8) may be written in the form of (6-2-1) and (6-2-2), but the coefficients  $K_{ij}$  ( $i = 1, 2; j = 1, 2$ ), in this case, will be equal to

$$K_{11} = a_{m2}/(1 - \varepsilon) = a_m; \quad K_{12} = \frac{a_{m2} \delta_2}{1 - \varepsilon} = a_m \delta \quad (6-2-11)$$

$$\left. \begin{aligned} K_{22} &= a + \varepsilon \frac{r_{21}}{c} \frac{a_{m2} \delta_2}{(1 - \varepsilon)} = a + \varepsilon \frac{r_{21}}{c} a_m \delta \\ K_{21} &= \varepsilon \frac{r_{21}}{c} \frac{a_{m2}}{(1 - \varepsilon)} = \varepsilon \frac{r_{21}}{c} a_m \end{aligned} \right\} \quad (6-2-12)$$

The vapourous moisture source  $J_1$  for unsteady-state moisture transfer during drying is thus expressed by

$$J_2 = -J_1 = \varepsilon \rho_0 \frac{\partial u}{\partial \tau} \quad (6-2-13)$$

which is similar to (6-1-37) except that the evaporation coefficient  $\varepsilon$  is substituted by the ice-content coefficient  $\varepsilon_3$ .

Relation (6-2-13) allows us to express the coefficient  $\varepsilon$  in terms of absolute values of liquid  $|\vec{j}_2|$  and vapour  $|\vec{j}_1|$  flows. Indeed,  $\rho_0 \frac{\partial u}{\partial \tau}$  is substituted by an appropriate expression

$$\rho_0 \frac{\partial u}{\partial \tau} = -\operatorname{div} \vec{j}_1 - \operatorname{div} \vec{j}_2$$

and the equality

$$J_2 = -J_1 = -\operatorname{div} \vec{j}_1$$

is used.

Now we have

$$J_2 = \varepsilon \rho_0 \frac{\partial u}{\partial \tau} = -\varepsilon (\operatorname{div} \vec{j}_1 + \operatorname{div} \vec{j}_2) = -\operatorname{div} \vec{j}_1 \quad (6-2-14)$$

hence

$$\varepsilon = \frac{\operatorname{div} \vec{j}_1}{\operatorname{div} \vec{j}_1 + \operatorname{div} \vec{j}_2} = \left( 1 + \frac{\operatorname{div} \vec{j}_2}{\operatorname{div} \vec{j}_1} \right)^{-1} \quad (6-2-15)$$

For one-dimensional problems with parallel or antiparallel vectors  $\vec{j}_1$  and  $\vec{j}_2$ , we have

$$\frac{\operatorname{div} \vec{j}_2}{\operatorname{div} \vec{j}_1} = \frac{(\partial/\partial x) j_2}{(\partial/\partial x) j_1} \quad (6-2-16)$$

If  $\varepsilon = \text{const}$  is assumed, relation (6-2-16) should certainly be constant and equal to  $|\vec{j}|/|\vec{j}_1| = \text{const}$ , and from formula (6-2-15) we get

$$\varepsilon = \frac{|\vec{j}_1|}{|\vec{j}_1| + |\vec{j}_2|} \quad (6-2-17)$$

For drying moist materials relation (6-2-17) may be written as

$$\varepsilon = \frac{a_{m1}(\nabla^2 u + \delta_1 \nabla^2 T)}{a_m(\nabla^2 u + \delta \nabla^2 T)} \quad (6-2-18)$$

If according to formula (6-2-9) we assume that  $\delta = \delta_1 = \delta_2$ , then we get an equality  $\varepsilon = a_{m1}/a_m$ . This fact has been established earlier.

Thus, the introduction of coefficient  $\varepsilon$  as a ratio of liquid to vapour flows during unsteady-state moisture transfer in the drying process requires that equality (6-2-9) or (6-2-17) should hold. These equalities hold in the range of hygroscopic state of moist materials. If the coefficient  $\varepsilon$  is introduced by the aid of relation (6-2-4), then a number of requirements, including constancy of the coefficient with respect to the coordinates, are eliminated.

In the derivation of differential moisture-transfer equations formula (6-2-17) may be taken as the basis as was done in [6-4].

From (6-2-17) it follows that

$$|\vec{j}_1| = \frac{\varepsilon}{1 - \varepsilon} |\vec{j}_2| \quad (6-2-19)$$

We have

$$\vec{j}_1 = \vec{l}_{n1} |\vec{j}_1|; \quad \vec{j}_2 = \vec{l}_{n2} |\vec{j}_2| \quad (6-2-20)$$

where  $\vec{l}_{n1}$  and  $\vec{l}_{n2}$  are unit vectors directed along the vectors  $\vec{j}_1$  and  $\vec{j}_2$ . Then from the differential equation

$$\rho_0 \frac{\partial u}{\partial \tau} = - \operatorname{div} \vec{j}_1 - \operatorname{div} \vec{j}_2 \quad (6-2-21)$$

we get

$$\rho_0 \frac{\partial u}{\partial \tau} = - \operatorname{div} \vec{l}_{n1} |\vec{j}_1| - \operatorname{div} \vec{j}_2 = - \operatorname{div} \frac{\varepsilon}{1 - \varepsilon} |\vec{j}_2| \vec{l}_{n1} - \operatorname{div} \vec{j}_2$$

If we assume that  $\vec{l}_{n1} = \vec{l}_{n2}$ , it means the vectors  $\vec{j}_1$  and  $\vec{j}_2$  have the same direction, then

$$\rho_0 \frac{\partial u}{\partial \tau} = - \operatorname{div} \frac{\varepsilon}{1 - \varepsilon} \vec{j}_2 - \operatorname{div} \vec{j}_2 \quad (6-2-22)$$

Further, by assuming that the coefficient  $\varepsilon$  is independent of the coordinates ( $\varepsilon = \text{const}$ ), from (6-2-22) we get

$$\rho_0 \frac{\partial u}{\partial \tau} = - \operatorname{div} \vec{j}_2 + \varepsilon \rho_0 \frac{\partial u}{\partial \tau} \quad (6-2-23)$$

From this we get the moisture source expression

$$J_2 = \varepsilon \rho_0 \frac{\partial u}{\partial \tau} \quad (6-2-24)$$

If the assumption  $\vec{l}_{n1} = \vec{l}_{n2}$  is applied to a drying process, it means that the vectors

$$\vec{j}_1 = -a_{m1}(\vec{\nabla}u + \delta_1 \vec{\nabla}T)\rho_0 \quad (6-2-25)$$

$$\vec{j}_2 = -a_{m2}(\vec{\nabla}u + \delta_2 \vec{\nabla}T)\rho_0 \quad (6-2-26)$$

should have the same direction. For this it is necessary that  $\delta_1 = \delta_2$ . This requirement is met in a hygroscopic range.

It should be pointed out that expression (6-2-24) for moisture sources holds only during unsteady-state moisture transfer when  $\frac{\partial u}{\partial \tau} \neq 0$ . For a steady-state moisture transfer  $\frac{\partial u}{\partial \tau} = 0$ . The coefficient  $\varepsilon = \frac{d_i u}{du} \rightarrow \infty$ , because  $du = 0$ . Therefore,  $J_2 = \infty \cdot 0$  is an uncertainty.

To estimate this uncertainty, use will be made of equation (6-2-5) from which it follows that

$$\varepsilon \frac{\partial u}{\partial \tau} = \frac{\partial u}{\partial \tau} - a_{m2} \nabla^2 u - a_{m2} \delta_2 \nabla^2 T \quad (6-2-27)$$

In a steady state  $\partial u / \partial \tau = 0$ , therefore

$$\varepsilon \frac{\partial u}{\partial \tau} = -a_{m2}(\nabla^2 u + \delta_2 \nabla^2 T) = \frac{1}{\rho_0} \operatorname{div} \vec{j}_2 \quad (6-2-28)$$

On the other hand, from the differential equation when  $\partial u / \partial \tau = 0$  we get

$$\operatorname{div} \vec{j}_2 = -\operatorname{div} \vec{j}_1 \quad (6-2-29)$$

Thus, for a moisture source in the steady state we get the expression

$$J_2 = \varepsilon \rho_0 \frac{\partial u}{\partial \tau} = \operatorname{div} \vec{j}_2 = -\operatorname{div} \vec{j}_1 \quad (6-2-30)$$

i.e. an expression similar to (6-2-15).

Hence, the moisture source equation

$$J_2 = -\operatorname{div} \vec{j}_1 = \operatorname{div} \rho_0(a_{m1} \vec{\nabla}u + a_{m1}^T \vec{\nabla}T) \quad (6-2-31)$$

is a more general one valid both for unsteady-state and steady-state moisture transfer processes.

Thus the system of differential equations (6-2-1) and (6-2-2) remains unchanged, only the coefficient  $K_{ij}$  is determined from formulas (6-2-11) and (6-2-12).

From this it is also inferred that the coefficient  $\varepsilon$  is the characteristic of unsteady-state moisture transfer, which is similar to the characteristic  $m$  for heating or cooling at the regular regime stage.

The heating rate

$$m = - \left( \frac{1}{T_c - T} \right) \frac{\partial T}{\partial \tau} \quad (6-2-32)$$

is finite and equals

$$m = \frac{a}{R_v^2} \text{Bi} \psi \quad (6-2-33)$$

where  $\psi$  describe nonuniformity of the temperature field and varies between 0 and 1 ( $0 \leq \psi \leq 1$ );  $R_v$  is the hydraulic radius of the body; Bi is the Biot number. In the steady state  $\partial T / \partial \tau = 0$  and  $\left( \frac{1}{T_c - T} \right) = \infty$  ( $T = T_c$ ).

Consequently, similar to the coefficient  $\varepsilon$  the heating rate  $m$  is the characteristic of unsteady-state heat transfer. This, however, does not imply that during unsteady-state heat transfer  $m$  will be a variable varying from 0 to  $\infty$ .

It should be pointed out that for the calculation of unsteady-state temperature fields and the amount of moisture present in the body when dried, it is necessary to know the coefficients  $a$ ,  $a_{m1}$ ,  $a_{m1}^T$ ,  $\varepsilon$  and the thermodynamic characteristics  $c$  and  $r$ .

But if a more general relation (6-2-30) is used, the thermophysical characteristics will be as follows:  $a$ ,  $c$ ,  $r$ ,  $a_{m1}$ ,  $a_{m1}^T$ ,  $a_m$ ,  $a_m^T$ . Thus, the introduction of  $\varepsilon$  results in a reduction of parameters from 7 to 6. In all the cases the system of differential moisture-transfer equations (6-2-1) and (6-2-2) remains the same. Only in the solutions of this system the expressions of coefficients  $K_{ij}$  will be different depending upon thermophysical characteristics.

#### (d) Diffusion and Filtration Moisture Transfer

During an intense drying process (high-frequency drying, conductive drying, etc.) total pressure gradient of humid air arises within the moist material. As a result an additional transfer of moisture and heat takes place due to the hydrodynamic (filtration) motion of liquid and vapour.

The total pressure gradient within the material appears as a result of evaporation and the resistance of the porous skeleton during vapour motion. This is favoured by a system of microcapillaries, through which air from the environment flows in a molecular (effusion) way, and by slip diffusion in the microcapillary system [6-51].

In this case the system of differential moisture-transfer equations, as applied to a drying process, takes the form (for details see [6-1])

$$\frac{\partial u}{\partial \tau} = K_{11} \nabla^2 u + K_{12} \nabla^2 T + K_{13} \nabla^2 P \quad (6-2-34)$$

$$\frac{\partial T}{\partial \tau} = K_{21} \nabla^2 u + K_{22} \nabla^2 T + K_{23} \nabla^2 P \quad (6-2-35)$$

$$\frac{\partial P}{\partial \tau} = K_{31} \nabla^2 u + K_{32} \nabla^2 T + K_{33} \nabla^2 P \quad (6-2-36)$$

where the coefficients  $K_{ij}$  ( $ij = 1, 2, 3$ ) are

$$K_{11} = a_m; \quad K_{12} = a_m \delta = (a_{m1}^T + a_{m2}^T); \quad K_{13} = k_p \rho_0 \quad (6-2-37)$$

$$K_{21} = \frac{r\varepsilon}{c} a_m; \quad K_{22} = a + \frac{\varepsilon r}{c} a_m \delta; \quad K_{23} = \varepsilon r \frac{a_m}{c} \delta_p \quad (6-2-38)$$

$$K_{31} = -\frac{\varepsilon a_m}{c_p}; \quad K_{32} = -\frac{\varepsilon a_m}{c_p} \delta; \quad K_{33} = \left( a_p - \frac{\varepsilon a_m}{c_p} \delta_p \right) \quad (6-2-39)$$

where  $k_p$  is the filtration moisture-transfer coefficient defined by equation  $\vec{j}_p = -k_p \vec{\nabla} P$ ;  $\delta_p$  is the dimensionless coefficient of filtration moisture flow,  $\delta_p = k_p / a_m \rho_0$ ;  $a_p$  is the convective filtration diffusion coefficient,  $a_p = \frac{k_p}{c_p \rho_0}$ ;  $c_p$  is the coefficient of humid air capacity in a porous body, which is defined by the relation  $d(u_1 + u_4) = c_p dp$ . If the variation of  $(u_1 + u_4)$  with temperature is accounted for, then the coefficients  $K_{3i}$  ( $i = 1, 2, 3$ ) should be supplemented by

$$\frac{\varepsilon r}{cT} p a_m; \quad \left( a + \frac{r}{c} \varepsilon a_m \delta \right) \frac{p}{T}; \quad \frac{\varepsilon r}{cT} p a_m \delta_p \quad (6-2-40)$$

respectively.

But in the case of filtration transfer, terms characterizing heat transfer by convection should be included in heat-transfer equation (6-2-35)

$$\sum_i c_i \vec{j}_i \vec{\nabla} T \quad (6-2-41)$$

which in certain cases are considerably greater than the diffusional terms.

The above moisture-transfer equations are valid for a homogeneous moist body. They also hold for a system of moist bodies, but it should be kept in mind that at the line of contact of moist bodies their moisture contents are different. This is due to the fact that the moisture content of a body is not a moisture transfer potential; it is similar to enthalpy rather than temperature.



The difference in moisture contents at the contact line is determined from the moisture capacity  $c_m$  of the bodies in contact. At the contact line of moist bodies the moisture transfer potential equals the difference in moisture contents  $\theta_i = \theta_{(i+1)}$

$$u_i = \frac{c_{mi}}{c_{m(i+1)}} u_{(i+1)} \quad (6-2-42)$$

The system of equations (6-2-34) through (6-2-39) will hold if the moisture content  $u$  is substituted by moisture transfer potential  $\theta$ . In this case the coefficients  $K_{ij}$  are determined from similar relations, for example the coefficient  $\delta$  should be substituted by  $\delta/c_m$  and  $\delta_p$  by  $\delta_p/c_m$  and so on.

Let us now consider limit transitions.

I. Let us assume that in a moisture-transfer process the temperature of a moist capillary-porous body remains unchanged with time, e.g.  $\frac{\partial T}{\partial \tau} = 0$ .

Two cases are then possible:

(1) From equation (6-1-47) it follows that  $\nabla^2 T = 0$  and  $\nabla^2 u = 0$ . This is a trivial case of an equilibrium state (temperature and moisture content are independent both of time and coordinates,  $u = \text{const}$ ,  $T = \text{const}$ ) or of a steady state (temperature and moisture content are linear functions of coordinates).

(2) Since  $T \equiv T(x_i, \tau)$  and  $u = u(x_i, \tau)$ , then at  $\frac{\partial T}{\partial \tau} = 0$  from (6-1-47) it follows that

$$\nabla^2 u = -\frac{K_{22}}{K_{21}} \nabla^2 T \neq 0; \quad \left. \begin{matrix} u^* \\ T^* \end{matrix} \right\} = f(x_i) \quad (6-2-43)$$

i.e. the moisture content and temperature fields are similar. In (6-2-43)  $u^*$  and  $T^*$  are dimensionless moisture content and temperature, respectively. As the form of the distribution function  $f(x_i)$  is independent of time, then according to equation (6-1-46) the local rate  $\frac{\partial u}{\partial \tau}$  is also independent of time.

In a particular case at

$$\frac{\partial u}{\partial \tau} = \text{const} \quad (6-2-44)$$

the temperature and moisture content distributions in one-dimensional problems are described by a simple parabola. A similar situation prevails when the drying rate is constant. The moisture content at any point is a linear time function  $\left( \frac{\partial u}{\partial \tau} = \text{const} \right)$ , and the moisture content distribution across

the body (infinite plate and cylinder, sphere) is described by a parabola. The temperature at any point of the body does not change with time  $\left(\frac{\partial T}{\partial \tau} = 0\right)$ .

If evaporation within the body takes place ( $\varepsilon \neq 0$ ;  $a_{m1} \neq 0$ ) at a constant drying rate, then according to (6-2-43) temperature distribution in one-dimensional problems is described by a parabolic law as  $K_{21} \neq 0$ .

(3) In a particular case, when evaporation does not take place within the body during constant rate period ( $\varepsilon = 0$  or  $a_{m1} = 0$ ), temperature at all points of the body is the same and equals the wet-bulb temperature  $T_m$  ( $T = T_m = \text{const}$ ,  $\nabla^2 T = 0$ ) in the first approximation. This does not contradict relation (6-2-43) as at  $\nabla^2 T = 0$   $\varepsilon = 0$  and  $K_{21} = 0$ , while  $K_{22} \neq 0$ ,  $\nabla^2 u \neq 0$ . Actually, at  $\frac{\partial T}{\partial \tau} = 0$  from (6-1-47), which may be expressed as

$$\frac{\partial T}{\partial \tau} = a \nabla^2 T + \frac{\varepsilon r_{12}}{c} \frac{\partial u}{\partial \tau} \quad (6-1-47a)$$

it follows that at  $\varepsilon = 0$   $\nabla^2 T = 0$ . Hence, from equation (6-1-46) we obtain the classical diffusion equation

$$\frac{\partial u}{\partial \tau} = K_{11} \nabla^2 u \quad (6-1-46a)$$

that implies that in the constant rate period the moisture transfer occurs at isothermal conditions.

At a constant drying rate ( $d\bar{u}/d\tau = \text{const}$ ) the local rate  $\frac{\partial u}{\partial \tau}$  is also constant. Then from (6-1-46 a)

$$\nabla^2 u = \text{const} \quad \text{at} \quad \frac{\partial u}{\partial \tau} = \text{const} \quad (6-2-43a)$$

This is observed upon drying colloidal materials like gelatine.

All the described cases of moisture content and temperature distributions are observed in the period when the drying rate of capillary-porous materials is constant, and are at present experimental facts equivalent to experimental laws.

II. Let us assume that during heat and moisture transfer the moisture content remains unchanged with time (a quasi-steady state upon heating a moist body), i.e.  $\frac{\partial u}{\partial \tau} = 0$ .

Then from (6-1-46) it follows that

$$\nabla^2 u = - \frac{K_{12}}{K_{11}} \nabla^2 T \quad \text{at} \quad \frac{\partial u}{\partial \tau} = 0 \quad (6-2-45)$$

If  $K_{11}=a_m \neq 0$  and  $K_{12}=a_m^T \neq 0$ , then the moisture content field is similar to the temperature field. However, at  $\frac{\partial u}{\partial \tau} = 0$  the local rate  $\frac{\partial T}{\partial \tau} \neq 0$  according to equation (6-2-35), i.e. the temperature profile within the body does not change with time, as the temperature at any point varies with time following one and the same law. Such cases may be observed in thermophysical experiments, for example, in the quasi-steady-state heating of moist bodies when determining thermogradient coefficient, thermal diffusivity and thermal conductivity of moist bodies. According to this method the moist body is heated at a constant rate (the ambient temperature is a linear function of time). From a certain moment onwards the temperature at any point of the body is a linear time function, and for one-dimensional symmetrical problems the temperature distribution is described by a parabolic law [6-49]. Parabolic distribution of temperature corresponds to parabolic distribution of moisture content. The coefficients  $\alpha$  and  $\delta$  are found from the temperature and moisture content drops, and if the heat-transfer coefficient is known, then the thermal diffusivity can be easily determined. It is quite natural that in case (2), i.e. at  $\frac{\partial u}{\partial \tau} = 0$ , the trivial

case may occur:  $\frac{\partial T}{\partial \tau} = 0$  at  $\nabla^2 u = 0$  and  $\nabla^2 T = 0$  (steady or equilibrium state).

To conclude the section we shall consider transition of equations (6-2-34), (6-2-35) to the heat-conduction equation, as the transition to the diffusion equation has been discussed earlier. In a perfectly dry body no mass-transfer takes place (at  $u = 0$ ,  $K_{11} = K_{12} = K_{21} = 0$ ), then from equations (6-2-34), (6-2-35) the classical heat-conduction equation\*

$$\frac{\partial T}{\partial \tau} = K_{22} \nabla^2 T = a \nabla^2 T \quad (6-2-46)$$

is obtained.

The same equation is obtained from (6-1-46), (6-1-47) for a moist body with a maximum moisture content  $u_{\max}$  (swelling moisture). Then at small temperature drops moisture transfer is absent (at  $u = u_{\max}$ ,  $K_{11} = K_{12} = K_{21} = 0$ ), consequently  $\frac{\partial u}{\partial \tau} = 0$  and from (6-1-46) and (6-1-47) we get

(6-2-46). In this case the moist body is heated similarly to a dry body without redistribution and evaporation of moisture (the moisture content at all points of the body is constant and equal:  $u = u_m$ ). A similar analysis is also applicable to heat and moisture transfer equations given a total pressure gradient ( $\vec{\nabla} P \neq 0$ ). The above conclusions will hold. For this it

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\* The equality  $u = 0$  does not imply that  $\nabla^2 T = 0$ , according to equation (6-1-46, as  $K_{12} = 0$ .

is necessary to assume that the moisture temperature is equal to that of the solid skeleton and the Darcy law (convective diffusion) holds. All other assumptions on thermal and physical properties also remain valid.

In conclusion it should be pointed out that the analysis of equations (6-1-46) and (6-1-47) has not only revealed the details of the heat and moisture transfer mechanism upon drying various materials, but also has made possible the development of express methods for experimental determination of thermal properties of moist capillary-porous materials.

### 6-3. GENERALIZED SYSTEM OF DIFFERENTIAL HEAT- AND MASS-TRANSFER EQUATIONS

For high-rate heat- and mass-transfer processes, heat and mass flows are not described by the Onzager equations

$$\vec{j}_k = \sum_i L_{ki} \vec{X}_i \quad (6-3-1)$$

but by the generalized system of equations

$$\vec{j}_k = \sum_i L_{ki} \vec{X}_i + L_k^{(r)} \frac{\partial \vec{j}_k}{\partial \tau} \quad (6-3-2)$$

For example, the Fourier heat-conduction equation  $\vec{q} = -\lambda \vec{\nabla} T$  will be replaced by the generalized relation

$$\vec{j}_q = \vec{q} = -\lambda \vec{\nabla} T - \tau_{rq} \frac{\partial \vec{q}}{\partial \tau} \quad (6-3-3)$$

Strictly speaking, relation (6-3-3) is valid only for one-dimensional problems. For multi-dimensional problems relation (6-3-3) may be used as an approximation, where the relaxation period  $\tau_{rq}$  of thermal stress is defined as a certain experimental constant.

A similar relation is used for moisture diffusion.

Terms  $L_k^{(r)} \frac{\partial \vec{j}_k}{\partial \tau}$  correspond to finite propagation velocity of a certain substance. The stress relaxation period  $\tau_{rk}$  of substance  $k$  (mass, heat, etc.) is defined by

$$\tau_{rk} = \frac{a_k}{v_k^2} \quad (6-3-4)$$

where  $v_k$  is the finite propagation velocity of substance  $k$  and  $a_k$  is the diffusivity of substance  $k$ .

Using energy and mass conservation laws and also the system of generalized Onzager relations for a gradient relation between thermodynamic forces and corresponding transfer potentials, we obtain a system of diffe-

potential transfer equations

$$c'_k \rho_0 \tau_{rk} \frac{\partial^2 \theta_k}{\partial \tau^2} + c'_k \rho_0 \left( \frac{\partial \theta_k}{\partial \tau} + \bar{v} \nabla \theta_k \right) = \operatorname{div} \sum_{i=1}^n L_{ki} n_i \nabla \theta_{ki} + J_k; \quad k = 1, 2, \dots, n \quad (6-3-5)$$

where  $\tau$  is the time;  $c'_k \rho_0$  is the generalized volume capacity of corresponding potential  $\theta_k$ ;  $\bar{v}$  is the macroscopic velocity;  $J_k$  is the potential source or sink;  $L_{ki}$  are kinetic transfer potentials related by the Onzager reciprocal relation  $L_{ik} = L_{ki}$ ;  $n_i$  is the proportionality factor between the force and the potential gradient. Given only two potentials and thermodynamic forces ( $i, k = 1, 2$ ), the system of equations (6-2-34)-(6-2-36) describes the following: heat and mass transfer in a mobile binary gaseous mixture, heat and mass transfer in molecular solutions, two-phase laminar flows of immiscible liquids in porous bodies, electrodiffusion phenomena in electrolytic solutions, drying processes in capillary-porous bodies and other processes.

In the case of constant coefficients and transfer characteristics, equation (6-3-5) becomes

$$\tau_{rk} \frac{\partial^2 \theta_k}{\partial \tau^2} + \frac{\partial \theta_k}{\partial \tau} + \bar{v} \nabla \theta_k = \sum_{i=1}^n K_{ki} \nabla^2 \theta_i + \Pi_k; \quad k = 1, 2, \dots, n \quad (6-3-6)$$

where  $K_{ki}$  are the transfer coefficients at the corresponding potentials,  $K_{ki} \neq K_{ik}$  which are no longer related by the Onzager reciprocal relations;  $\Pi_k = J_k / c_k \rho_0$  is the specific power of additional sources. With reference to drying processes in capillary-porous bodies at moderate temperatures ( $k, i = 1, 2$ ) and at  $\tau_{rk} = 0$  the quantities appearing in (6-3-6) take the following form

$$\left. \begin{aligned} \theta_1 = T; \quad \theta_2 = \theta; \quad \Pi_1 = \Pi_2 = 0; \quad \bar{v} = 0 \\ K_{11} = a'_q + a'_m \delta \varepsilon r \frac{1}{c_q}; \quad K_{12} = \varepsilon r \frac{c_m}{c_q} a'_m \\ K_{21} = a'_m \delta / c_m; \quad K_{22} = a'_m \end{aligned} \right\} \quad (6-3-7)$$

where  $T$  is the body temperature;  $\theta$  is the mass transfer potential;  $a'_q$  and  $a'_m$  are heat and mass diffusion coefficients;  $c_q$  and  $c_m$  are the reduced specific heat of the material and isothermal mass capacity, respectively;  $r$  is the specific heat of phase transition;  $\varepsilon$  is the phase change criterion. In the transition from moisture content to potential  $\theta$  relation  $du = c_m d\theta$  was used, which, strictly speaking, holds only for isothermal conditions. At nonisothermal conditions this relation is approximate. Therefore, it is better to use the moisture content as the moisture-transfer potential, i.e. to use the system of equations (6-2-34)-(6-2-36).

With reference to high-temperature drying ( $k, i=1, 2, 3$ ) parameters entering into equation (6-3-6) may be written as

$$\begin{aligned}
 \theta_1 &= T; \quad \theta_2 = \theta; \quad \theta_3 = P; \quad \Pi_1 = \Pi_2 = \Pi_3 = 0; \quad \bar{v} = 0; \\
 K_{11} &= a'_q + a'_m \delta \varepsilon r \frac{1}{c_q}; \quad K_{12} = \varepsilon r \frac{c_m}{c_q} a'_m; \quad K_{13} = a'_m \varepsilon r \delta_p \frac{1}{c_q}; \\
 K_{21} &= a'_m \delta / c_m; \quad K_{22} = a'_m; \quad K_{23} = a'_m \delta_p / c_m \\
 K_{31} &= -\varepsilon \frac{c_m}{c_p} a'_m \delta_\theta; \quad K_{32} = -\varepsilon \frac{c_m}{c_p} a'_m \\
 K_{33} &= a_p - \varepsilon \frac{1}{c_p} a'_m \delta_p; \quad \delta_\theta c_m = \delta
 \end{aligned} \quad (6-3-8)$$

where  $P$  is the filtration mass-transfer potential;  $a_p$  is the mass diffusivity of vapour-gas mixture filtration;  $c_p$  is the capacity of the capillary-porous body with respect to humid air or vapour in the process of its molar transfer.

#### 6-4. MASS TRANSFER SIMILARITY NUMBERS

From the system of differential moisture transfer equations the following similarity numbers and generalized variables may be obtained.

1. Homochronism numbers of transfer field potentials referred to as the Fourier numbers

$$Fo_q = \frac{a_q \tau}{R^2}; \quad Fo_m = \frac{a_m \tau}{R^2}; \quad Fo_p = \frac{a_p \tau}{R^2} \quad (6-4-1)$$

where  $R$  is the characteristic dimension. These numbers are related by the criteria  $Lu$  and  $Lu_p$ .

The mass-transfer relaxation Fourier number

$$Fo_{rm} = \frac{a_m \tau_{rm}}{R^2} \quad (6-4-2)$$

in which all quantities ( $a_m, R, \tau_{rm}$ ) are known. The relaxation period of concentrated mass stress is about  $10^3$ - $10^5$  times the thermal stress relaxation. The Fourier mass-transfer number is, therefore, many times greater than the Fourier heat-transfer relaxation number

$$Fo_{rq} = a_q \tau_{rq} / R^2$$

2. The diffusion moisture-transfer number with respect to heat diffusion, or the moisture-transfer diffusion number

$$Lu = \frac{a_m}{a_q}; \quad Lu_p = \frac{a_p}{a_q} \quad (6-4-3)$$

Number  $Lu$  is the ratio of the mass-diffusion coefficient to heat-diffusion coefficient.

$Lu_p$  is the ratio of convective diffusivity to thermal diffusivity. An alternative explanation may be given to  $Lu$  number.

$Lu$  is the ratio of the propagation velocity of the isoconcentration surface to that of isothermal surface, in other words it characterizes the temperature field inertia with respect to the moisture content field (the heat- and moisture-transfer inertia number).

The filtration number  $Lu_p$  is the ratio of filtration field (pressure field) potential propagation to temperature field propagation.

For a number of moist materials  $Lu$  increases with the moisture content following a linear law. If  $Lu > 1$ , the mass-transfer potential  $\theta$  propagates more rapidly than the temperature field.

The filtration number  $Lu_p$  is much in excess of unity (of the order of 100-1000), therefore, the total pressure relaxation of a vapour-gas mixture in a capillary-porous body takes place at a rate higher by 2-3 orders than temperature field relaxation.

The relation between the Fourier numbers may be expressed in terms of  $Lu$  and  $Lu_p$

$$Fo_m = FoLu; \quad Fo_p = FoLu_p \quad (6-4-4)$$

where  $Fo = Fo_q$  is the heat-transfer Fourier number.

3. The Kossovich number is defined by

$$Ko = \frac{rc_m \Delta \theta}{c_q \Delta t} = \frac{r \Delta u}{c_q \Delta t} \quad (6-4-5)$$

The number  $Ko$  shows the relation between the amounts of heat spent on liquid evaporation ( $r \Delta u$ ) and on heating of moist body ( $c_q \Delta t$ ).

4. Posnov number for diffusion and filtration transfer

$$\left. \begin{aligned} Pn &= \frac{\delta \Delta T}{\Delta u} = \frac{a_m^T \rho_0 \Delta T}{\rho_0 a_m \Delta u} = \frac{a_m^T \rho_0 \Delta T}{\lambda_m \Delta \theta} \\ Pn_p &= \frac{\delta_p \Delta p}{\Delta u} = \frac{k_p \Delta p}{\rho_0 a_m \Delta u} \end{aligned} \right\} \quad (6-4-6)$$

$P_n$  is the ratio of thermodiffusion moisture-transfer rate to moisture diffusion rate or the ratio of thermal-moisture diffusivity to moisture diffusivity.  $P_{n_p}$  is the ratio of filtration moisture transfer rate to diffusion transfer rate.

5. Analytical solutions very often involve a generalized variable which is the product of  $\varepsilon P_n K_o$ . This is sometimes referred to as the Fedorov number

$$Fe = \varepsilon K_o P_n = \frac{\varepsilon \delta r}{c_q} \quad (6-4-7)$$

The Fedorov number is independent of the choice of heat- and mass-transfer potentials and is defined only by coefficients  $\varepsilon$  and  $\delta$  and thermodynamic characteristics  $r$  and  $c_q$ .

If  $B$  is used to denote the dimensionless temperature coefficient of drying

$$B = b \frac{\Delta \bar{u}}{\Delta T} = \frac{\partial(\bar{T}/\Delta T)}{\partial(\bar{u}/\Delta \bar{u})} \quad (6-4-8)$$

( $b = \partial \bar{T} / \partial \bar{u}$ ), then the ratio  $B/K_o$  yields a new number called the Rebinder number (Rb)

$$Rb = \frac{B}{K_o} = \frac{c_q b}{r} \quad (6-4-9)$$

The Rebinder number enters into the fundamental heat-balance equation.

Unlike the Posnov number, the temperature drying coefficient describes the change in integral mean temperature  $\bar{T}$  and integral mean moisture content  $\bar{u}$ , i.e. it is a kinetic property of integral heat- and moisture-transfer properties. The Posnov number is concerned with local changes in  $u$  and  $T$ .

6. From differential equation (6-2-36), we obtain the criterion

$$K = \frac{c_B \Delta p}{c_m \Delta \theta} = \frac{Bu}{K_o} \quad (6-4-10)$$

where  $Bu$  is the Bulygin number defined by

$$Bu = \frac{rc_B \Delta p}{c_q \Delta T} \quad (6-4-11)$$

7. From boundary conditions of the third kind two Biot numbers may be obtained

the heat-transfer Biot number

$$Bi_q = \frac{R}{\lambda_q} = \frac{\alpha_h R}{a_q} \quad (6-4-12)$$



and mass-transfer Biot number

$$\text{Bi}_m = \frac{\beta_\theta R}{\lambda_m} = \frac{\beta_u R}{a_m} \quad (6-4-13)$$

The Biot numbers vary in the range from 0 to  $\infty$  ( $0 < \text{Bi} < \infty$ ). When boundary conditions of the third kind do not hold for heat  $q(\tau)$  and mass  $j(\tau)$  flows, then  $\text{Bi}_q$  and  $\text{Bi}_m$  are substituted by two Kirpichev numbers

$$\text{Ki}_q = \frac{q(\tau)R}{\lambda \Delta T}; \quad \text{Ki}_m = \frac{j(\tau)R}{\lambda_m \Delta \theta} \quad (6-4-14)$$

When fluxes  $q(\tau)$  and  $j(\tau)$  are defined by Newton laws, these numbers are related by a simple relation

$$\text{Ki}_q = \text{Bi}_q \frac{T_c - T_n}{\Delta T}; \quad \text{Ki}_m = \text{Bi}_m \frac{\theta_n - \theta_c}{\Delta \theta} \quad (6-4-15)$$

Quantities  $\Delta T$ ,  $\Delta u$ ,  $\Delta \theta$  and  $\Delta p$ , appearing in heat- and mass-transfer similarity numbers, are chosen in accordance with the problem conditions.

#### 6-5. SOLUTION OF HEAT- AND MASS-TRANSFER EQUATIONS AT GENERALIZED BOUNDARY CONDITIONS

The system of heat- and mass-transfer equations in a dimensionless form for one-dimensional fixed bodies or stagnant fluids is of the form

$$\frac{\partial \theta_k}{\partial \text{Fo}_k} = \sum_{i=1}^n K_{ki}^* \left( \frac{\partial^2 \theta_i}{\partial X^2} + \frac{\Gamma}{X} \frac{\partial \theta_i}{\partial X} \right) + \Pi_k^* (k, i = 1, 2, \dots, n) \quad (6-5-1)$$

where  $\theta_k$  is a dimensionless transfer potential;  $K_{ki}^*$  is a dimensionless simple or complex number which characterizes transfer of the  $k$ -th interconnected substance by the motive force with index  $i$ ;  $\Pi_k^*$  is a dimensionless source (or sink) of the  $k$ -th transfer;  $\text{Fo}_k = \frac{a_k \tau}{R^2}$  is the homochronism number;  $X$  is the dimensionless coordinate;  $\Gamma$  is the shape factor equal to zero for an infinite plate, one for an infinite cylinder, and two for a sphere.

For diffusion heat and mass transfer in moist bodies ( $k, i = 1, 2$ ) the dimensionless parameters appearing in equation (6-5-1) are of the form

$$\left. \begin{aligned} \theta_1 &= T; \quad \theta_2 = \theta; \quad \Pi_1^* = \Pi_2^* = 0 \\ K_{11}^* &= 1 + \varepsilon \text{KoPnLu}; \quad K_{12}^* = -\varepsilon \text{KoLu} \\ K_{21}^* &= -\text{LuPn}; \quad K_{22}^* = \text{Lu} \end{aligned} \right\} \quad (6-5-2)$$

where  $T = \frac{t - t_*}{t_c - t_*}$ ,  $\theta = \frac{\theta_* - \theta}{\theta_* - \theta_p}$  are dimensionless temperature and mass-transfer potential (subscript \* refers to a fixed initial potential value).

For molar-and-molecular heat and mass transfer ( $k, i = 1, 2, 3$ ) the dimensionless parameters  $\theta_k$  and  $K_{ki}^*$  become

$$\theta_1 = T; \quad \theta_2 = \theta; \quad \theta_3 = P; \quad \Pi_i^* = 0 \quad (i = 1, 2, 3);$$

$$K_{11}^* = 1 + \varepsilon \text{KoPn Lu}; \quad K_{12}^* = -\varepsilon \text{Ko Lu};$$

$$K_{13}^* = \varepsilon \text{Bu Lu}_p; \quad (6-5-3)$$

$$K_{21}^* = -\text{Lu Pn}; \quad K_{22}^* = \text{Lu}; \quad K_{23}^* = -\text{Lu}_p \frac{\text{Bu}}{\text{Ko}};$$

$$K_{31}^* = -\text{Lu} \frac{\varepsilon \text{Ko Pn}}{\text{Bu}}; \quad K_{32}^* = \text{Lu} \frac{\varepsilon \text{Ko}}{\text{Bu}}; \quad K_{33}^* = \text{Lu}_p(1 - \varepsilon)$$

where  $P = \frac{p - p_0}{p_0}$  is the dimensionless potential of filtration mass transfer.

The system of equations (6-5-1) may be rewritten as

$$\frac{\partial \theta_1(X, \text{Fo})}{\partial \text{Fo}} = \frac{\partial^2 \theta_1(X, \text{Fo})}{\partial X^2} + \frac{\Gamma}{X} \frac{\partial \theta_1(X, \text{Fo})}{\partial X} - \text{Ko}^* \frac{\partial \theta_2(X, \text{Fo})}{\partial \text{Fo}} \quad (6-5-4)$$

$$\begin{aligned} \frac{\partial \theta_2(X, \text{Fo})}{\partial \text{Fo}} = & \text{Lu} \left[ \frac{\partial^2 \theta_2(X, \text{Fo})}{\partial X^2} + \frac{\Gamma}{X} \frac{\partial \theta_2(X, \text{Fo})}{\partial X} \right] \\ & - \text{Lu Pn} \left[ \frac{\partial^2 \theta_1(X, \text{Fo})}{\partial X^2} + \frac{\Gamma}{X} \frac{\partial \theta_1(X, \text{Fo})}{\partial X} \right] \end{aligned} \quad (6-5-5)$$

$$\begin{aligned} & - \text{Lu}_p \frac{\text{Bu}}{\text{Ko}} \left[ \frac{\partial^2 \theta_3(X, \text{Fo})}{\partial X^2} + \frac{\Gamma}{X} \frac{\partial \theta_3(X, \text{Fo})}{\partial X} \right] \\ \frac{\partial \theta_3(X, \text{Fo})}{\partial \text{Fo}} = & \text{Lu}_p \left[ \frac{\partial^2 \theta_3(X, \text{Fo})}{\partial X^2} + \frac{\Gamma}{X} \frac{\partial \theta_3(X, \text{Fo})}{\partial X} \right] \\ & - \frac{\text{Ko}^*}{\text{Bu}} \frac{\partial \theta_2(X, \text{Fo})}{\partial \text{Fo}} \end{aligned} \quad (6-5-6)$$

where  $\text{Ko}^* = \varepsilon \text{Ko}$ .

The following boundary conditions are adopted

$$\theta_k(X, 0) = 0, \quad k = 1, 2 \quad (6-5-7)$$

which imply a uniform initial distribution.

For symmetrical temperature and moisture content fields

$$\frac{\partial \theta_k(0, Fo)}{\partial X} = 0; \quad \theta_k(0, Fo) \neq \infty \quad (6-5-8)$$

The boundary conditions are as follows

$$\frac{\partial \theta_1(1, Fo)}{\partial X} - Ki_1(Fo) + (1 - \varepsilon) KoLu Ki_2(Fo) = 0 \quad (6-5-9)$$

$$- \frac{\partial \theta_2(1, Fo)}{\partial X} + Pn \frac{\partial \theta_1(1, Fo)}{\partial X} + Ki_2(Fo) \quad (6-5-10)$$

$$+ \frac{Bu}{Ko} \frac{Lu_p}{Lu} \frac{\partial \theta_3(1, Fo)}{\partial X} = 0$$

$$\theta_3(1, Fo) = \varphi_3(Fo)$$

For boundary conditions of the third kind the numbers  $Ki_1(Fo)$ ,  $Ki_2(Fo)$  are of the form

$$Ki_k(Fo) = Bi_k[\theta_{fk}(Fo) - \theta_k(1, Fo)] \quad (6-5-11)$$

For the drying process of moist materials, the boundary conditions of the third kind for mass transfer do not hold, and are substituted by the integral relation

$$\frac{\partial \bar{u}}{\partial \tau} = K(\bar{u} - u_p)$$

which may be written as

$$Ki_2(Fo) = K \left[ 1 - (\Gamma + 1) \int_0^1 X^\Gamma \theta_2(X, Fo) dX \right] \quad (6-5-12)$$

It is of interest to solve the problem for generalized boundary conditions

$$\begin{aligned} & K_{k1} \theta_1(1, Fo) + K_{k2} \theta_2(1, Fo) + K_{k3} \frac{\partial \theta_1(1, Fo)}{\partial X} \\ & + K_{k4} \frac{\partial \theta_2(1, Fo)}{\partial X} + K_{k5} (\Gamma + 1) \int_0^1 X^\Gamma \\ & \times \theta_1(X, Fo) dx + K_{k6} (\Gamma + 1) \int_0^1 X^\Gamma \theta_2(X, Fo) dx = \varphi_k(Fo) \end{aligned} \quad (6-5-13)$$

where  $\varphi_k(Fo)$  are arbitrary limit integrable time functions.

The system of equations (6-5-4)-(6-5-6) with boundary conditions (6-5-7)-(6-5-13) was solved by M. D. Mikhailov with the use of the Laplace integral transformations. The solution involves the following characteristic numbers

$$v_1^2 = \frac{1}{2} \left[ \left( 1 + \text{Ko}^* \text{Pn} + \frac{1}{\text{Lu}} \right) + (-1)^i \right. \\ \left. \times \sqrt{\left( 1 + \text{Ko}^* \text{Pn} + \frac{1}{\text{Lu}} \right)^2 - \frac{4}{\text{Lu}}} \right]; \quad i = 1, 2 \quad (6-5-14)$$

which characterize the interrelation of heat and mass transfer.

For the analysis of solutions the following relations are necessary

$$\left. \begin{aligned} v_1^2 + v_2^2 &= 1 + \text{Ko}^* \text{Pn} + \frac{1}{\text{Lu}}; & v_1^2 v_2^2 &= \frac{1}{\text{Lu}} \\ (1 - v_1^2)(1 - v_2^2) &= -\text{Ko}^* \text{Pn} \end{aligned} \right\} \quad (6-5-15)$$

$$\left. \begin{aligned} \frac{v_1^6 - v_2^6}{v_1^2 - v_2^2} &= \left( 1 + \text{Ko}^* \text{Pn} + \frac{1}{\text{Lu}} \right)^2 - \frac{1}{\text{Lu}} \\ v_1^2 - v_2^2 - (v_1^4 - v_2^4) &= (v_2^2 - v_1^2) \left( \text{Ko}^* \text{Pn} + \frac{1}{\text{Lu}} \right) \end{aligned} \right\} \quad (6-5-16)$$

In Table 6-1 are presented the values of  $\text{Lu } v_i^2$  as functions of  $\text{Lu}$  and  $\text{Ko}^* \text{Pn}$  for the range of  $\text{Lu}$  from 0.01 to 1.0 and of  $\text{Ko}^* \text{Pn}$  from 0.1 to 1.0.

Table 6-1

Values of  $\text{Lu } v_i^2$  [6-6]

Lu	Lu $v_i^2$	Ko* Pn					
		0.1	0.2	0.4	0.6	0.8	1.0
0.01	Lu $v_1^2$	0.00999	0.00998	0.00996	0.00994	0.00992	0.00990
	Lu $v_2^2$	1.00101	1.00202	1.00404	1.00606	1.00808	1.01010
0.05	Lu $v_1^2$	0.04974	0.04948	0.04897	0.04847	0.04798	0.04751
	Lu $v_2^2$	1.00526	1.01052	1.02103	1.03153	1.04202	1.05249
0.1	Lu $v_1^2$	0.09890	0.09783	0.09576	0.09379	0.09190	0.09010
	Lu $v_2^2$	1.0110	1.02217	1.04424	1.06621	1.08810	1.10990
0.2	Lu $v_1^2$	0.19515	0.19058	0.18218	0.17461	0.16775	0.16148
	Lu $v_2^2$	1.02485	1.04942	1.09782	1.14539	1.19225	1.23852
0.4	Lu $v_1^2$	0.37591	0.35581	0.32349	0.29808	0.27725	0.25969
	Lu $v_2^2$	1.06409	1.12419	1.23651	1.34192	1.44275	1.54031
0.6	Lu $v_1^2$	0.53184	0.48637	0.42361	0.37967	0.34603	0.31898
	Lu $v_2^2$	1.12816	1.23363	1.41639	1.58033	1.73397	1.88102
0.8	Lu $v_1^2$	0.65086	0.57950	0.49114	0.43318	0.39030	0.35660
	Lu $v_2^2$	1.22914	1.38050	1.62886	1.84682	2.04970	2.24340
1.0	Lu $v_1^2$	0.72984	0.64174	0.53668	0.46934	0.42020	0.38197
	Lu $v_2^2$	1.37016	1.55826	1.86332	2.13066	2.37980	2.61803

Let us now introduce the notations

$$\alpha_{ik} = K_{i1} + \frac{v_k^2 - 1}{K_{0*}} K_{i2}; \quad \beta_{ik} = K_{i3} + \frac{v_k^2 - 1}{K_{0*}} K_{i4} \quad (6-5-17)$$

$$\gamma_{ik} = K_{i5} - \frac{v_k^2 - 1}{K_{0*}} K_{i6} \quad (6-5-18)$$

The solution of the system of heat- and mass-transfer equations becomes

$$\begin{aligned} \theta_1(X, Fo) = & \varphi_1(0) \left\{ B_{11} - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \psi_n} \begin{vmatrix} W_r(v_1 \mu_n X) & W_r(v_2 \mu_n X) \\ Q_{21}(\mu_n) & Q_{22}(\mu_n) \end{vmatrix} \exp(-\mu_n^2 Fo) \right\} \\ & + B_{11} \int_0^{Fo} \varphi_1'(Fo - Fo^*) dFo^* - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \psi_n} \\ & \times \begin{vmatrix} W_r(v_1 \mu_n X) & W_r(v_2 \mu_n X) \\ Q_{21}(\mu_n) & Q_{22}(\mu_n) \end{vmatrix} \\ & \times \int_0^{Fo} \varphi_1'(Fo - Fo^*) \exp(-\mu_n^2 Fo^*) dFo^* + \varphi_2(0) \left\{ B_{12} - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \psi_n} \right. \\ & \times \begin{vmatrix} Q_{11}(\mu_n) & Q_{12}(\mu_n) \\ W_r(v_1 \mu_n X) & W_r(v_2 \mu_n X) \end{vmatrix} \exp(-\mu_n^2 Fo) \left. \right\} \\ & + B_{12} \int_0^{Fo} \varphi_2'(Fo - Fo^*) dFo^* - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \psi_n} \begin{vmatrix} Q_{11}(\mu_n) & Q_{12}(\mu_n) \\ W_r(v_1 \mu_n X) & W_r(v_2 \mu_n X) \end{vmatrix} \\ & \times \int_0^{Fo} \varphi_2'(Fo - Fo^*) \exp(-\mu_n^2 Fo^*) dFo^* \end{aligned} \quad (6-5-19)$$

$$\begin{aligned} \theta_2(X, Fo) = & \varphi_1(0) \left\{ B_{21} - \frac{1}{K_{0*}} \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \psi_n} \right. \\ & \times \begin{vmatrix} (v_1^2 - 1) W_r(v_1 \mu_n X) & (v_2^2 - 1) W_r(v_2 \mu_n X) \\ Q_{21}(\mu_n) & Q_{22}(\mu_n) \end{vmatrix} \exp(-\mu_n^2 Fo) \left. \right\} \\ & + B_{21} \int_0^{Fo} \varphi_1'(Fo - Fo^*) dFo^* - \frac{1}{K_{0*}} \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \psi_n} \end{aligned}$$

$$\begin{aligned}
& \times \left| \frac{(v_1^2 - 1) W_r(v_1 \mu_n X)}{Q_{21}(\mu_n)} \frac{(v_2^2 - 1) W_r(v_2 \mu_n X)}{Q_{22}(\mu_n)} \right| \\
& \times \int_0^{Fo} \varphi'_1(Fo - Fo^*) \exp(-\mu_n^2 Fo^*) dFo^* + \varphi_2(0) \\
& \times \left\{ B_{22} - \frac{1}{K_{0*}} \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \psi_n} \left| \frac{Q_{11}(\mu_n)}{(v_1^2 - 1) W_r(v_1 \mu_n X)} \frac{Q_{12}(\mu_n)}{(v_2^2 - 1) W_r(v_2 \mu_n X)} \right| \right. \\
& \times \exp(-\mu_n^2 Fo) \left. \right\} + B_{22} \int_0^{Fo} \varphi'_2(Fo - Fo^*) dFo^* - \frac{1}{K_{0*}} \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \psi_n} \\
& \times \left| \frac{Q_{11}(\mu_n)}{(v_1^2 - 1) W_r(v_1 \mu_n X)} \frac{Q_{12}(\mu_n)}{(v_2^2 - 1) W_r(v_2 \mu_n X)} \right| \\
& \times \int_0^{Fo} \varphi'_2(Fo - Fo^*) \exp(-\mu_n^2 Fo^*) dFo^* \quad (6-5-20)
\end{aligned}$$

where

$$Q_{ki}(\mu_n) = \alpha_{ki} W_r(v_i \mu_n) - \beta_{ki} v_i \mu_n V_r(v_i \mu_n) + \gamma_{ki} (\Gamma + 1) \frac{V_r(v_i \mu_n)}{v_i \mu_n} \quad (6-5-21)$$

By prescribing the shape of function  $\varphi_k(Fo)$  particular solutions can be obtained from the general solution.

For a majority of problems mean transfer potentials are not introduced into the boundary conditions, i.e.

$$K_{15} = K_{16} = K_{25} = K_{26} = 0 \quad (6-5-22)$$

Then in the solution of the problem the functions  $Q_{ki}$  and  $B_{ki}$  become

$$\begin{aligned}
Q_{ki}(\mu_n) = & \left[ K_{ki} + \frac{v_k^2 - 1}{K_{0*}} K_{k2} \right] W_r(v_i \mu_n) \\
& - \left[ K_{k3} + \frac{v_i^2 - 1}{K_{0*}} K_{ki} \right] v_i \mu_n V_r(v_i \mu_n) \quad (6-5-23)
\end{aligned}$$

$$B_{11} = \frac{K_{22}}{\begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix}}; \quad B_{12} = \frac{-K_{12}}{\begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix}}; \quad B_{21} = \frac{-K_{21}}{\begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix}} \quad (6-5-24)$$

$$\begin{aligned}
\psi_n = & \frac{v_2^2 - v_1^2}{K_{0*}} \left[ \left\{ \begin{aligned} & \left| K_{0*} K_{11} - K_{12} K_{13} \right| \\ & \left| K_{0*} K_{21} - K_{22} K_{23} \right| \end{aligned} \right. \right. \\
& + \left. \left\{ \begin{aligned} & \left( K_{0*} P_n + \frac{1}{Lu} \right) K_{11} - P_n K_{12} K_{14} \\ & \left( K_{0*} P_n + \frac{1}{Lu} \right) K_{21} - P_n K_{22} K_{24} \end{aligned} \right. + \frac{1 - \Gamma}{\mu_n^2} \left| \begin{aligned} & K_{11} K_{12} \\ & K_{21} K_{23} \end{aligned} \right| \right\} \\
& \times W_I(v_1 \mu_n) W_I(v_2 \mu_n) + \frac{\mu_n^2}{Lu} \left| \begin{aligned} & K_{13} K_{14} \\ & K_{23} K_{24} \end{aligned} \right| \left| \frac{W_I(v_1 \mu_n)}{V_I(v_1 \mu_n)} - \frac{W_I(v_2 \mu_n)}{V_I(v_2 \mu_n)} \right| \\
& + \left| \begin{aligned} & K_{11} K_{12} \\ & K_{21} K_{22} \end{aligned} \right| \left| \frac{W_I(v_1 \mu_n)}{v_1^2 V_I(v_1 \mu_n)} - \frac{W_I(v_2 \mu_n)}{v_2^2 V_I(v_2 \mu_n)} \right| + \frac{\mu_n^2}{Lu} \\
& \times \left\{ \left| \begin{aligned} & K_{12} + (1 - \Gamma) K_{14} K_{13} \\ & K_{22} + (1 - \Gamma) K_{24} K_{23} \end{aligned} \right| - \left| \begin{aligned} & K_{11} K_{14} \\ & K_{21} K_{24} \end{aligned} \right| \right\} \frac{V_I(v_1 \mu_n)}{v_1 \mu_n} \frac{V_I(v_2 \mu_n)}{v_2 \mu_n} \quad (6-5-25)
\end{aligned}$$

$\mu_n$  are the roots of the characteristic equation

$$\begin{aligned}
& (v_1^2 - v_2^2) \left| \begin{aligned} & K_{11} K_{12} \\ & K_{21} K_{22} \end{aligned} \right| W_I(v_1 \mu_n) W_I(v_2 \mu_n) + (v_1^2 - v_2^2) \frac{\mu_n^4}{Lu} \\
& \times \left| \begin{aligned} & K_{13} K_{14} \\ & K_{23} K_{24} \end{aligned} \right| \frac{V_I(v_1 \mu_n)}{v_1 \mu_n} \frac{V_I(v_2 \mu_n)}{v_2 \mu_n} + \frac{\mu_n^2}{Lu} \left| \begin{aligned} & K_{12} K_{13} \\ & K_{22} K_{23} \end{aligned} \right| \\
& \times \left| \frac{W_I(v_1 \mu_n)}{v_1 \mu_n} \frac{W_I(v_2 \mu_n)}{v_2 \mu_n} \right| + \mu_n^2 \left| \begin{aligned} & K_{11} K_{14} \\ & K_{21} K_{24} \end{aligned} \right| \left| \frac{W_I(v_1 \mu_n)}{v_1^4 V_I(v_1 \mu_n)} - \frac{W_I(v_2 \mu_n)}{v_2^4 V_I(v_2 \mu_n)} \right| \\
& + \mu_n^2 \left\langle \left| \begin{aligned} & K_{0*} K_{11} - K_{12} K_{13} \\ & K_{0*} K_{21} - K_{22} K_{23} \end{aligned} \right| \right. \\
& \left. - \left| \begin{aligned} & K_{11} + P_n K_{12} K_{14} \\ & K_{21} + P_n K_{22} K_{24} \end{aligned} \right| \right\rangle \left| \frac{W_I(v_1 \mu_n)}{v_1^2 V_I(v_1 \mu_n)} - \frac{W_I(v_2 \mu_n)}{v_2^2 V_I(v_2 \mu_n)} \right| = 0 \quad (6-5-26)
\end{aligned}$$

Solutions (6-5-19)-(6-5-20) are correct only provided that

$$\begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} \neq 0$$

Let us now consider the solution on condition that

$$\begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} = 0 \quad (6-5-27)$$

From characteristic equation (6-5-26) it may be seen that  $\mu = 0$  is also a root. The solutions (6-5-19)-(6-5-20) involve the following constants

$$\begin{aligned} B_{11} = P & \left[ K_{22}F + \frac{1}{2(\Gamma + 1)} \left\{ X^2 \begin{vmatrix} 1 & K_{0*} \\ K_{21} & K_{22} \end{vmatrix} \right. \right. \\ & \left. \left. + \begin{vmatrix} K_{0*} P_n + \frac{1}{Lu} - K_{0*} \\ K_{21} + 2K_{23}K_{22} + 2K_{24} \end{vmatrix} \right\} - K_{22}QP \right] \end{aligned} \quad (6-5-28)$$

$$\begin{aligned} B_{12} = P & \left[ -K_{12}F + \frac{1}{2(\Gamma + 1)} \left\{ X^2 \begin{vmatrix} K_{11} & K_{12} \\ 1 & K_{0*} \end{vmatrix} \right. \right. \\ & \left. \left. + \begin{vmatrix} K_{11} + 2K_{13} & K_{12} + K_{14} \\ K_{0*} P_n + \frac{1}{Lu} - K_{0*} \end{vmatrix} \right\} + K_{12}QP \right] \end{aligned} \quad (6-5-29)$$

$$\begin{aligned} B_{21} = P & \left[ -K_{21}F + \frac{1}{2(\Gamma + 1)} \left\{ X^2 \begin{vmatrix} P_n K_{0*} & P_n + \frac{1}{Lu} \\ K_{21} & K_{22} \end{vmatrix} \right. \right. \\ & \left. \left. + \begin{vmatrix} -P_n & 1 \\ K_{21} + 2K_{23} & K_{22} + 2K_{24} \end{vmatrix} \right\} + K_{21}QP \right] \end{aligned} \quad (6-5-30)$$



$$B_{22} = P \left[ K_{11}F + \frac{1}{2(\Gamma + 1)} \left\{ X^2 \left| \begin{array}{cc} K_{11} & K_{12} \\ \text{Pn Ko}^* & \text{Pn} \end{array} \right| + \frac{1}{\text{Lu}} \right. \right. \\ \left. \left. + \left| \begin{array}{cc} K_{11} + 2K_{13} & K_{12} + 2K_{14} \\ -\text{Pn} & 1 \end{array} \right| \right\} - K_{11}QP \right] \quad (6-5-31)$$

where

$$P = \frac{\Gamma + 1}{\left| \begin{array}{cc} \text{Ko}^* K_{11} - K_{12} K_{13} + \text{Pn} K_{14} \\ \text{Ko}^* K_{21} - K_{22} K_{23} + \text{Pn} K_{24} \end{array} \right| + \frac{1}{\text{Lu}} \left| \begin{array}{cc} K_{11} & K_{14} \\ K_{21} & K_{24} \end{array} \right|} \quad (6-5-32)$$

$$Q = \frac{1}{2(\Gamma + 1)} \left[ \frac{1}{\text{Lu}} \left\{ \left| \begin{array}{cc} K_{11} + K_{13}K_{14} \\ K_{21} + K_{23}K_{24} \end{array} \right| + \left| \begin{array}{cc} K_{13}K_{12} + K_{14} \\ K_{23}K_{22} + K_{24} \end{array} \right| \right\} \right. \\ \left. - \frac{1}{\text{Lu}} \left\{ \left| \begin{array}{cc} K_{12} K_{13} \\ K_{22} K_{23} \end{array} \right| - \left| \begin{array}{cc} K_{11} K_{14} \\ K_{21} K_{24} \end{array} \right| \right\} \right. \\ \left. + \frac{1 + \text{Ko}^* \text{Pn} + \frac{1}{\text{Lu}}}{\Gamma + 3} \left\{ \left| \begin{array}{cc} \text{Ko}^* K_{11} - K_{12} K_{13} \\ \text{Ko}^* K_{21} - K_{22} K_{23} \end{array} \right| - \left| \begin{array}{cc} K_{11} + \text{Pn} K_{12} K_{14} \\ K_{21} + \text{Pn} K_{22} K_{24} \end{array} \right| \right\} \right. \\ \left. + \frac{\left( 1 + \text{Ko}^* \text{Pn} + \frac{1}{\text{Lu}} \right)^2}{\Gamma + 3} \left| \begin{array}{cc} K_{11} & K_{14} \\ K_{21} & K_{24} \end{array} \right| \right] \quad (6-5-33)$$

The obtained solution (6-5-28) through (6-5-33) is valid provided that

$$\left| \begin{array}{cc} \text{Ko}^* K_{11} - K_{12} K_{13} + \text{Pn} K_{14} \\ \text{Ko}^* K_{21} - K_{22} K_{23} + \text{Pn} K_{24} \end{array} \right| + \frac{1}{\text{Lu}} \left| \begin{array}{cc} K_{11} & K_{14} \\ K_{21} & K_{24} \end{array} \right| \neq 0 \quad (6-5-34)$$

Constraint (6-5-34) is not fulfilled when  $K_{11} = K_{12} = K_{21} = K_{22} = 0$ . For this case the solution becomes [6-2]

$$\theta_1(X, \text{Fo}) = \frac{1}{\left| \begin{array}{cc} K_{13} & K_{14} \\ K_{23} & K_{24} \end{array} \right|} \left\{ \varphi_1(0) \left\{ (\Gamma + 1) \text{Lu Fo} \left[ \text{Ko}^* K_{23} + \left( \text{Ko}^* \text{Pn} \right. \right. \right. \right.$$

$$\begin{aligned}
& + \frac{1}{\text{Lu}} \Big) K_{24} \Big] - \frac{1}{2} \left( \frac{\Gamma + 1}{\Gamma + 3} - X^2 \right) K_{24} - \sum_{n=1}^{\infty} \sum_{i=1}^2 S_{1i}^1 \frac{2W_{\Gamma}(\mu_n X)}{\mu_n^2 W_{\Gamma}(\mu_n)} \\
& \exp [-\mu_n^2 \text{Lu } v_i^2 \text{Fo}] \Big\} + (\Gamma + 1) \text{Lu} \left[ K_{0*} K_{23} + \left( K_{0*} \text{Pn} + \frac{1}{\text{Lu}} \right) K_{24} \right] \int_0^{\text{Fo}} \varphi_1'(\text{Fo} \\
& - \text{Fo}^*) \text{Fo}^* d \text{Fo}^* - \frac{1}{2} \left( \frac{\Gamma + 1}{\Gamma + 3} - X^2 \right) K_{24} \int_0^{\text{Fo}} \varphi_1'(\text{Fo} - \text{Fo}^*) d \text{Fo}^* \\
& - \sum_{n=1}^{\infty} \sum_{i=1}^2 S_{1i}^1 \frac{2W_{\Gamma}(\mu_n X)}{\mu_n^2 W_{\Gamma}(\mu_n)} \times \int_0^{\text{Fo}} \exp [-\mu_n^2 \text{Lu } v_i^2 \text{Fo}^*] \varphi_1'(\text{Fo} - \text{Fo}^*) d \text{Fo}^* \\
& - \varphi_2(0) \left\{ (\Gamma + 1) \text{Lu } F \times \left[ K_{0*} K_{13} + \left( K_{0*} \text{Pn} + \frac{1}{\text{Lu}} \right) K_{14} \right] \right. \\
& \left. - \frac{1}{2} \left( \frac{\Gamma + 1}{\Gamma + 3} - X^2 \right) K_{14} - \sum_{n=1}^{\infty} \sum_{i=1}^2 S_{2i}^1 \frac{2W_{\Gamma}(\mu_n X)}{\mu_n^2 W_{\Gamma}(\mu_n)} \exp [-\mu_n^2 \text{Lu } v_i^2 \text{Fo}] \right\} \\
& - (\Gamma + 1) \text{Lu} \left[ K_{0*} K_{13} + \left( K_{0*} \text{Pn} + \frac{1}{\text{Lu}} \right) K_{14} \right] \int_0^{\text{Fo}} \varphi_2'(\text{Fo} \\
& - \text{Fo}^*) \text{Fo}^* d \text{Fo}^* + \frac{1}{2} \left( \frac{\Gamma + 1}{\Gamma + 3} - X^2 \right) K_{14} \int_0^{\text{Fo}} \varphi_2'(\text{Fo} - \text{Fo}^*) d \text{Fo}^* \\
& + \sum_{n=1}^{\infty} \sum_{i=1}^2 S_{2i}^1 \frac{2W_{\Gamma}(\mu_n X)}{\mu_n^2 W_{\Gamma}(\mu_n)} \int_0^{\text{Fo}} \varphi_2'(\text{Fo} - \text{Fo}^*) \exp [-\mu_n^2 \text{Lu } v_i^2 \text{Fo}^*] d \text{Fo}^* \Big\}
\end{aligned}$$

(6-5-35)

$$\theta_2 = (X, \text{Fo}) = \frac{1}{\begin{vmatrix} K_{13} & K_{14} \\ K_{23} & K_{24} \end{vmatrix}} \left[ -\varphi_1(0) \left\{ (\Gamma + 1) \text{Lu } F (K_{23} + \text{Pn } K_{24}) \right. \right.$$

$$\begin{aligned}
& - \frac{1}{2} \left( \frac{\Gamma + 1}{\Gamma + 3} - X^2 \right) K_{23} - \sum_{n=1}^{\infty} \sum_{i=1}^2 S_{2i}^2 \frac{2W_{\Gamma}(\mu_n X)}{\mu_n^2 W_{\Gamma}(\mu_n)} \exp \left[ - \frac{\mu_n^2}{v_i^2} \text{Fo}^* \right] \Bigg\} \\
& - (\Gamma + 1) \text{Lu} [K_{23} + \text{Pn } K_{24}] \int_0^{\text{Fo}} \varphi_1'(\text{Fo} - \text{Fo}^*) d\text{Fo}^* + \frac{1}{2} \left( \frac{\Gamma + 1}{\Gamma + 3} \right. \\
& \left. - X^2 \right) K_{23} \times \int_0^{\text{Fo}} \varphi_1'(\text{Fo} - \text{Fo}^*) d\text{Fo}^* + \sum_{n=1}^{\infty} \sum_{i=1}^2 S_{2i}^2 \frac{2W_{\Gamma}(\mu_n X)}{\mu_n^2 W_{\Gamma}(\mu_n)} \int_0^{\text{Fo}} \varphi_1'(\text{Fo} \\
& - \text{Fo}^*) \times \exp \left[ - \frac{\mu_n^2}{v_i^2} - \text{Fo}^* \right] d\text{Fo}^* + \varphi_2(0) \left\{ (\Gamma + 1) \text{Lu } \text{Fo} [K_{13} \right. \\
& \left. + \text{Pn } K_{14}] - \frac{1}{2} \left( \frac{\Gamma + 1}{\Gamma + 3} - X^2 \right) \times K_{13} - \sum_{n=1}^{\infty} \sum_{i=1}^2 S_{1i}^2 \frac{2W_{\Gamma}(\mu_n X)}{\mu_n^2 W_{\Gamma}(\mu_n)} \right. \\
& \left. \exp \left[ - \frac{\mu_n^2}{v_i^2} \text{Fo} \right] \right\} + (\Gamma + 1) \text{Lu}(K_{13} + \text{Pn } K_{14}) \times \int_0^{\text{Fo}} \varphi_2'(\text{Fo} - \text{Fo}^*) \text{Fo}^* d\text{Fo}^* \\
& - \frac{1}{2} \left( \frac{\Gamma + 1}{\Gamma + 3} - X^2 \right) K_{13} \int_0^{\text{Fo}} \varphi_2'(\text{Fo} - \text{Fo}^*) d\text{Fo}^* \\
& - \sum_{n=1}^{\infty} \sum_{i=1}^2 S_{1i}^2 \frac{2W_{\Gamma}(\mu_n X)}{\mu_n^2 W_{\Gamma}(\mu_n)} \int_0^{\text{Fo}} \varphi_2'(\text{Fo} - \text{Fo}^*) \exp \left[ - \frac{\mu_n^2}{v_i^2} \text{Fo}^* \right] d\text{Fo}^* \quad (6-5-36)
\end{aligned}$$

where

$$S_{ki}^1 = \frac{(-1)^{i+1}}{v_1^2 - v_2^2} \{K_{0k} K_{k3} + (v_i^2 - 1) K_{k4}\} \quad (k = 1, 2) \quad (6-5-37)$$

$$S_{ki}^2 = \frac{(-1)^i}{v_1^2 - v_2^2} \{K_{k3}(v_i^2 - 1) - \text{Pn } K_{k4}\} \quad (k = 1, 2) \quad (6-5-38)$$

$\mu_n$  are the roots of the characteristic equation

$$V_{\Gamma}(\mu_n) = 0 \quad (6-5-39)$$

The averaged transfer potentials are defined by the formula

$$\bar{\theta}_k(\text{Fo}) = (\Gamma + 1) \int_0^1 X^{\Gamma} \theta_k(X, \text{Fo}) dX \quad (6-5-40)$$

## 6-6. BOUNDARY CONDITIONS OF THE SECOND KIND

Many high-temperature processes involve complicated relation between specific heat and mass flows at the surface and transfer potentials. For example, during radiation the heat flow is proportional to the difference

of the fourth powers of absolute temperatures of the surfaces involved in heat transfer. In this case it is extremely difficult to get a closed solution of heat- and mass-transfer equations. This difficulty may be overcome if in the boundary conditions the flows are chosen as functions of time alone. The constitutive equations are then solved under the second-kind boundary conditions, which for one-dimensional problems are as follows

$$\frac{\partial \theta_1(1, Fo)}{\partial X} = Ki_1(Fo) \quad (6-6-1)$$

$$- Pn \frac{\partial \theta_2(1, Fo)}{\partial X} + \frac{\partial \theta_2(1, Fo)}{\partial X} = Ki_2(Fo) \quad (6-6-2)$$

If at the body surface along with a heat source there exists a sink, a case similar to the one observed during dynamic cooling of irradiated materials, or during radiative drying of moist disperse materials, condition (6-6-1) should be generalized to the form

$$\frac{\partial \theta_1(1, Fo)}{\partial X} = Ki_1(Fo) - (1 - \varepsilon) Lu Ko Ki_2(Fo) \quad (6-6-3)$$

When there is no sink at the surface ( $\varepsilon = 1$ ), the generalized condition (6-6-3) changes into (6-6-1).

Comparison of boundary conditions (6-6-2) and (6-6-3) with generalized boundary conditions (6-5-13) shows that the similarity numbers  $K_{km}$  have the following values

$$\begin{aligned} K_{11} = 0, \quad K_{12} = 0, \quad K_{13} = 1, \quad K_{14} = 0, \quad K_{15} = 0, \quad K_{16} = 0 \\ K_{21} = 0, \quad K_{22} = 0, \quad K_{23} = -Pn, \quad K_{24} = 1, \quad K_{25} = 0, \quad K_{26} = 0. \end{aligned} \quad (6-6-4)$$

and also

$$\varphi_1(Fo) = Ki_1(Fo) - (1 - \varepsilon) Lu Ko Ki_2(Fo) \quad (6-6-5)$$

$$\varphi_2(Fo) = Ki_2(Fo) \quad (6-6-6)$$

After necessary algebraic manipulations, solutions (6-5-35) and (6-5-36) take the following form [6-2]

$$\begin{aligned} \theta_1(X, Fo) = & (\Gamma + 1) [\varphi_1(0) - Lu Ko^* \varphi_2(0)] Fo - \frac{1}{2} \left( \frac{\Gamma + 1}{\Gamma + 3} - X^2 \right) \varphi_1(0) \\ & - \sum_{n=1}^{\infty} \sum_{l=1}^2 (-1)^l \frac{(1 + Ko^* Pn - v_l^2) \varphi_1(0) + Ko^* \varphi_2(0)}{v_1^2 - v_2^2} \\ & \times \frac{2W_r(\mu_n X)}{\mu_n^2 W_r(\mu_n)} \exp(-\mu_n^2 Lu v_1^2 Fo) + (\Gamma + 1) \int_0^{Fo} [\varphi_1'(Fo - Fo^*) - Lu Ko^* \\ & \times \varphi_2'(Fo - Fo^*)] Fo^* dFo^* - \frac{1}{2} \left( \frac{\Gamma + 1}{\Gamma + 3} - X^2 \right) \int_0^{Fo} \varphi_1'(Fo - Fo^*) dFo^* \end{aligned}$$

$$\begin{aligned}
& - \sum_{n=1}^{\infty} \sum_{i=1}^2 \frac{2W_r(\mu_n X)}{\mu_n^2 W_r(\mu_n)} (-1)^i \times \int_0^{Fo} [(1 + Ko^* Pn - v_i^2) \varphi_1'(Fo - Fo^*) + Ko^* \\
& \times \frac{\varphi_2'(Fo - Fo^*)] \exp(-\mu_n^2 Lu v_i^2 Fo^*) d Fo^*}{v_1^2 - v_2^2} \quad (6-6-7)
\end{aligned}$$

$$\begin{aligned}
\theta_2(X, Fo) = & (\Gamma + 1) Lu \varphi_2(0) Fo - \frac{1}{2} \left( \frac{\Gamma + 1}{\Gamma + 3} - X^2 \right) [Pn \varphi_1(0) + \varphi_2(0)] \\
& - \sum_{n=1}^{\infty} \sum_{i=1}^2 \frac{2W_r(\mu_n X)}{\mu_n^2 W_r(\mu_n)} (-1)^i \frac{v_i^2 Pn \varphi_1(0) + (v_i^2 - 1) \varphi_2(0)}{v_1^2 - v_2^2} \exp\left(-\frac{\mu_n^2}{v_i^2} Fo\right) \\
& + (\Gamma + 1) Lu \int_0^{Fo} \varphi_2'(Fo - Fo^*) Fo^* d Fo^* - \frac{1}{2} \left( \frac{\Gamma + 1}{\Gamma + 3} - X^2 \right) \int_0^{Fo} [Pn \varphi_1'(Fo \\
& - Fo^*) + \varphi_2'(Fo - Fo^*)] d Fo^* - \sum_{n=1}^{\infty} \sum_{i=1}^2 \frac{2W_r(\mu_n X)}{\mu_n^2 W_r(\mu_n)} \frac{(-1)^i}{v_1^2 - v_2^2} \\
& \times \int_0^{Fo} [v_i^2 Pn \varphi_1'(Fo - Fo^*) + (v_i^2 - 1) \varphi_2'(Fo \\
& - Fo^*)] \exp\left(-\frac{\mu_n^2}{v_i^2} Fo^*\right) d Fo^* \quad (6-6-8)
\end{aligned}$$

where  $\mu_n$  are the roots of the characteristic equation

$$V_r(\mu) = 0 \quad (6-6-9)$$

If the heat and mass flows at the surface are constant, i.e.  $Ki_1 = \text{const}$  and  $Ki_2 = \text{const}$ , then solutions (6-6-7) and (6-6-8) become

$$\begin{aligned}
\theta_1(X, Fo) = & (\Gamma + 1) (N - Lu Ko^* Ki_2) Fo - \frac{1}{2} \left( \frac{\Gamma + 1}{\Gamma + 3} - X^2 \right) N \\
& - \sum_{n=1}^{\infty} \sum_{i=1}^2 \frac{2W_r(\mu_n X)}{\mu_n^2 W_r(\mu_n)} C_i^1 \exp(-\mu_n^2 Lu v_i^2 Fo) \quad (6-6-10)
\end{aligned}$$

$$\begin{aligned}
\theta_2(X, Fo) = & (\Gamma + 1) Lu Ki_2 Fo - \frac{1}{2} \left( \frac{\Gamma + 1}{\Gamma + 3} - X^2 \right) (Pn N + Ki_2) \\
& - \sum_{n=1}^{\infty} \sum_{i=1}^2 \frac{2W_r(\mu_n X)}{\mu_n^2 W_r(\mu_n)} C_i^2 \exp(-\mu_n^2 Lu v_i^2 Fo) \quad (6-6-11)
\end{aligned}$$

where

$$N = Ki_1 - (1 - \varepsilon) Lu Ko Ki_2 \quad (6-6-12)$$

$$C_i^1 = (-1)^i \frac{(1 + Ko^* Pn - v_i^2 N + Ko^* Ki_2)}{(v_1^2 - v_2^2)} \quad (6-6-13)$$

$$C_i^2 = (-1)^i \frac{Pn N - (Lu v_i^2 - 1) Ki_2}{(v_1^2 - v_2^2) Lu v_i^2} \quad (6-6-14)$$

Coefficients  $C_i^1$  and  $C_i^2$ , as functions of different similarity criteria are presented in [6-1].

In the absence of thermodiffusion mass transfer in the test material, i.e. when  $Pn = 0$ , and taking into consideration that  $v_1^2 = \frac{1}{Lu}$  and  $v_2^2 = 1$ , from (6-6-10) and (6-6-11) we get

$$\begin{aligned} \theta_1(X, Fo) = & (\Gamma + 1) (Ki_1 - Lu Ko^* Ki_2) Fo - \frac{1}{2} \left( \frac{\Gamma + 1}{\Gamma + 3} - X^2 \right) Ki_1 \\ & - \sum_{n=1}^{\infty} \frac{2W_r(\mu_n X)}{\mu_n^2 W_r(\mu_n)} \left\{ Ki_1 \exp(-\mu_n^2 Fo) + \frac{Ko^* Lu Ki_2}{Lu - 1} \right. \\ & \times [\exp(-\mu_n^2 Fo) - \exp(-\mu_n^2 Lu Fo)] \} \end{aligned} \quad (6-6-15)$$

$$\begin{aligned} \theta_2(X, Fo) = & Ki_2 \left[ (\Gamma + 1) Lu Fo - \frac{1}{2} \left( \frac{\Gamma + 1}{\Gamma + 3} - X^2 \right) \right. \\ & \left. - \sum_{n=1}^{\infty} \frac{2W_r(\mu_n X)}{\mu_n^2 W_r(\mu_n)} \exp(-\mu_n^2 Lu Fo) \right] \end{aligned} \quad (6-5-16)$$

Solution (6-6-15) may be written as follows [6-2]

$$\theta_1(X, Fo) = \theta_1^I(X, Fo) + \theta_1^{II}(X, Fo) + \theta_1^{III}(X, Fo) \quad (6-6-17)$$

where

$$\begin{aligned} \theta_1^I(X, Fo) = & Ki_1 \left[ (\Gamma + 1) Fo - \frac{1}{2} \left( \frac{\Gamma + 1}{\Gamma + 3} - X^2 \right) \right. \\ & \left. - \sum_{n=1}^{\infty} \frac{2W_r(\mu_n X)}{\mu_n^2 W_r(\mu_n)} \exp(-\mu_n^2 Fo) \right] \end{aligned} \quad (6-6-18)$$

$$\begin{aligned} \theta_1^{II}(X, Fo) = & - Ko^* Lu Ki_2 \left[ (\Gamma + 1) Fo + \frac{1}{Lu - 1} \right. \\ & \left. \times \sum_{n=1}^{\infty} \frac{2W_r(\mu_n X)}{\mu_n^2 W_r(\mu_n)} \exp(-\mu_n^2 Fo) \right] \end{aligned} \quad (6-6-19)$$

$$\theta_1^{III}(X, Fo) = Ko^* Lu Ki_2 \frac{1}{Lu - 1} \sum_{n=1}^{\infty} \frac{2W_r(\mu_n X)}{\mu_n^2 W_r(\mu_n)} \exp(-\mu_n^2 Lu Fo) \quad (6-6-20)$$

Hence, in the absence of thermodiffusion mass transfer in the test material the temperature in the body is determined by superimposing three temperature fields. The first field characterizes material heating without

taking into consideration mass transfer and phase changes. The second field accounts for the effects of fixed heat sinks (sources). And the last field describes the effect of moving heat sinks due to mass transfer.

If in addition to all this we assume that no phase changes take place in the body ( $Ko^* = 0$ ), the temperature field is described only by formula (6-6-18), which gives classical solution of the equation of heat conduction at the second-kind boundary conditions and without sources of heat. It should be pointed out that a dimensionless mass-transfer potential, determined from formula (6-6-16), is described by the classical solution after the substitution  $Fo_m = Lu Fo$ .

### 6-7. BOUNDARY CONDITIONS OF THE THIRD KIND

As a first approximation convective heat transfer may be described by the Newton law, i.e.

$$Ki_1 = Bi_1 \{\theta_{f1}(Fo) - \theta_1(1, Fo)\}, \quad Ki_2 = Ki_2(Fo) \quad (6-7-1)$$

Then the boundary conditions in the dimensionless form are written as

$$\theta_1(1, Fo) + \frac{1}{Bi_1} \frac{\partial \theta_1(1, Fo)}{\partial X} = \theta_{fi}(F) - (1 - \varepsilon) \frac{Ko Lu}{Bi_1} Ki_2(Fo) \quad (6-7-2)$$

$$- Pn \frac{\partial \theta_1(1, Fo)}{\partial X} + \frac{\partial \theta_2(1, Fo)}{\partial X} = Ki_2(Fo) \quad (6-7-3)$$

After making necessary algebraic manipulations, solutions (6-5-19) and (6-5-20) take the following form [6-2]

$$\begin{aligned} \theta_1(X, Fo) = & \varphi_1(0) - \frac{1}{2} Ko^* Lu \left( 1 + \frac{2}{Bi_1} - X^2 \right) \varphi_2(0) + \frac{Ko^* Lu}{v_2^2 - v_1^2} \varphi_2(0) \\ & \times \sum_{m=1}^{\infty} \sum_{i=1}^2 C_{mi} W_r(v_i \mu_m X) \exp(-\mu_m^2 Fo) - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \psi_n} \\ & \times \left\{ \begin{vmatrix} \varphi_1(0) & Q_{12}(\mu_n) \\ \varphi_2(0) & Q_{21}(\mu_n) \end{vmatrix} W_r(v_1 \mu_n X) + \begin{vmatrix} Q_{11}(\mu_n) & \varphi_1(0) \\ Q_{21}(\mu_n) & \varphi_2(0) \end{vmatrix} W_r(v_2 \mu_n X) \right\} \\ & \times \exp(-\mu_n^2 Fo) + \int_0^{Fo} \left[ \varphi_1'(Fo - Fo^*) - \frac{1}{2} Ko^* Lu \left( 1 + \frac{2}{Bi_1} - X^2 \right) \right. \\ & \left. \times \varphi_2'(Fo - Fo^*) \right] dFo^* + \frac{Ko^* Lu}{v_2^2 - v_1^2} \sum_{m=1}^{\infty} \sum_{i=1}^2 C_{mi} W_r(v_i \mu_m X) \int_0^{Fo} \varphi_2'(Fo - Fo^*) \end{aligned}$$

$$\begin{aligned} & \times \exp(-\mu_m^2 \text{Fo}^*) d\text{Fo}^* - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \psi_n} \int_0^{\text{Fo}} \left\{ \left| \frac{\varphi_1'(\text{Fo} - \text{Fo}^*) Q_{12}(\mu_n)}{\varphi_2'(\text{Fo} - \text{Fo}^*) Q_{22}(\mu_n)} \right| \right. \\ & \times W_r(v_1 \mu_n X) + \left. \left| \frac{Q_{11}(\mu_n) \varphi_1'(\text{Fo} - \text{Fo}^*)}{Q_{21}(\mu_n) \varphi_2'(\text{Fo} - \text{Fo}^*)} \right| \right\} \exp(-\mu_n^2 \text{Fo}^*) d\text{Fo}^* \quad (6-7-4) \end{aligned}$$

$$\begin{aligned} \theta_2(X, \text{Fo}) = & \left[ (\Gamma + 1) \text{Lu} \text{Fo} - \frac{1}{2} (1 + \text{Ko}^* \text{Pn} \text{Lu}) \left( \frac{\Gamma + 1}{\Gamma + 3} - X^2 \right) \right] \varphi_2(0) \\ & + \frac{\text{Lu}}{v_2^2 - v_1^2} \varphi_2(0) \sum_{m=1}^{\infty} \sum_{i=1}^2 C_{mi} (v_i^2 - 1) W_r(v_i \mu_m X) \exp(-\mu_m^2 \text{Fo}) \\ & - \frac{1}{\text{Ko}^*} \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \psi_n} \left\{ \left| \frac{\varphi_1(0) Q_{12}(\mu_n)}{\varphi_2(0) Q_{22}(\mu_n)} \right| (v_1^2 - 1) W_r(v_1 \mu_n X) \right. \\ & + \left. \left| \frac{Q_{11}(\mu_n) \varphi_1(0)}{Q_{21}(\mu_n) \varphi_2(0)} \right| (v_2^2 - 1) W_r(v_2 \mu_n X) \right\} \exp(-\mu_n^2 \text{Fo}) \\ & + \int_0^{\text{Fo}} \left[ (\Gamma + 1) \text{Lu} \text{Fo}^* - \frac{1}{2} (1 + \text{Ko}^* \text{Pn} \text{Lu}) \left( \frac{\Gamma + 1}{\Gamma + 3} - X^2 \right) \right] \varphi_2'(\text{Fo} \\ & - \text{Fo}^*) d\text{Fo}^* + \frac{\text{Lu}}{v_2^2 - v_1^2} \times \sum_{m=1}^{\infty} \sum_{i=1}^2 C_{mi} (v_i^2 - 1) \int_0^{\text{Fo}} \varphi_2'(\text{Fo} \\ & - \text{Fo}^*) \exp(-\mu_m^2 \text{Fo}^*) d\text{Fo}^* - \frac{1}{\text{Ko}^*} \\ & \times \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \psi_n} \int_0^{\text{Fo}} \left\{ \left| \frac{\varphi_1'(\text{Fo} - \text{Fo}^*) Q_{12}(\mu_n)}{\varphi_2'(\text{Fo} - \text{Fo}^*) Q_{22}(\mu_n)} \right| (v_1^2 - 1) W_r(v_1 \mu_n X) \right. \\ & + \left. \left| \frac{Q_{11}(\mu_n) \varphi_1'(\text{Fo} - \text{Fo}^*)}{Q_{21}(\mu_n) \varphi_2'(\text{Fo} - \text{Fo}^*)} \right| (v_2^2 - 1) W_r(v_2 \mu_n X) \right\} \exp(-\mu_n^2 \text{Fo}^*) d\text{Fo}^* \quad (6-7-5) \end{aligned}$$

where

$$\begin{aligned} \psi_n = & -v_1 v_2 \frac{v_2^2 - v_1^2}{\text{Ko}^*} \left\{ \left( 1 + \frac{1 - \Gamma}{\text{Bi}_1} \right) V_r(v_1 \mu_n) V_r(v_2 \mu_n) \right. \\ & + \frac{\mu_n}{\text{Bi}_1} \langle v_1 W_r(v_1 \mu_n) V_r(v_2 \mu_n) + v_2 W_r(v_2 \mu_n) V_r(v_1 \mu_n) \rangle \\ & \left. - v_1 v_2 W_r(v_1 \mu_n) W_r(v_2 \mu_n) \right\} \quad (6-7-6) \end{aligned}$$



$$Q_{1i}(\mu_n) = W_r(v_i\mu_n) - \frac{1}{\text{Bi}_1} v_i\mu_n V_r(v_i\mu_n) \quad (6-7-7)$$

$$Q_{2i}(\mu_n) = \frac{1 - v_i^2}{v_i^2 \text{Lu Ko}^*} v_i\mu_n V_r(v_i\mu_n) \quad (6-7-8)$$

$\mu_m$  are the roots of the characteristic equation

$$V_r(v_1\mu) = 0; \quad V_r(v_2\mu) = 0 \quad (6-7-9)$$

and  $\mu_n$  are the roots of the characteristic equation

$$v_1(1 - v_2^2) \frac{W_r(v_1\mu)}{v_1\mu} - v_2(1 - v_2^2) \frac{W_r(v_2\mu)}{v_2\mu} = (v_1^2 - v_2^2) \frac{\mu}{\text{Bi}} \quad (6-7-9a)$$

If the ambient temperature and mass flow are constant, i.e.

$$\theta_f(\text{Fo}) = 1 \text{ and } \text{Ki}_2 = \text{const}$$

then solutions (6-7-4) and (6-7-5) are rewritten as

$$\begin{aligned} \theta_1(X, \text{Fo}) &= 1 - \text{Ko}^* \bar{\text{Ki}}_1 - \frac{1}{2} \text{Ko}^* \text{Lu Ki}_2 \left( 1 + \frac{2}{\text{Bi}_1} - X^2 \right) \\ &- \frac{\text{Ko}^* \text{Lu Ki}_2}{v_2^2 - v_1^2} \sum_{m=1}^{\infty} \sum_{i=1}^2 C_{mi} W_r(v_i\mu_m X) \exp(-\mu_m^2 \text{Fo}) \\ &- \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} W_r(v_i\mu_n X) \exp(-\mu_n^2 \text{Fo}) \end{aligned} \quad (6-7-10)$$

$$\begin{aligned} \theta_2(X, \text{Fo}) &= \left[ (\Gamma + 1) \text{Lu Fo} - \frac{1}{2} (1 + \text{Ko}^* \text{Pn Lu}) \left( \frac{\Gamma + 1}{\Gamma + 3} \right) - X^2 \right. \\ &- \left. \frac{\text{Lu}}{v_2^2 - v_1^2} \sum_{m=1}^{\infty} \sum_{i=1}^2 C_{mi} (v_2^2 - 1) W_r(v_i\mu_m X) \exp(-\mu_m^2 \text{Fo}) \right] \text{Ki}_2 \\ &- \frac{1}{\text{Ko}^*} \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} (v_i^2 - 1) W_r(v_i\mu_n X) \exp(-\mu_n^2 \text{Fo}) \end{aligned} \quad (6-7-11)$$

$$\bar{K}_1 = \frac{1 - \varepsilon}{\varepsilon} \text{Lu} \frac{\text{Ki}_2}{\text{Bi}_1} \quad (6-7-12)$$

$$C_{mi} = (-1)^i \frac{2}{\mu_m^2 W_r(v_i\mu_m)} \quad (6-7-13)$$

$$C_{ni} = (-1)^i \frac{2}{\mu_n^2 \psi_n} \{ \text{Ki}_2 Q_{1j} - (1 - \text{Ko}^* \bar{K}_1) Q_{2j} \} \begin{pmatrix} i=1, j=2 \\ i=2, j=1 \end{pmatrix} \quad (6-7-14)$$

Substitution of solutions (6-7-10) and (6-7-11) into (6-5-40) gives the mean dimensionless potentials

$$\bar{\theta}_1(\text{Fo}) = 1 - \text{Ko}^* \bar{\text{Ki}}_1 - \text{Ko}^* \text{Lu} \text{Ki}_2 \left( \frac{1}{\Gamma + 3} + \frac{1}{\text{Bi}_1} \right) \quad (6-7-15)$$

$$- \frac{\text{Ko}^* \text{Lu} \text{Ki}_2}{v_2^2 - v_1^2} \sum_{m=1}^{\infty} \sum_{i=1}^2 D_{mi} \exp(-\mu_m^2 \text{Fo}) - \sum_{n=1}^{\infty} \sum_{i=1}^2 D_{ni} \exp(-\mu_n^2 \text{Fo})$$

$$\bar{\theta}_2(\text{Fo}) = \left[ (\Gamma + 1) \text{Lu} \text{Fo} - \frac{\text{Lu}}{v_2^2 - v_1^2} \sum_{m=1}^{\infty} \sum_{i=1}^2 D_{mi} (v_i^2 - 1) \right.$$

$$\left. \times \exp(-\mu_m^2 \text{Fo}) \right] \text{Ki}_2 - \frac{1}{\text{Ko}^*} \sum_{n=1}^{\infty} \sum_{i=1}^2 D_{ni} (v_i^2 - 1) \exp(-\mu_n^2 \text{Fo}) \quad (6-7-16)$$

where

$$D_{mi} = (\Gamma + 1) C_{mi} \frac{V_r(v_i \mu_m)}{v_i \mu_m}, \quad D_{ni} = (\Gamma + 1) C_{ni} \frac{V_r(v_i \mu_n)}{v_i \mu_n} \quad (6-7-17)$$

Analysis of the solutions obtained demonstrates that infinite sums converge quickly enough. As the Fourier number decreases, the number of terms necessary for calculation increases. The first six roots of the characteristic equations are given in [6-1]. The values of constant  $C_{ni}$  for various combinations of similarity criteria are also given in [6-1].

The infinite sum with respect to index  $m$  converges even more quickly; from  $\text{Fo} = 0.05$  on it may be neglected in calculations.

The first terms of solutions (6-7-10) and (6-7-11) give steady-state distributions of temperature and mass-transfer potential reported in F. M. Polonskaya's work [6-5]. Under steady-state conditions the distributions of dimensionless transfer potentials in the material are parabolic.

Let us now consider the case when the Kirpichev heat- and mass-transfer numbers are equal

$$\text{Ki}_1 = \text{Bi}_1 \{ \theta_{f1}(\text{Fo}) - \theta_1(1, \text{Fo}) \}$$

$$\text{Ki}_2 = \text{Bi}_2 \{ \theta_{f2}(\text{Fo}) - \theta_2(1, \text{Fo}) \}$$

The dimensionless boundary conditions (6-5-9) and (6-5-10) are written as

$$\theta_1(1, \text{Fo}) - K_1 \text{Ko}^* \theta_2(1, \text{Fo}) + \frac{1}{\text{Bi}_1} \frac{\partial \theta_1(1, \text{Fo})}{\partial X} = \theta_{f1}(\text{Fo}) - K_1 \text{Ko}^* \theta_{f2}(\text{Fo}) \quad (6-7-18)$$

$$\text{Bi}_2 \theta_2(1, \text{Fo}) - \text{Pn} \frac{\partial \theta_1(1, \text{Fo})}{\partial X} + \frac{\partial \theta_2(1, \text{Fo})}{\partial X} = \text{Ki}_2(\text{Fo}) \quad (6-7-19)$$

Comparison of boundary conditions (6-7-18) and (6-7-19) with generalized boundary conditions (6-5-13) shows that sets of similarity criteria  $K_{km}$  have the following values

$$\left. \begin{aligned} K_{11} &= 1, & K_{12} &= -K_1 K_o^*, & K_{13} &= \frac{1}{Bi_i}, & K_{14} &= 0 \\ K_{15} &= 0, & K_{16} &= 0 \\ K_{21} &= 0, & K_{22} &= Bi_2, & K_{23} &= -Pn, & K_{24} &= 1 \\ K_{25} &= 0, & K_{26} &= 0 \end{aligned} \right\} \quad (6-7-20)$$

and also

$$\left. \begin{aligned} \varphi_1(Fo) &= \theta_{f1}(Fo) - K_1 K_o^* \theta_{f2}(Fo) \\ \varphi_2(Fo) &= Bi_2 \theta_{f2}(Fo) \end{aligned} \right\} \quad (6-7-21)$$

$$K_1 = \frac{1 - \varepsilon}{\varepsilon} Lu \frac{Bi_2}{Bi_1} \quad (6-7-22)$$

After making necessary algebraic manipulations, solutions (6-7-10) and (6-7-11) take the following form [6-2]

$$\begin{aligned} \theta_1(X, Fo) &= \theta_{f1}(Fo) - \varphi_1(0) \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \psi_n} \left| \begin{array}{cc} W_r(v_1 \mu_n X) & W_r(v_2 \mu_n X) \\ Q_{21}(\mu_n) & Q_{22}(\mu_n) \end{array} \right| \\ &\times \exp(-\mu_n^2 Fo) - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \psi_n} \left| \begin{array}{cc} W_r(v_1 \mu_n X) & W_r(v_2 \mu_n X) \\ Q_{21}(\mu_n) & Q_{22}(\mu_n) \end{array} \right| \int_0^{Fo} \varphi_1'(Fo - Fo^*) \\ &\times \exp(-\mu_n^2 Fo^*) dFo^* - \varphi_2(0) \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \psi_n} \left| \begin{array}{cc} Q_{11}(\mu_n) & Q_{12}(\mu_n) \\ W_r(v_1 \mu_n X) & W_r(v_2 \mu_n X) \end{array} \right| \\ &\times \exp(-\mu_n^2 Fo) - \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \psi_n} \left| \begin{array}{cc} Q_{11}(\mu_n) & Q_{12}(\mu_n) \\ W_r(v_1 \mu_n X) & W_r(v_2 \mu_n X) \end{array} \right| \int_0^{Fo} \varphi_2'(Fo - Fo^*) \\ &\times \exp(-\mu_n^2 Fo^*) dFo^* \quad (6-7-23) \\ \theta_2(X, Fo) &= \theta_{f2}(Fo) - \varphi_1(0) \frac{1}{K_o^*} \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \psi_n} \\ &\times \left| \begin{array}{cc} (v_1^2 - 1) W_r(v_1 \mu_n X) & (v_2^2 - 1) W_r(v_2 \mu_n X) \\ Q_{21}(\mu_n) & Q_{22}(\mu_n) \end{array} \right| \exp(-\mu_n^2 Fo) \end{aligned}$$

$$\begin{aligned}
& - \frac{1}{\text{Ko}^*} \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \psi_n} \left| \frac{(v_1^2 - 1) W_r(v_1 \mu_n X)}{Q_{21}(\mu_n)} \frac{(v_2^2 - 1) W_r(v_2 \mu_n X)}{Q_{22}(\mu_n)} \right| \\
& \times \int_0^{\text{Fo}} \varphi_1'(\text{Fo} - \text{Fo}^*) \exp(-\mu_n^2 \text{Fo}^*) d\text{Fo}^* - \varphi_2(0) \frac{1}{\text{Ko}^*} \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \psi_n} \\
& \times \left| \frac{Q_{11}(\mu_n)}{(v_1^2 - 1) W_r(v_1 \mu_n X)} \frac{Q_{12}(\mu_n)}{(v_2^2 - 1) W_r(v_2 \mu_n X)} \right| \exp(-\mu_n^2 \text{Fo}) \\
& - \frac{1}{\text{Ko}^*} \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \psi_n} \left| \frac{Q_{11}(\mu_n)}{(v_1^2 - 1) W_r(v_1 \mu_n X)} \frac{Q_{12}(\mu_n)}{(v_2^2 - 1) W_r(v_2 \mu_n X)} \right| \\
& \times \int_0^{\text{Fo}} \varphi_2'(\text{Fo} - \text{Fo}^*) \exp(-\mu_n^2 \text{Fo}^*) d\text{Fo}^* \tag{6-7-24}
\end{aligned}$$

where

$$\begin{aligned}
\psi_n = & \frac{v_1^2 - v_2^2}{\text{Ko}^*} \left\{ \left( 1 + \frac{1 - \Gamma}{\text{Bi}_1} + \frac{\text{Bi}_2}{\text{Bi}_1} - K_1 \text{Ko}^* \text{Pn} \right) v_1 v_2 V_r(v_1 \mu_n) V_r(v_2 \mu_n) \right. \\
& - \left( \frac{1}{\text{Lu}} + \frac{\text{Bi}_2}{\text{Bi}_1} + \frac{1 - \Gamma}{\mu_n^2} \text{Bi}_2 \right) W_r(v_1 \mu_n) W_r(v_2 \mu_n) + \frac{\mu_n}{\text{Bi}_1} \left[ v_2 \left( v_1^2 - \frac{\text{Bi}_1 \text{Bi}_2}{\mu_n^2} \right) \right. \\
& \times W_r(v_1 \mu_n) V_r(v_2 \mu_n) + v_1 \left( v_2^2 - \frac{\text{Bi}_1 \text{Bi}_2}{\mu_n^2} \right) W_r(v_2 \mu_n) V_r(v_1 \mu_n) \left. \right] \left. \right\} \tag{6-7-25}
\end{aligned}$$

$$Q_{1i}(\mu_n) = \{1 + (1 - v_i^2) K_1\} W_r(v_i \mu_n) - \frac{1}{\text{Bi}_1} v_i \mu_n V_r(v_i \mu_n) \tag{6-7-26}$$

$$Q_{2i}(\mu_n) = \frac{1 - v_i^2}{\text{Ko}^*} \left\{ \frac{v_i \mu_n}{v_i^2 \text{Lu}} V_r(v_i \mu_n) - \text{Bi}_2 W_r(v_i \mu_n) \right\} \tag{6-7-27}$$

where  $\mu_n$  are the roots of the characteristic equation

$$\begin{aligned}
& v^2 \mu \left\{ v_1^2 (1 - v_2^2) + (1 - v_1^2) \left[ v_1^2 (1 - v_2^2) K_1 - \frac{\text{Bi}_2}{\text{Bi}_1} \right] \right\} W_r(v_1 \mu) V_r(v_2 \mu) \\
& - v_1 \mu \left\{ v_2^2 (1 - v_1^2) + (1 - v_2^2) \left[ v_2^2 (1 - v_1^2) K_1 - \frac{\text{Bi}_2}{\text{Bi}_1} \right] \right\} W_r(v_2 \mu) V_r(v_1 \mu) \\
& = (v_1^2 - v_2^2) \left\{ \text{Bi}_2 W_r(v_1 \mu) W_r(v_2 \mu) + \frac{\mu_2}{\text{Bi}_1} v_1 v_2 V_r(v_1 \mu) V_r(v_2 \mu) \right\} \tag{6-7-28}
\end{aligned}$$

If the ambient potentials and mass flow are constant, i.e.  $\theta_{f1}(\text{Fo}) = 1$  and  $\theta_{f2}(\text{Fo}) = 1$ , solutions (6-7-23) and (6-7-24) are rewritten as

$$\theta_1(X, \text{Fo}) = 1 - \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} W_r(v_i \mu_n X) \exp(-\mu_n^2 \text{Fo}) \quad (6-7-29)$$

$$\theta_2(X, \text{Fo}) = 1 - \frac{1}{\text{Ko}^*} \sum_{n=1}^{\infty} \sum_{i=1}^2 C_{ni} (v_i^2 - 1) W_r(v_i \mu_n X) \exp(-\mu_n^2 \text{Fo}) \quad (6-7-30)$$

where

$$C_{ni} = (-1)^i \frac{2}{\mu_n^2 \psi_n} \left\{ \text{Bi}_2 Q_{1j} - (1 - K_1 \text{Ko}^*) Q_{2j} \right\} \begin{pmatrix} i=1, j=2 \\ i=2, j=1 \end{pmatrix} \quad (6-7-31)$$

An analysis shows that infinite series (6-7-29) and (6-7-30) converge rather quickly. As the Fourier number increases, the error due to neglect of the series terms with  $n = 2, 3$ , etc. decreases. Beginning from a certain initial value of Fo the first two terms of the sum give a correct result with specified degree of accuracy. Initial values of Fo for simplified calculations change with the variation of sets of criteria entering into the series. The values of Lu have maximum effect on initial Fo. With the increase in the initial value of Lu Fo decreases. The effect of  $\varepsilon$  and  $\text{Ko}^* \text{Pn}$  on the final result is much less. The effect of  $\text{Bi}_1$  on the dimensionless heat-transfer potential is appreciable. In a similar way, the initial value of Fo for a dimensionless mass-transfer potential depends essentially on  $\text{Bi}_2$ . In general, for heat-transfer parameters the initial values of Fo are smaller than for mass-transfer ones.

In the absence of thermodiffusion mass transfer in the material characteristic equation (6-7-28) as a result of Pn-transition ( $\text{Pn} = 0$ ) takes the form

$$\left\{ \frac{W_r(\mu)}{V_r(\mu)} - \frac{\mu}{\text{Bi}_1} \right\} \left\{ \frac{W_r\left(\frac{\mu}{\sqrt{\text{Lu}}}\right)}{V_r\left(\frac{\mu}{\sqrt{\text{Lu}}}\right)} - \frac{\mu}{\text{Bi}_2} \right\} = 0 \quad (6-7-32)$$

i.e. the old single form of root  $\mu_n$ , contained in the solution of complete set of equations, has attained two forms:  $\mu_n = \mu_p$  and  $\mu_n/\sqrt{\text{Lu}} = \mu_q$ . Therefore, infinite sums of the solution should be repeated twice: first by writing  $\mu_n = \mu_p$  instead of  $\mu_n$  and then  $\mu_n/\sqrt{\text{Lu}} = \mu_q$ . Solutions (6-7-29) and (6-7-30)

become

$$\begin{aligned}\theta_1(X, Fo) = & 1 - \sum_{p=1}^{\infty} A_p \left( 1 + \frac{\varphi_p}{\psi_p} \right) W_r(\mu_p X) \exp(-\mu_p^2 Fo) \\ & - \sum_{q=1}^{\infty} A_q Ko \left[ \frac{\varepsilon Lu}{1 - Lu} W_r(\mu_q X) + \frac{\varphi_q}{\psi_q} W_r(\mu_q \sqrt{Lu} X) \right] \exp(-\mu_q^2 Lu Fo)\end{aligned}\quad (6-7-33)$$

$$\theta_2(X, Fo) = 1 - \sum_{q=1}^{\infty} A_q W_r(\mu_q X) \exp(-\mu_q^2 Lu Fo) \quad (6-7-34)$$

where

$$\begin{aligned}\varphi_p = & Ko \left\{ \frac{\varepsilon Lu Bi_2}{Lu - 1} W_r \left( \frac{\mu_p}{\sqrt{Lu}} \right) + \left( 1 - \varepsilon \right. \right. \\ & \left. \left. - \frac{\varepsilon}{Lu - 1} \right) \frac{Bi_2}{Bi_1} \mu_p \sqrt{Lu} V_r \left( \frac{\mu_p}{\sqrt{Lu}} \right) \right\}\end{aligned}\quad (6-7-35)$$

$$\psi_p = Bi_2 W_r \left( \frac{\mu_p}{\sqrt{Lu}} \right) - \frac{\mu_p}{\sqrt{Lu}} V_r \left( \frac{\mu_p}{\sqrt{Lu}} \right) \quad (6-7-36)$$

$$\varphi_q = \frac{\varepsilon Bi_1}{Lu - 1} W_r(\mu_q) + \left( 1 - \varepsilon - \frac{\varepsilon}{Lu - 1} \right) \mu_q Lu V_r(\mu_q) \quad (6-7-37)$$

$$\psi_q = Bi_1 W_r(\mu_q \sqrt{Lu}) - \mu_q \sqrt{Lu} V_r(\mu_q \sqrt{Lu}) \quad (6-7-38)$$

$$\begin{aligned}A_p = & \frac{2}{\mu_p V_r(\mu_p)} \frac{Bi_1^2}{\mu_p^2 + (1 - \Gamma) Bi_1 + Bi_1^2} \\ A_q = & \frac{2}{\mu_q V_r(\mu_q)} \frac{Bi_2^2}{\mu_q^2 + (1 - \Gamma) Bi_2 + Bi_2^2}\end{aligned}\quad (6-7-39)$$

Here  $\mu_p$  and  $\mu_q$  are the roots of characteristic equations

$$\frac{W_r(\mu_p)}{V_r(\mu_p)} = \frac{\mu_p}{Bi_1}, \quad \frac{W_r(\mu_q)}{V_r(\mu_q)} = \frac{\mu_q}{Bi_2} \quad (6-7-40)$$

If in the material under test, besides thermodiffusion mass-transfer phase changes are also absent ( $\varepsilon = 0$ ), then the solutions retain the form of (6-7-33) and (6-7-34) with  $\varphi_p^*$  and  $\varphi_q^*$  substituted for  $\varphi_p$  and  $\varphi_q$ , respectively. The former are defined as follows:

$$\varphi_p^* = Ko \frac{Bi_2}{Bi_1} \mu_p \sqrt{Lu} V_r \left( \frac{\mu_p}{\sqrt{Lu}} \right); \quad \varphi_q^* = \mu_q Lu V_r(\mu_q) \quad (6-7-41)$$

The characteristic roots and coefficients  $A_p$  and  $A_q$  are found from the above equations.

In the absence of mass transfer ( $Ko = 0$ ) the earlier given solutions change to ordinary classical solutions.

The above solutions involve the following coefficients:

$$H_{ki} = (-1)^i \frac{(K-1)Ko^* - (K-2)Pn}{(v_1^2 - v_2^2)} \quad (6-7-42)$$

$$G_{ki} = \frac{(-1)^i}{(v_1^2 - v_2^2)} \left[ (K-1) \left( \frac{1}{Lu v_1^2} - 1 \right) - (K-2)Pn \right] \quad (6-7-43)$$

The numerical values of these coefficients varying between 0.1 and 1 are presented in Tables 6-2, 6-3, 6-4 as functions of  $Lu$ ,  $Pn$  and  $Ko^* Pn$ .

#### 6-8. DIFFERENTIAL EQUATIONS OF FILTRATION THROUGH POROUS MATERIALS

In Sections 6-1 through 6-7 the phenomena of mass transfer in capillary-porous materials during phase changes have been considered. Mass transfer in such materials is induced by diffusion and thermodiffusion. Here, the term "mass diffusion" covers not only diffusion of vapour, gas or liquid, but also the capillary motion of liquid. Physically capillary motion of liquid is a kind of molar motion governed by the laws of hydrodynamics, but because of polycapillary structure of material the capillary motion is conventionally identified with random motion called capillary diffusion. However, for a monocapillary structure of material the capillary diffusion degenerates into an ordinary hydrodynamic flow in an equivalent capillary tube. This flow may be laminar or turbulent.

Vapour and noncondensable gases may be transferred not only by molecular diffusion (concentration diffusion and thermal diffusion) but also through filtration by Darcy's law. Physically, this kind of filtration motion

Table 6-2

Values of the coefficient  $H_{11} = G_{22}$  [6-6]

Lu	Ko* Pn					
	0.1	0.2	0.4	0.6	0.8	1.0
0.01	0.0000	0.0000	0.0000	0.0001	0.0001	0.0001
0.05	0.0003	0.0005	0.0011	0.0016	0.0020	0.0025
0.1	0.0012	0.0023	0.0045	0.0064	0.0081	0.0097
0.2	0.0058	0.0110	0.0195	0.0262	0.0315	0.0358
0.4	0.0350	0.0575	0.0838	0.0976	0.1053	0.1096
0.6	0.1143	0.1521	0.1777	0.1835	0.1830	0.1799
0.8	0.2579	0.2753	0.2715	0.2595	0.2469	0.2350
1.0	0.4219	0.3909	0.3492	0.3194	0.2959	0.2764

Table 6-3

Values of the Coefficient  $G_{11}$  [6-6]

Lu	Pn	Ko* Pn					
		0.1	0.2	0.4	0.6	0.8	1.0
0.01	0.1	0.0010	0.0010	0.0010	0.0010	0.0010	0.0010
	0.2	0.0020	0.0020	0.0020	0.0020	0.0020	0.0020
	0.4	0.0040	0.0040	0.0040	0.0040	0.0040	0.0040
	0.6	0.0060	0.0060	0.0060	0.0060	0.0060	0.0060
	0.8	0.0081	0.0081	0.0080	0.0080	0.0080	0.0080
	1.0	0.0101	0.0101	0.0101	0.0100	0.0100	0.0100
0.05	0.1	0.0052	0.0052	0.0051	0.0051	0.0050	0.0050
	0.2	0.0105	0.0104	0.0103	0.0102	0.0101	0.0100
	0.4	0.0209	0.0208	0.0206	0.0203	0.0201	0.0199
	0.6	0.0314	0.0312	0.0309	0.0305	0.0302	0.0298
	0.8	0.0419	0.0416	0.0411	0.0407	0.0402	0.0398
	1.0	0.0523	0.0520	0.0514	0.0509	0.0503	0.0498
0.1	0.1	0.0110	0.0108	0.0105	0.0103	0.0100	0.0098
	0.2	0.0219	0.0216	0.0211	0.0206	0.0201	0.0196
	0.4	0.0438	0.0433	0.0422	0.0411	0.0402	0.0392
	0.6	0.0658	0.0649	0.0633	0.0617	0.0602	0.0588
	0.8	0.0877	0.0865	0.0843	0.0823	0.0803	0.0784
	1.0	0.1096	0.1082	0.1054	0.1028	0.0104	0.0981
0.2	0.1	0.0241	0.0233	0.0218	0.0206	0.0195	0.0186
	0.2	0.0482	0.0466	0.0437	0.0412	0.0390	0.0371
	0.4	0.0964	0.0931	0.0874	0.0824	0.0781	0.0743
	0.6	0.1446	0.1397	0.1311	0.1236	0.1171	0.1114
	0.8	0.1928	0.1863	0.1747	0.1648	0.1562	0.1486
	1.0	0.2410	0.2329	0.2184	0.2060	0.1952	0.1857
0.4	0.1	0.0581	0.0521	0.0438	0.0383	0.0343	0.0312
	0.2	0.1162	0.1041	0.0876	0.0766	0.0686	0.0625
	0.4	0.2325	0.2082	0.1752	0.1533	0.1373	0.1249
	0.6	0.3487	0.3123	0.2629	0.2299	0.2059	0.1874
	0.8	0.4650	0.4165	0.3505	0.3066	0.2746	0.2499
	1.0	0.5812	0.5206	0.4381	0.3832	0.3432	0.3123
0.6	0.1	0.1006	0.0803	0.0604	0.0500	0.0432	0.0384
	0.2	0.2012	0.1606	0.1209	0.0999	0.0865	0.0768
	0.4	0.4025	0.3212	0.2417	0.1999	0.1729	0.1530
	0.6	0.6037	0.4818	0.3626	0.2998	0.2594	0.2305
	0.8	0.8049	0.6423	0.4835	0.3998	0.3458	0.3073
	1.0	1.0062	0.8029	0.6044	0.4997	0.4323	0.3841
0.8	0.1	0.1383	0.0999	0.0703	0.0566	0.0482	0.0424
	0.2	0.2767	0.1998	0.1406	0.1132	0.0964	0.0848
	0.4	0.5534	0.3995	0.2813	0.2264	0.1928	0.1696
	0.6	0.8301	0.5992	0.4219	0.3395	0.2893	0.2544
	0.8	1.1067	0.7990	0.5625	0.4527	0.3857	0.3392
	1.0	1.3834	0.9987	0.7032	0.5659	0.4821	0.4240
1.0	0.1	0.1562	0.1091	0.0754	0.0602	0.0510	0.0447
	0.2	0.3123	0.2182	0.1507	0.1204	0.1021	0.0894
	0.4	0.6247	0.4364	0.3015	0.2408	0.2041	0.1789
	0.6	0.9370	0.6546	0.4523	0.3612	0.3062	0.2683
	0.8	1.2494	0.8729	0.6030	0.4815	0.4082	0.3578
	1.0	1.5617	1.0911	0.7538	0.6019	0.5103	0.4472



Table 6-4

Values of the Coefficient  $H_{21}$  [6-6]

Lu	Pn	Ko* Pn					
		0.1	0.2	0.4	0.6	0.8	1.0
0.01	0.1	0.0101	0.0202	0.0402	0.0602	0.0801	0.1000
	0.2	0.0050	0.0101	0.0201	0.0301	0.0401	0.0500
	0.4	0.0025	0.0050	0.0101	0.0151	0.0200	0.0250
	0.6	0.0017	0.0034	0.0067	0.0100	0.0134	0.0167
	0.8	0.0013	0.0025	0.0050	0.0075	0.0100	0.0125
	1.0	0.0010	0.0020	0.0040	0.0060	0.0080	0.0100
0.05	0.1	0.0523	0.1040	0.2057	0.3052	0.4024	0.4975
	0.2	0.0262	0.0520	0.1029	0.1520	0.2012	0.2488
	0.4	0.0131	0.0260	0.0514	0.0763	0.1006	0.1244
	0.6	0.0087	0.0173	0.0343	0.0509	0.0671	0.0829
	0.8	0.0065	0.0130	0.0257	0.0381	0.0503	0.0622
	1.0	0.0052	0.0104	0.0206	0.0305	0.0402	0.0498
0.1	0.1	0.1096	0.2164	0.4217	0.6170	0.8031	0.9806
	0.2	0.0548	0.1082	0.2109	0.3085	0.4015	0.4903
	0.4	0.0274	0.0541	0.1054	0.1642	0.2008	0.2451
	0.6	0.0183	0.0361	0.0703	0.1028	0.1338	0.1634
	0.8	0.0137	0.0270	0.0527	0.0771	0.1004	0.1226
	1.0	0.0110	0.0216	0.0422	0.0617	0.0803	0.0981
0.2	0.1	0.2411	0.4657	0.8737	1.2361	1.5617	1.8570
	0.2	0.1205	0.2329	0.4368	0.6181	0.7809	0.9283
	0.4	0.0603	0.1164	0.2184	0.3090	0.3904	0.4642
	0.6	0.0402	0.0776	0.1456	0.2060	0.2603	0.3095
	0.8	0.0301	0.0582	0.1092	0.1545	0.1952	0.2321
	1.0	0.0241	0.0466	0.0874	0.1236	0.1562	0.1857
0.4	0.1	0.5812	1.0412	1.7524	1.2992	2.7456	3.1235
	0.2	0.2906	0.5206	0.8762	0.1496	1.3728	1.5617
	0.4	0.1453	0.2603	0.4381	0.5748	0.6864	0.7809
	0.6	0.0969	0.1735	0.2921	0.3832	0.4576	0.5206
	0.8	0.0726	0.1301	0.2190	0.2874	0.3432	0.3904
	1.0	0.0581	0.1041	0.1752	0.2299	0.2746	0.3123
0.6	0.1	1.0062	1.6059	2.4174	2.9983	3.4583	3.8411
	0.2	0.5031	0.8029	1.2087	1.4992	1.7292	1.9200
	0.4	0.2515	0.4015	0.6044	0.7496	0.8646	0.9603
	0.6	0.1677	0.2676	0.4029	0.4997	0.5764	0.6402
	0.8	0.1258	0.2007	0.3022	0.3748	0.4323	0.4801
	1.0	0.1006	0.1606	0.2417	0.2993	0.3458	0.3841
0.8	0.1	1.3834	1.9975	2.8126	3.3955	3.8568	4.2400
	0.2	0.6917	0.9988	1.4063	1.6977	1.9284	2.1200
	0.4	0.3459	0.4994	0.7032	0.8489	0.9642	1.0600
	0.6	0.2306	0.3329	0.4688	0.5659	0.6428	0.7067
	0.8	0.1729	0.2497	0.3516	0.4244	0.4821	0.5300
	1.0	0.1383	0.1998	0.2813	0.3395	0.3857	0.4240
1.0	0.1	1.5617	2.1822	3.0151	3.6116	4.0825	4.4721
	0.2	0.7809	1.0911	1.5076	1.8058	2.0412	2.2361
	0.4	0.3904	0.5455	0.7538	0.9029	1.0206	1.1180
	0.6	0.2603	0.3637	0.5025	0.6019	0.6804	0.7454
	0.8	0.1952	0.2728	0.3769	0.4514	0.5103	0.5590
	1.0	0.1562	0.2182	0.3015	0.3612	0.4082	0.4472

is also a hydrodynamic flow, but in the case of filtration through capillary-porous materials, where the mass flow path is complex and tortuous, such a filtration may conventionally be classed with filtration diffusion. Thus, mass is transferred by diffusion if diffusion is considered as random motion involving not only molecular, but capillary and filtration diffusions also.

As a result diffusional mass transfer is described by parabolic type differential equations, which have been considered earlier in detail.

Porous materials are bodies having complicated network of pores. The structures of such materials and mass transfer in them have been considered in Chapter 5.

In porous materials the flow of liquid and gas caused by pressure drop (filtration flow) cannot be considered as filtration diffusion. Only in one particular case, in the so-called pressure filtration, the flow may be identified as filtration diffusion. Such problems are usually treated in the filtration theory of soils, and are adequately described in [6-7].

The differential equation of pressure filtration is a particular case of the differential equation of mass transfer in capillary-porous bodies. If we assume that  $\rho_i = \rho$ ,  $b_i = 1$ , then from equation (6-1-11) we obtain the differential equation for pressure filtration

$$\Pi \frac{\partial \rho}{\partial \tau} = \text{div} \left[ \frac{\rho k_a}{\eta} \left( \vec{\nabla} p + \rho \vec{\nabla} h \right) \right] \quad (6-8-1)$$

Equation (6-8-1) describes the motion of a compressible fluid in an isothermal medium.

Coefficient  $k_a$  for gases is equal to

$$k_a = k \left( 1 + \frac{a}{p} \right) \quad (6-8-2)$$

Let us denote the isothermal compression factor by  $\beta$ , i.e.

$$\beta = - \frac{1}{V} \left( \frac{\partial V}{\partial p} \right)_T = \frac{1}{\rho} \left( \frac{\partial \rho}{\partial p} \right)_T \quad (6-8-3)$$

If the coefficient  $\beta$  for a compressible fluid is assumed constant, then

$$\rho = \rho_0 \exp [\beta(p - p_0)] \quad (6-8-4)$$

where  $\rho_0$  is the density at pressure  $p_0$ .

Then for isothermal conditions, from relation (6-8-4) we get

$$\vec{\nabla} \rho = \beta \rho \vec{\nabla} p \quad (6-8-5)$$

Substitution of (6-8-5) into (6-8-1) gives:

$$\Pi \frac{\partial \rho}{\partial \tau} = \text{div} \left( \frac{k}{\eta \beta} \vec{\nabla} \rho + \frac{k \rho^2}{\eta} \vec{\nabla} h \right) \quad (6-8-6)$$

If the force of gravity remains unchanged ( $h = \text{const}$ ), then

$$\Pi \frac{\partial \rho}{\partial \tau} = \text{div} \left( \frac{k}{\eta \beta} \vec{\nabla} \rho \right) \quad (6-8-7)$$

In case of filtration of gases equation (6-8-7) may be written in a different form.

Let us denote the compressibility factor of a gas by  $\pi$ , i.e.

$$\rho = \frac{pm}{\pi RT} \quad (6-8-8)$$

where  $m$  is the molecular mass. In a general case the compressibility factor depends on  $p$  and  $T$ , i.e.  $\pi(p, T)$ .

With allowance made for the Klinberg effect we have

$$\Pi \frac{\partial}{\partial \tau} \left( \frac{p}{\pi T} \right) = \text{div} \left[ \left( \frac{k}{\eta \pi T} \right) (p + a) \left( \vec{\nabla} p + \frac{mp}{\pi RT} \vec{\nabla} h \right) \right] \quad (6-8-9)$$

If  $h$  is constant, then

$$\Pi \frac{\partial}{\partial \tau} \left( \frac{p}{\pi} \right) = \text{div} \left[ \frac{k}{\eta \pi} (p + a) \vec{\nabla} p \right] \quad (6-8-10)$$

If  $\mathcal{P}$  is used to denote,  $p + a$ , e.g.

$$\mathcal{P} = p + a \quad (6-8-11)$$

then

$$\Pi \frac{\partial}{\partial \tau} \left( \frac{\mathcal{P} - a}{\pi} \right) = \text{div} \left( \frac{k}{2\eta \pi} \vec{\nabla} p^2 \right) \quad (6-8-12)$$

If  $\pi$ ,  $b$  and  $k$  are constant, then

$$\frac{\partial p}{\partial \tau} = \frac{k}{2\eta \Pi} \nabla^2 p^2 \quad (6-8-13)$$

Equation (6-8-13) may be rewritten as

$$\frac{\partial p^2}{\partial \tau} = \frac{kp}{\Pi \eta} \nabla^2 p^2 \quad (6-8-14)$$

As a first approximation,  $kp$  may be taken constant assuming  $kp \approx kp_m$ , then

$$\frac{\partial p^2}{\partial \tau} = \frac{kp_m}{\Pi \eta} \nabla^2 p^2 \quad (6-8-15)$$

For illustration, in Fig. 6-1 are plotted dimensionless pressure profiles in an infinite plate of thickness  $L$ . These profiles have been obtained from the solution of differential equation (6-8-15) at the first-kind boundary

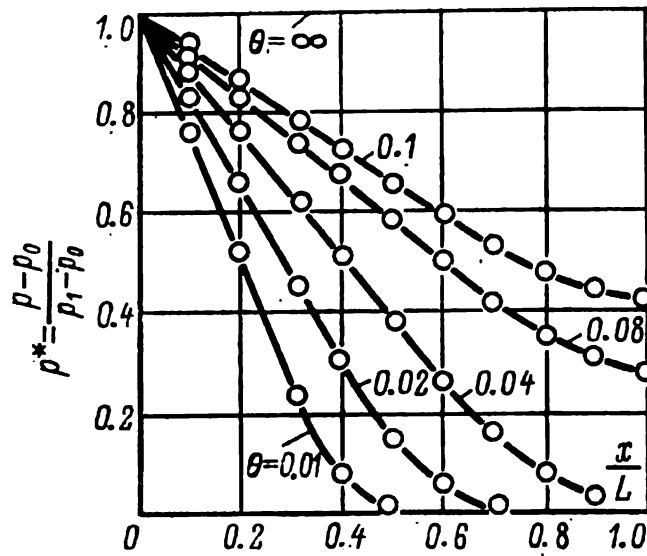


Fig. 6-1. Plot of dimensionless pressure  $p^*$  [ $p^* = (p - p_0)/(p_1 - p_0)$ ] versus coordinate  $x/L$  in an infinite plate for the case

$$p_1/p_0 = 5 \quad [6-8]. \quad \theta = \frac{k p_0 \tau}{\Pi \eta L^2}$$

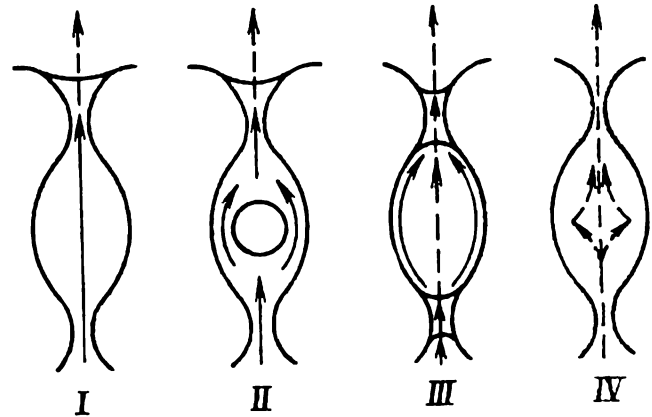


Fig. 6-2. Mechanism of moisture migration in a capillary-porous material

conditions (at  $x = 0$ ,  $p = p_1 = \text{const}$ ). A survey of filtration solutions is given in [6-7], [6-9], their analysis is out of the scope of this book.

## 6-9. DIFFUSION THROUGH POROUS MATERIALS

As described earlier, mass transfer in capillary-porous bodies is given by a system of parabolic differential equations. Concentration ( $\nabla \omega_i = \rho_0 \nabla u_i$ ) and temperature ( $\nabla T$ ) gradients are thermodynamic motive forces of mass transfer for homogeneous bodies of polycapillary structures.

In a very simple model liquid migration in a capillary-porous body [6-52] occurs as capillary motion and film flow of liquid (Fig. 6-2, Stage I). In the process of capillary motion of liquid a part of the surrounding air enters the capillary (see Fig. 6-2, Stage II). The bubble size increases and moisture migration proceeds in the form of evaporation — condensation (see Fig. 6-2, stage III), and finally, vapour diffusion through the capillary takes place (Fig. 6-2, Stage IV). A temperature gradient however complicates the vapour migration process in capillaries and pores of materials.

A schematic of a possible moisture-transfer process occurring in a heated capillary-porous material layer is shown in Fig. 6-3 (the heat flux has a leftward direction) [6-50]. In large cavities (caverns) air may circulate in a manner similar to that considered in Chapter 3 (Figs. 3-41 and 3-43). The Stefan and thermal slip flows are directed opposite to the heat flux. Besides these flows hydrodynamic and diffusion slip flows prevail.

At high rates of evaporation and when air effusion from the ambient along microcapillaries in coarse pores communicating with microcapillaries takes place, a total pressure gradient develops. Under the effect of this gradient vapour and air come into motion similar to filtration. In this

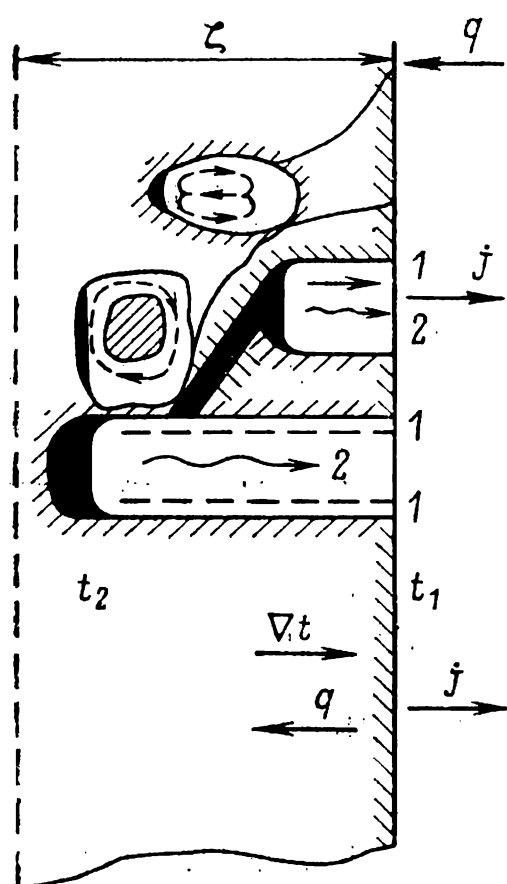


Fig. 6-3. Schematic of moisture migration and heat propagation in a surface layer of a capillary-porous material

1 — slip flow; 2 — Stefan flow;  $q$  — heat flux,  $j$  — moisture flow

case filtration of vapourous moisture is a dominating mode of transfer, so the other modes of transfer may be neglected. It is therefore of interest to consider in detail the filtration transfer mechanism.

### (a) Transfer Mechanism

At isothermal conditions the following modes of mass transfer in porous materials may be considered. Let us assume that a liquid containing dissolved substance or a gaseous mixture flows through a porous medium. Here the following mass-transfer mechanisms may be present:

1. Molecular diffusion, if the time scales are large; mass transfer results from molecular diffusion.
2. Vortices. If the flow in a channel of the porous medium becomes turbulent, mass transfer arises from vortex migration.
3. Mixing caused by various obstacles. A porous material has various types of zigzag channels, therefore two liquid elements  $A$  and  $B$  (Fig. 6-4), which start their journey lying at a certain distance from each other and having equal velocities, will not remain at the same distance from each other after sometime, mixing will take place.

4. With self-correlation in a flow: mass transfer may take place because the liquid element has access to all the pores, or in other words, mass transfer may result from the medium incoherence (Fig. 6-5).

5. Recirculation due to low pressure local spots. Mass transfer may be induced by recirculation which occurs due to the presence of narrowed

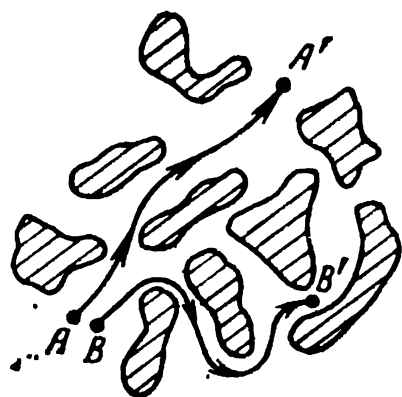


Fig. 6-4. Flow mixing induced by various bars

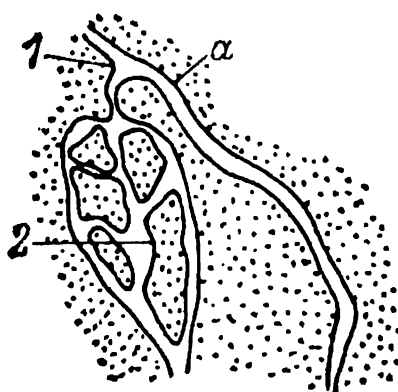


Fig. 6-5. Incomplete compendency of flow

(a) liquid inlet; 1 — no access for the liquid from reticulum 2

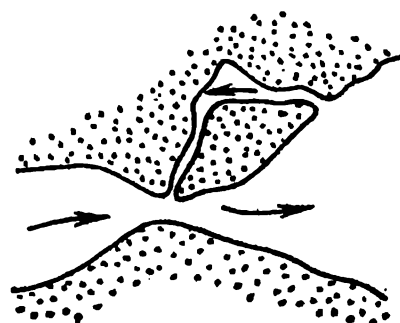


Fig. 6-6. Liquid recirculation due to low pressure local spots

sections along flow channels. Transition of pressure energy to kinetic energy gives rise to low pressure local spots. If such a spot is accessible for the liquid, which has earlier passed through it, then recirculation sets in, as in the case shown in Fig. 6-6.

6. Closed pores. Closed volumes give rise to mass transfer in unsteady flows (variable profiles of concentration) because of molecular diffusion arising as the liquid passes the front rich in the dissolved substance. After the front has been passed, reverse diffusion of the dissolved substance and mass transfer in the system occur. The pore volume introduces certain uncertainty in the experimental determination of transfer properties, because the pore volume is measured in the same manner as porosity, but this porosity has no effect on the flow cross-section.

7. Adsorption. Mass transfer by adsorption is an unsteady process. Similarly to closed pores, adsorption increases or decreases the amount of dissolved substance in the flow, therefore it tends to level the concentration profiles in the moving medium.

8. Hydrodynamic mass transfer. Macroscopic transfer in a capillary takes place even in the absence of molecular diffusion because of the presence of a velocity profile produced by liquid adhesion to the wall. This compels liquid particles at different radial positions to move relative to each other so that mass transfer occurs as the liquid moves along the capillary.

9. Macroscopic dispersion due to imperfections which change general flow lines.

10. Molar motion due to concentration drop near the pore walls caused by slip diffusion.

The above listed modes of mass transfer may conventionally be termed mass diffusion. In the British and American literature such a combined mass transfer is called dispersion that evidently means random mass scatter within porous material. At isothermal conditions the volume concentration gradient ( $\nabla \omega$ ) is a motive force of such mass transfer. For one-dimensional problems such a mass-transfer mode is described by differential equation

$$\frac{\partial \omega}{\partial \tau} = D \frac{\partial^2 \omega}{\partial x^2} - v_x \frac{\partial \omega}{\partial x} \quad (6-9-1)$$

where the coefficient  $D$  is called the dispersion coefficient;  $v_x$  is the filtration velocity defined by the Darcy law. If  $v_x$  is assumed constant, then equation (6-9-1) may be written as:

$$\frac{\partial \omega}{\partial \tau} = - \frac{\partial}{\partial x} \left( - D \frac{\partial \omega}{\partial x} + v_x \omega \right) \quad (6-9-2)$$

Hence, the mass flow

$$j_m = - D \frac{\partial \omega}{\partial x} + v_x \omega \quad (6-9-3)$$

Formula (6-9-3) is similar to that for a mass flow consisting of diffusion flow  $\left( - D \frac{\partial \omega}{\partial x} \right)$  and convective flow  $(v_x \omega)$ . Therefore, in our opinion, it is advisable to refer to the total mass transfer in a porous medium, described by an equation of the type (6-9-1), as diffusion, implying all the earlier mentioned mass-transfer modes. In this case the dispersion coefficient  $D$  is a coefficient of diffusion in a porous medium. This is advisable also in other respect that mass transfer in a porous medium is similar to the mass transfer in capillary-porous bodies.

If use is made of the Darcy formula, then we may write

$$j_m = - D \frac{\partial \omega}{\partial x} - \frac{k}{\eta} \omega \frac{\partial p}{\partial x} \quad (6-9-4)$$

this is a relation for mass transfer in a capillary-porous body.

In some papers, for example [6-15], equation (6-9-1) has been reduced by transformation to an ordinary equation of diffusion

$$x_1 = x - v_x \tau \quad (6-9-5)$$

Equation (6-9-1) then becomes

$$\frac{\partial \omega}{\partial \tau} = D \frac{\partial^2 \omega}{\partial x_1^2} \quad (6-9-6)$$

If an equivalent capillary tube is substituted for a porous material, then the propagation of dissolved substance (liquid) characterizes mass-transfer mechanism. In this case the diffusion coefficient will be equal to

$$D = \frac{4R^2 v_m^2}{192 D_m} \quad (6-9-7)$$

where  $R$  is the tube radius;  $v_m$  is the mean flow velocity of liquid and  $D_m$  is the molecular diffusion coefficient.

The numerator in formula (6-9-7) is a measure of axial molar diffusion, the denominator takes account of both molecular mixing and total effect of transverse molecular and molar mixing.

Hence, any mechanism, which increases the rate of cross mixing, say, turbulent flows, lowers the diffusion coefficient  $D$ . This conclusion is qualitatively supported by mass-transfer experiments in porous bodies.

In [6-11] the generalized expression for diffusion coefficients is presented

$$D = D_m + \frac{4R^2 v_m^2}{192 D_m} \quad (6-9-8)$$

This expression is called the Taylor-Aris formula.

Using an equivalent capillary tube instead of a porous medium, some authors [6-12] take the effective diameter  $d_e$  of pores as a structural parameter. Coefficient  $D$  is then considered proportional to the product of the structural parameter by the fluid velocity in the pores raised to the  $n$ -th power

$$D \approx d_e v^n \quad (6-9-9)$$

The exponent  $n$  ranges from 1 to 2 and depends on  $d_e$ . The value  $n = 1$  corresponds to that case when sufficient time is available for liquid mixing, at  $n = 2$  mixing is complete.

In two-dimensional diffusion problems the diffusion coefficient  $D$  has different values for different directions

$$\frac{\partial \omega}{\partial \tau} = D_L \frac{\partial^2 \omega}{\partial x^2} + D_T \frac{\partial^2 \omega}{\partial y^2} - v_x \frac{\partial \omega}{\partial x} \quad (6-9-10)$$

where  $D_L$  and  $D_T$  are the coefficients of longitudinal and transverse diffusion, respectively.

Because of the complexity of porous materials such simplified methods do not prove useful and give values considerably different from the experimental data.

A microscopic approach to the description of fluid flow phenomenon in a porous material is very complicated. In principle, having information about the system, the path of each fluid particle can be calculated. However such an approach is practically useless for the reason that it is impossible



to determine exactly the fluid-solid interface; besides, the interface, even if it would have been known, is so complicated that the problem would not have lent itself to mathematical treatment. Even if it would have been possible to obtain a solution for the paths of individual fluid particles, the solution would have been of no practical value, because of the need of macroscopic or integral characteristics.

### (b) Statistical Approach

The statistical approach to the description of porous materials is more fruitful. A flow through a porous medium may be considered as random motion, similar to a turbulent flow, by using for its description the developed mathematical formulas. Beside this, by simulating the inter pore space as mixing cells, the mathematical methods of statistical thermodynamics including the concepts of Hamiltonians can be used. However, this does not exclude the method for the use of which the porous medium is simulated by a network of capillaries consisting of interconnected sub-elements or a network of capillaries connected in series.

Although the considered processes, in principle, are not strictly Markovian processes, nevertheless the Markovian assumption is ordinarily used for simplicity. In other words, consideration is usually restricted to processes wherein the transition probability of fluid element from one state into another remains constant, in particular, it is independent of time (for example, swelling materials are not considered). In practice, not only the Markovian assumption is made, but it is also assumed that autocorrelation is completely absent, i.e. the probability for the fluid to enter a pore of given size and location is independent of the size and location of the pore vacated by the fluid. This is not valid for materials characterized by any uniform, say, lamellar, mechanism.

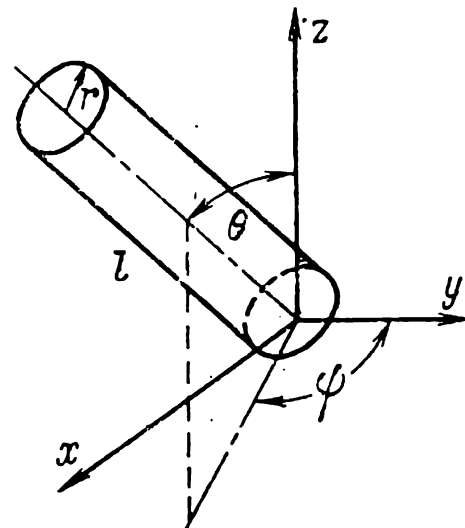


Fig. 6-7. Elementary pore in the statistical model of porous medium

Using the statistical method the capillary pressure, permeability, longitudinal and transverse diffusions can be calculated with the use of parameters  $a$ ,  $b$ ,  $\alpha$  and  $\beta$ . This model involves an elementary pore (Fig. 6-7).

If the dimensionless length of the pore is designated by  $l^*$  ( $l^* = l/L$ ), where  $L$  is the length of the longest pore,  $r^* = r/R$  is the dimensionless radius and  $R$  is the largest pore radius, then the distribution functions  $f(l^*)$  and  $F(r^*)$  are written as [6-13]:

$$f(l^*) = \frac{(a + b + 1)!}{a!b!} (l^*)^a (1 - l^*)^b \quad (6-9-11)$$

$$F(r^*) = \frac{(\alpha + \beta + 1)!}{\alpha!\beta!} (r^*)^\alpha (1 - r^*)^\beta \quad (6-9-12)$$

Using these relations, we can calculate all the structural parameters, including the longitudinal  $D_L$  and transverse  $D_T$  diffusion coefficients (Table 6-5):

$$D_L = f(a, b, \alpha, \beta) \quad (6-9-13)$$

Table 6-5

**Diffusion Coefficients (cm<sup>2</sup>/s) Calculated by the Statistical Model at Velocity  $v_m = 0.1$  cm/s**

$a$	$b$	$\alpha$	$\beta$	$D_L \cdot 10^3$	$D_T \cdot 10^4$	$D_L/D_T$
2	2	0	0	1.77	4.85	3.66
2	2	-1/2	-1/2	7.28	7.87	9.37
2	2	-1/2	1	33.2	16.8	19.8
2	2	2	4	4.82	6.54	7.40
2	2	2	2	3.78	5.87	6.42
2	2	4	2	2.43	4.89	4.96
2	2	1	1	4.54	6.32	7.18
2	4	2	4	3.96	5.15	7.70
4	2	2	4	4.30	5.98	7.20
2	4	1	1	4.78	4.96	9.65
4	2	1	1	7.95	11.60	6.85
2	4	4	2	2.02	4.00	5.05
4	2	4	2	4.26	8.95	4.76
1	1	4	2	1.98	4.28	4.68
1	1	2	4	3.54	5.75	6.16

In the described model fluid particles move along the most probable path with velocity  $\tilde{v}$ :

$$\tilde{v} = \frac{v}{J} \quad (6-9-14)$$

where  $\tilde{v}$  is the velocity defined by the Darcy law, and  $J$  equals:

$$J = \frac{(\alpha + 1)(\alpha + 2)(\alpha + \beta + 4)(\alpha + \beta + 5)}{(\alpha + 3)(\alpha + 4)(\alpha + \beta + 2)(\alpha + \beta + 3)} \quad (6-9-15)$$

*(c) Turbulent Approach*

Using the concepts and mathematical procedures used in the turbulent flow theory, all the properties of diffusion fluid flows through porous materials may be calculated.

Most of the authors, following this method, use the equation

$$\frac{\partial \omega}{\partial \tau} + v_i \frac{\partial \omega}{\partial x_i} = \frac{\partial}{\partial x_i} \left( D_{jk} \frac{\partial \omega}{\partial x_k} \right) \quad (6-9-16)$$

where  $D_{jk}$  is the diffusion tensor.

A similar equation has been obtained in [6-14]. If the fluid tracer travels with mean velocity  $v_i$ , then the dimensionless concentration  $\omega^*$  is found from the solution of differential equation

$$\frac{\partial \omega^*}{\partial \tau} = \sum_{i=1}^3 \frac{\partial}{\partial x_i} \left( D_i \frac{\partial \omega^*}{\partial x_i} \right) \quad (6-9-17)$$

where  $D_i$  is the axial component of diffusion tensor.

Diffusion coefficient  $D$  is the second order tensor. If  $D$  is assumed to be invariant upon rotation towards the direction of mean velocity, and to be a specular reflection with respect to the planes, then

$$D_{ij} = A u_i^0 u_j^0 + B \delta_{ij} \quad (6-9-18)$$

where  $u_i^0$  is a component of the mean velocity fluctuation;  $\delta_{ij}$  is a unit tensor;  $A$  and  $B$  are constants.

If the fluid flow follows the Darcy law, then, the coefficient  $D$  is proportional to the product of the mean velocity by the characteristic length.

In general the characteristic length is not a scalar quantity, but a fourth-order tensor, which for an isotropic case equals:

$$L_{ijkl} = H_1 \delta_{ij} \delta_{kl} + H_2 \delta_{ik} \delta_{jl} + H_3 \delta_{il} \delta_{jk} \quad (6-9-19)$$

where  $H_2 = H_3$  and

$$B = H_1(\bar{\eta}); \quad A = 2H_2(\bar{\eta}) \quad (6-9-20)$$

For formula (6-9-18) to be valid, it is necessary that the equality

$$D_{ij} = L_{ijkl} \frac{\bar{u}_k \bar{u}_l}{|\bar{u}|} \quad (6-9-21)$$

be satisfied.

Quantities  $A$ ,  $B$  or  $H_1$  and  $H_2$  are dependent upon the mixing mechanism in the porous material. At each point of the porous material the mean velocity vector changes arbitrarily with respect to the local velocity:

$$u_i = \Pi_{ij} \bar{u}_j \quad (6-9-22)$$

where  $\Pi_{ij}$  is the local tensor of the porous material.

For a one-dimensional flow

$$\left. \begin{aligned} D_{11} &= D(\Pi_{11}) l\bar{u} = \lambda_1 \bar{u} \\ D_{22} &= D_{33} = D(\Pi_{12}) l\bar{u} = \lambda_2 \bar{u} \end{aligned} \right\} \quad (6-9-23)$$

$$D_{ij} = 0, \quad \text{if } i \neq j.$$

Hence

$$D_{11}/D_{22} = \frac{\lambda_1}{\lambda_2} \quad (6-9-24)$$

where  $\lambda_1$  and  $\lambda_2$  are components of the diffusion tensor of isotropic and porous media, respectively.

Quantity  $l$  accounts for about one half of particle diameter in nonpacked materials and one half of mean length of the straight line connecting the pores in packed materials.

If the media are assumed identical, exception being  $l$ , then the components of tensor  $\Pi_{ij}$  will be equal, and hence

$$\frac{D_a}{D_b} = \frac{l_a}{l_b} \quad (6-9-25)$$

i.e. the diffusion coefficient is proportional to the particle size.

Similar results were obtained in [6-14].

The data reported by various authors are summarized in Table 6-6, where  $\lambda$  and  $M$  are invariants of tensor  $L_{ijkl}$ ;  $A_1 A_{11}$  are symmetrical tensors which are the functions of porous material structure and fluid properties. Constants  $\alpha_i$  ( $i = 1, 2, 3$ ) characterize the configuration of porous structure\*.

#### (d) Experimental Data

It is of interest to compare the calculated and measured values. In [6-20] experimental data are also reported for sand-stone with 20% porosity (mean grain diameter being 180  $\mu\text{m}$ , and mean pore diameter being 15  $\mu\text{m}$ .)

Glass beads of equal diameter were used as nonpacked material. For beads of diameter 1.5, 0.25 and 0.074 mm porosity amounted to 37.5, 41 and 38%, respectively. The results of calculations made with the aid of the statistical model [6-13] are plotted in Figs. 6-8, 6-9. From these figures it is evident that experimental data are in good agreement with calculated one in the range where molecular diffusion dominates. At small velocities  $v$  the diffusion coefficient  $D$  is independent of the velocity, which is indicative of the dominating effect of molecular diffusion.

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\* For details see [6-13]-[6-19].

Table 6-6

Longitudinal  $D_L$  to Transverse  $D_T$  diffusion  
Coefficient Ratios Reproduced from Data  
Reported by Various Authors

Experimental data	$3 \leq \frac{D_L}{D_T} \leq f$	Reference
Statistical model	$\frac{D_L}{D_T} = \frac{4}{95} \log_e \left[ \frac{27(a+b+2)^2}{(a+1)J} \frac{vT}{\langle l \rangle} \right]$	[6-13]
Continuum, turbulent approach	$\frac{D_L}{D_T} = \frac{\lambda_1}{\lambda_2}$	[6-14]
Ditto	$\frac{D_L}{D_T} = \frac{A_I}{A_{II}}$	[6-15] [6-16]
Ditto	$\frac{D_L}{D_T} = \frac{\lambda + 2M}{\lambda}$	[6-17]
Ditto	$\frac{D_L}{D_T} = \frac{\alpha_1 + (\alpha_2 + \alpha_3) \frac{l^2 v_1^2}{D_0}}{\alpha_1 + \alpha_2 + \frac{l^2 v_1^2}{D_0}}$	[6-18]
Ditto	$\frac{D_L}{D_T} = 3$	[6-19]

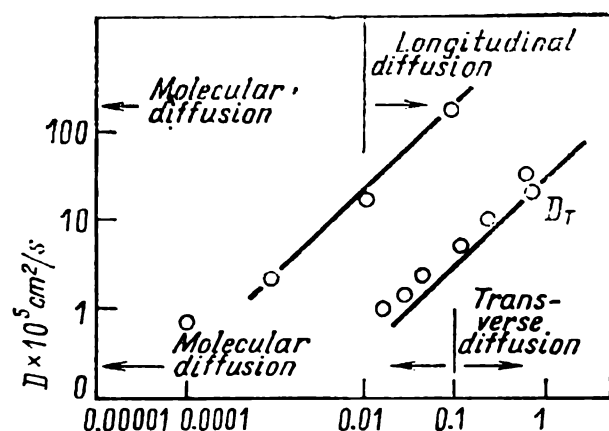


Fig. 6-8. Plot of diffusion coefficients  $D_L$  and  $D_T$ ,  $\text{cm}^2/\text{sec}$ , versus filtration velocity  $v$ ,  $\text{cm}/\text{sec}$ , for a porous material composed of glass beads

Solid lines correspond to values calculated by formulas listed in [6-13]; circles show experimental data [6-20]

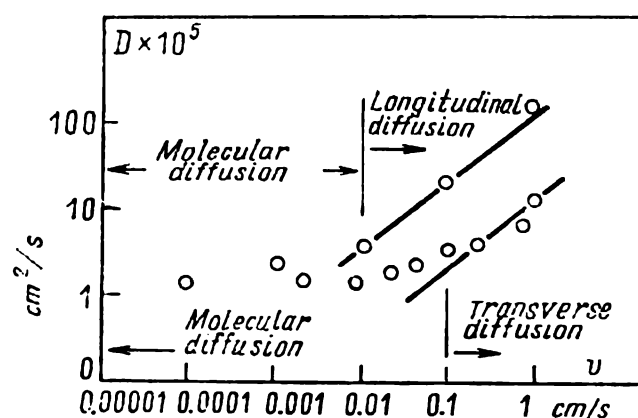


Fig. 6-9. Plot of  $D_L$  and  $D_T$  versus  $v$  for sandstone

Solid lines correspond to values calculated by formulas listed in [6-13]; circles show experimental data [6-20]

From Figures 6-8 and 6-9 it follows that the diffusion coefficient is a linear function of the filtration velocity  $v$ .

In [6-13] experimental data on diffusion in quartz sand were processed. As a result the following empirical formulas were obtained:  
for sand

$$\frac{D_L}{v} = 83 \text{Re}^{1.2} \quad (6-9-26)$$

for glass beads

$$\frac{D_L}{v} = 54 \text{Re}^{1.2} \quad (6-9-27)$$

where  $\text{Re} = \frac{v\sqrt{k}}{\nu}$ ;  $\nu$  is the coefficient of kinematic viscosity;  $k$  is the permeability.

Similar formulas in the form

$$\frac{D}{v} = A \text{Re}^n \quad (6-9-28)$$

where  $A$  and  $n$  are constants, have been obtained by other researchers. Here, the exponent at  $\text{Re}$  is independent of the grain diameter.

#### (e) Convective Diffusion in Capillary Tube Models

The method for the study of diffusion of a dissolved substance, i.e. a substance introduced into liquid which flows along a capillary tube, is widely employed to study the structure of a porous material. Such a capillary tube simulates of a porous material. This procedure has stemmed not only from the desire to investigate the structure of porous materials, but also from the practical value of obtained results for hydrodynamics of petroleum industry. For this reason numerous studies have been made in this direction, and it is difficult to give a comprehensive review of such studies.

In the study of diffusion of dissolved substances in a fluid flow, equations (6-9-1) and (6-9-6) are used. If the differential equation (6-9-6) forms the basis of study, then from its solution for most simple boundary conditions of the first kind (half-space with constant concentration at the free surface) the diffusion coefficient  $D$  (coefficient of dispersion) can be calculated.

This procedure involves the so-called mixing length  $L$  numerically equal to the distance along tube axis, over which the mean concentration of the dissolved substance varies from 10 to 90% with respect to the concentration at the inlet.

Coefficient  $D$  is found from equation (6-9-6) as

$$D = \frac{1}{\tau} \left( \frac{L}{3.625} \right)^2 \quad (6-9-29)$$

However, the calculations based on the solution of equation (6-9-6) do not always give satisfactory results. Therefore in [6-10] a method has been suggested, the main points of which are the following. The diffusion coefficients are calculated from the variations of mean concentration  $\omega_m$  of the substance introduced into liquid immiscible with the former (bulk liquid).

The relation between local  $\omega$  and mean  $\omega_m$  concentrations is taken in the form

$$\omega = \omega_m + \sum_{n=1}^{\infty} f_n(\tau, y) \frac{\partial^n \omega_m}{\partial x_1^n} \quad (6-9-30)$$

where  $f_n(\tau, y)$  is the so-called dispersion function.

Formula (6-9-30) is a Taylor series of the local concentration with respect to the mean concentration.

In this case equation (6-9-6) becomes

$$\frac{\partial^{n+1} \omega_m}{\partial \tau^{n+1}} = D \frac{\partial^{n+2} \omega_m}{\partial x_1^{n+2}} \quad (6-9-31)$$

The diffusion coefficient  $D$  is found from the first term of the series (6-9-30).

The first three terms of (6-9-30) are required for better agreement of calculations with experimental data in the mixing zone:

$$0.3 < \frac{x}{2v_m \tau} < 0.7 \quad (6-9-32)$$

and the higher-order terms introduce corrections for calculations in the edges of the mixing zone.

It should be pointed out that all the calculations based on diffusion equation are applicable only in the case of a fully developed laminar flow in the tube. This condition is analytically expressed as:

$$\tau \approx \frac{0.8R^2}{D} \quad (6-9-33)$$

where  $R$  is the tube radius.

Using the dimensionless mixing length,

$$L^* = \frac{DL}{2R^2 v_m} \quad (6-9-34)$$

we may write the conditions for a fully developed laminar flow as

$$L^* = 0.26 \sqrt{\frac{\tau D}{R^2}}; \quad \tau^* = \frac{\tau D}{R^2} > 0.8 \quad (6-9-35)$$

For small values of  $\tau^*$  ( $\tau^* < 0.01$ )  $L^* = 0.8\tau^*$ .

For small numbers of Peclet diffusion criterion  $Pe_D$  ( $Pe_D = \frac{2v_m R}{D}$ ), in [6-21] the following relations have been obtained:

$$\left. \begin{aligned} L^* &= 2.5 Pe_D^{-0.75} \tau^{*0.55} \\ \tau^* &> 0.05 \end{aligned} \right\} \quad (6-9-36)$$

$$\left. \begin{aligned} L^* &= 2.5 Pe_D^{-0.75} \tau^{*0.60} \\ \tau^* &< 0.05 \end{aligned} \right\} \quad (6-9-37)$$

Knowing  $L^*$  the diffusion coefficient may be calculated.

One-dimensional model of unsteady diffusion gives better results for a turbulent flow than for a laminar one, because time required for the establishment of such flow, which would have simulated fluid motion in a porous material, is very small.

The above given analytical solutions may be used for the analysis of experimental data on heat and mass transfer in capillary-porous bodies. For this it is necessary to have experimental data on transfer properties of capillary-porous materials. Unfortunately, few such data are at present available; this makes their review and analysis very difficult.

Besides, the analytical solutions may be used as a basis for the development of experimental methods of determining heat- and mass-transfer coefficients, including mass-transfer rate  $\omega_m$ . The availability of such methods would allow us to accumulate necessary experimental data on the structures of capillary-porous and porous bodies. This is a very urgent problem in the heat- and mass-transfer theory. The thing is that simulation of a porous material in the form of a capillary tube, from the viewpoint of hydrodynamics, will be possible only when the nature of fluid flow is similar in both the cases. To fulfil this condition it is necessary to achieve steady flow conditions in the capillary tube model.

Taylor-Aris' theory is applicable in that case when the dimensionless time  $\tau^*$  is not less than  $\tau_{\min}^*$

$$\tau_{\min}^* \approx \frac{0.6 Pe_D^2}{Pe_D^2 + 192} \quad (6-9-38)$$

Such conditions are quickly attained in a turbulent flow. The diffusion coefficient  $D$  in this case is defined by

$$D = - \frac{2v_m^2 R^2}{D_m} \int_0^1 y \left( \frac{v}{v_m} - 1 \right) f_{1s} dy \quad (6-9-39)$$



where  $f_{1s}$  is the function determined from formula (6-9-30). Function  $f_{1s}$  is obtained from the solution of differential equation

$$\frac{\partial \omega}{\partial \tau} + v(r) \frac{\partial \omega}{\partial x} = D \left( \frac{\partial^2 \omega}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial \omega}{\partial r} \right) \quad (6-9-40)$$

with allowance made for the substitution of (6-9-5). Here, the diffusion coefficient includes in it the turbulent diffusion coefficient. If the turbulent diffusion coefficient is much in excess of the molecular diffusivity, then  $D$  (dispersion coefficient) will approximately be equal to

$$D = 10.1 R v^* \quad (6-9-41)$$

where  $v^*$  is the flow velocity defined by

$$v^* = \sqrt{\frac{\sigma_w}{\rho}} = v_m \sqrt{\frac{1}{2} C_f} \quad (6-9-42)$$

where  $C_f$  is the friction factor and  $\sigma_w$  is the shear stress at the tube wall. Thus, for diffusion coefficient  $D$ , we obtain a relation

$$D = 7.14 R v_m \sqrt{C_f} \quad (6-9-43)$$

The accuracy of calculated values of  $D$  strongly depends on the accuracy of determining the velocity. When  $Re \geq 4.2 \times 10^4$  the obtained diffusion coefficients may differ from experimental values by 50 per cent.

From (6-9-43) it follows that the diffusion coefficient is proportional to mean velocity  $v_m$ .

Figure 6-10 is a plot of inverse Peclet criterion  $1/Pe_D$  versus  $Re$  ( $Re = 2v_m R/\nu$ ) based on the calculation by Taylor-Aris' formula (see formula (6-9-8)) and on the experimental data.

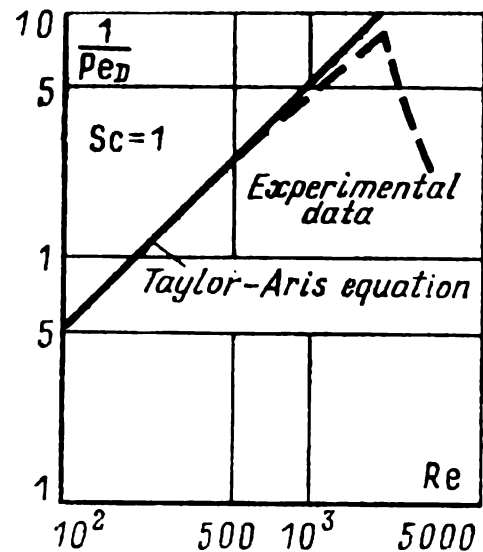


Fig. 6.10. Comparison of calculated with experimental data as a plot of  $1/Pe_D$  ( $D/2v_m R$ ) versus Reynolds number  $Re$  ( $Re = \frac{2v_m R}{\nu}$ );  $Sc = 1$

From Fig. 6-10 it is clear that in the range  $10^2 < Re < 10^3$  the calculated and experimental data are in adequate agreement, and thereafter sharp disparity between them is observed. This may be explained by the fact that

at  $Re > 10^3$  the flow becomes turbulent. This results in the sharp decrease of diffusion coefficient due to transverse turbulent mixing.

A more advanced model of porous structures affords substitution of a cylindrical capillary tube by tubes of various configurations, such as straight duct, straight slot, coaxial and concentric slots, in which the asymmetry effects of velocity profile, duct curvatures, converging and diverging ducts

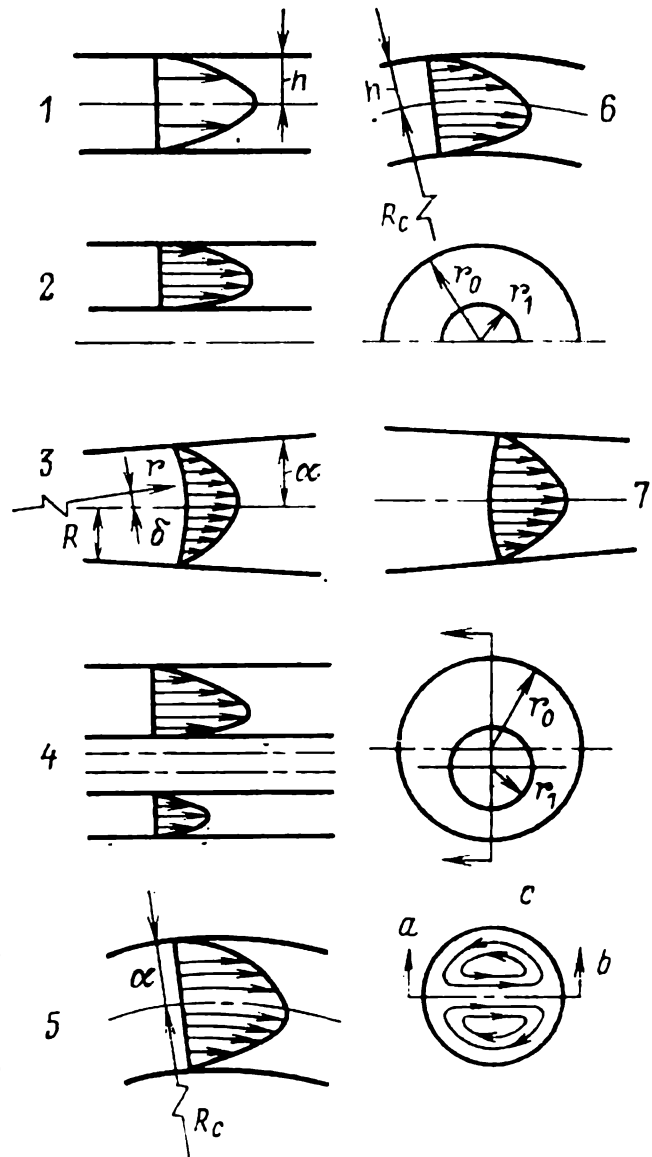


Fig. 6-11. Model capillary tubes with fluid velocity profiles

1 — straight channel; 2 — concentric annulus; 3 — expanding channel; 4 — eccentric annulus; 5 — curved tube ( $a$  — internal flowline,  $b$  — external flowline;  $c$  — flowline); 6 — curved channel; 7 — converging channel

(Fig. 6-11) can be studied. In particular, a mixing cell model is used to explain the fact that the diffusion coefficient for a cylindrical tube is directly proportional to squared velocity, but in a porous medium the diffusion coefficient is proportional to the first power of the linear flow velocity. However, it should be mentioned that in some papers a different relation between  $D$  and  $v_m$  has been obtained, particularly, the diffusion coefficient  $D$  is directly proportional to  $v_m^n$ , where  $n > 1$ .

Further we shall cite the results of some authors. For a parallel plane channel, when the laminar flow is fully developed ( $\tau^* \geq 1$ ), the diffusion coefficient equals

$$D = D_m + \frac{8}{945} \frac{v_{\max}^2 H^2}{D_m} \quad (6-9-44)$$

where  $H$  is the half width of the channel; and  $v_{\max}$  is the axial velocity.

Concentric and excentric channels are the extensions of a straight channel and straight tube, and take into account the effect of the asymmetric profile of velocity. The diffusion coefficient in this case is a function of the outer to inner radii ratio ( $1/K = R_0/R_i$ ). For small values of  $1/K$  (a narrow slot), when  $\frac{1}{K} \leq 1.5$ , the following relation was obtained:

$$D = D_m + \frac{8}{945} \left[ \frac{R_0^2 v_{\max}^2 (1 - K^2)}{4D_m} \right] \quad (6-9-45)$$

where  $R_0$  is the external cylinder radius.

From (6-9-45) it follows that  $D$  increases as  $1/K$  increases. Thus, the asymmetry effect in the velocity profile should result in large diffusion coefficients.

When  $\frac{1}{K} \geq 100$ , the diffusion coefficient  $D$  equals the diffusion coefficient for a cylindrical tube.

In [6-10] it has been shown that the curvature effect gives rise to a decrease in diffusion coefficients compared to their values for a straight tube with  $Sc \geq 0.124$ . Then it was demonstrated that the effect of cross-mixing at the cost of secondary flow and asymmetric axial velocity first results in the increase and then in the decrease of diffusion coefficient. Here, the diffusion coefficients are even smaller than those at large  $Re$  (see Fig. 6-10).

The flow in convergent and divergent slit channels is accelerated or decelerated due to the change in the cross-sectional area. Even small expansion angles considerably affect the diffusion coefficient, especially at large Peclet numbers [6-22].

Existence of stagnant fluid spots in porous media due to closed passages along the flow path results from the geometry of porous medium.

The effect of system capacity causes an increase in the mixing length. Many researchers used the considered model in their studies. A review of these studies is presented in [6-23] — [6-26].

Using the similarity theory methods as the base, it may be shown that a diffusion process depends on generalized variables, such as the Peclet numbers  $Pe_D$ , Reynolds numbers  $Re$ , Grashof number  $Gr$  and Froude number  $Fr$ .

Proceeding from formula (6-9-8), valid for a laminar flow in a cylindrical capillary tube, we may get a relation between the dimensionless diffusion coefficient  $D^*$  and the Peclet number:

$$D^* = \frac{D}{D_m Pe_D^2} = \frac{1}{Pe_D^2} + \frac{1}{48} \quad (6-9-46)$$

Many studies on immiscible fluid motion in porous materials, when the gravity field effect cannot be neglected, have been carried out. The review of these studies is out of the scope of this book.

Thus, by studying the picture of tracer diffusion in a fluid flow through a model capillary tube, an important information may be obtained about the effect of porous material structure on the diffusion mechanism taking place in it. But such a simulation is possible provided that the flow is fully developed.

In [6-27] it is shown that the equivalent capillary tube model can be used provided the inequality

$$\frac{L_T}{2R} > 0.04 \frac{2v_m R}{D} \quad (6-9-47)$$

holds, where  $L_T$  is the tube or system length in the direction of the main flow.

From the analysis of the studies made by using the capillary model with cavities (closed pores) it follows that the application of such a model requires significantly longer time.

Summing up, it may be pointed out that the diffusion law, stemming from concentration gradient, was used as the basis for the analysis of mass-transfer mechanism in porous medium. Here, the hydrodynamic mass transfer characterized by the mean fluid velocity  $v_m$  appeared as main parameter in the relation for diffusion coefficient (dispersion coefficient). Physically this means that the finite transfer rate  $v_m$  was indirectly introduced in mass-transfer calculations. The need for such calculation of diffusion coefficient is caused by the fact that the hypothesis about infinite velocity of mass propagation forms the basis of analytical theory of diffusion.

It is known that the analytical heat conduction and diffusion theory is based on infinite rates of heat and mass transfer. This assumption is however inconsistent with the physical essence of heat and mass transfer.

Therefore, a different method of investigating unsteady heat- and mass-transfer processes may be suggested, from the very beginning the finite rate of heat and mass transfer underlies this method. As may be seen from the above said, it is of particular importance for mass-transfer processes.

Such a method of analysis of mass-transfer processes in capillary-porous bodies and porous media was suggested by the present author.

## 6-10. HYPERBOLIC DIFFERENTIAL HEAT- AND MASS-TRANSFER EQUATIONS AND THEIR SOLUTIONS

In the phenomenological heat-conduction theory it is assumed that the heat propagation velocity  $\bar{w}_q$  is infinite ( $\bar{w}_q = \infty$ ). This assumption is confirmed by the results of calculations of temperature fields in different bodies at ordinary conditions encountered in practice.

However, at certain conditions of heat transfer it is necessary to assume that heat propagates with finite velocity. This was first noticed by P. Vernotte [6-28]. The present author independently proposed a hypothesis about the finite rate of heat and mass diffusion. Developing the Prigogine

principle [6-29] the author [6-31] suggested a generalized system of Onzager linear equations.

If  $\vec{j}_k$  is used to denote fluxes and  $\vec{X}_k$  the thermodynamic motive forces, then the generalized system becomes

$$\vec{j}_k = \sum_i (L_{ik} \vec{X}_k + L'_{ik} \dot{\vec{X}}_k) + L_{kr} \dot{\vec{j}}_k \quad (6-10-1)$$

where  $L_{ik}$ ,  $L'_{ik}$  and  $L_{kr}$  are kinetic coefficients ( $L_{ik} = L_{ki}$ ), the dot above the quantity indicates the time derivative.

In heat- and mass-transfer process the effects of force  $\dot{X}_k$  may be neglected. Without taking into account the thermodiffusion and diffusion thermo-effects, we may write

$$\vec{j}_k = \sum_i L_{ik} \vec{X}_k + L_{kr} \dot{\vec{j}}_k \quad (6-10-2)$$

For one-dimensional heat and mass fluxes we have

$$q = -\lambda \frac{\partial T}{\partial x} - \tau_{rq} \dot{q} \quad (6-10-3)$$

$$j_k = -D\rho \frac{\partial \rho_{k0}}{\partial x} - \tau_{rm} \dot{j}_k \quad (6-10-4)$$

where  $\tau_{rq}$ ,  $\tau_{rm}$  are relaxation (times) periods of heat and mass transfer. They are related with thermal diffusivity  $a$  and mass diffusivity  $D$  by relations

$$\left. \begin{aligned} \tau_{rq} &= \frac{a}{w_q^2} \\ \tau_{rm} &= \frac{D}{w_m^2} \end{aligned} \right\} \quad (6-10-5)$$

where  $w_m$  and  $w_q$  are heat and mass propagation velocities, respectively.

In accordance with (6-10-3) and (6-10-4) the differential heat-conduction and diffusion equations take the form

$$\frac{\partial T}{\partial \tau} + \tau_{rq} \frac{\partial^2 T}{\partial \tau^2} = a \frac{\partial^2 T}{\partial x^2} \quad (6-10-6)$$

$$\frac{\partial \rho_{k0}}{\partial \tau} + \tau_{rm} \frac{\partial^2 \rho_{k0}}{\partial \tau^2} = D \frac{\partial^2 \rho_{k0}}{\partial x^2} \quad (6-10-7)$$

These are hyperbolic heat-conduction and diffusion equations; they differ from ordinary parabolic heat-conduction equations by additional

terms  $\tau_{rq} \frac{\partial^2 T}{\partial \tau^2}$  and  $\tau_{rm} \frac{\partial^2 \rho_{k0}}{\partial \tau^2}$ , which account for the finite rate of heat and

mass transfer. At  $w \rightarrow \infty$  and  $\tau_r \rightarrow 0$  differential equations (6-10-6) and (6-10-7) change to ordinary differential heat- and mass-transfer equations of parabolic type.

Unlike the last solutions, equations (6-10-6), (6-10-7) have a distinct wave front travelling with finite velocity  $w$ . It should be mentioned that hyperbolic transfer equations, particularly the wave equation, were obtained by different researchers using different methods for the analysis of diffusion, heat conduction, turbulent diffusion, etc.

It should be pointed out that the finite velocity of disturbance propagation may be found from the solution of the nonlinear parabolic equation [6-39]—[6-43]:

$$\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x} \left[ a(T) \frac{\partial T}{\partial x} \right] \quad (6-10-8)$$

when thermal diffusivity  $a \rightarrow 0$ . The solution of equation (6-10-8) for this case is presented in [6-28]—[6-36] in the form of the so-called temperature waves.

Recently some works have been reported (for example [6-38]). They contain more general results for a finite transfer rate of substances (mass or heat), integro-differential equations derived for a three-dimensional case with consideration for the finite velocity and the formulation of a general nonlinear transfer problem.

Hyperbolic heat- and mass-transfer equations may have a different physical meaning. Let us imagine that evaporation of liquid within the body occurs at a surface which moves in the body with constant velocity  $w_{ph}$ . In this case  $w_q = w_m = w_{ph}$  and the travelling rate of phase change surface has a different physical meaning (see Sec. 1-11). Then heat and mass fluxes  $q$  and  $j$  are defined by ordinary Fourier and Fick formulas, and  $\frac{a}{w_{ph}^2} \frac{\partial^2 T}{\partial \tau^2}$

and  $\frac{D}{w_{ph}^2} \frac{\partial^2 \rho_{k0}}{\partial \tau^2}$  acquire the physical meaning of heat and mass sources.

Equations (6-10-6) and (6-10-7) then become

$$\frac{\partial T_i}{\partial \tau} = a_i \nabla^2 T_i - \frac{a_i \delta_{1i}}{w_{ph}^2} \frac{\partial^2 T_i}{\partial \tau^2}, \quad i = 1, 2 \quad (6-10-6a)$$

$$\frac{\partial \omega_i}{\partial \tau} = D_i \nabla^2 \omega_i - \frac{D_i \delta_{1i}}{w_{ph}^2} \frac{\partial^2 \omega_i}{\partial \tau^2} \quad (6-10-7a)$$

where  $\omega$  is the relative concentration ( $\omega \equiv \rho_{i0}$ );  $\delta_{1i}$  is the delta-function;  $\delta_{1i} = 1$  at  $i = 1$  and  $\delta_{1i} = 0$  at  $i = 2$ .

For a temperature field at  $\tau \leq \tau_{rq} \left( \tau_{rq} = \frac{a_1}{w_{ph}^2} \right) i=1$  (the evaporation zone); at  $\tau > \tau_{rm} i=2$  (the moist state zone).

For a concentration field at  $\tau \leq \tau_{rm}$  ( $\tau_{rm} = D_1/w_{ph}^2$ )  $i = 1$ ; at  $\tau > \tau_{rm}$   $i = 2$ .

*(a) Heat Transfer in a Semi-Infinite Solid with Finite Rate*

First we shall explain certain peculiarities of heat transfer in a solid with finite rate. For this purpose we shall consider the most simple case of heating a semi-infinite solid (half-space) at boundary conditions of the first kind. At the initial moment ( $\tau = 0$ ) the free surface of the solid is at temperature  $T_0$ , which is maintained constant during the heating process. The initial temperature is taken as the reference point, which is assumed to be uniform throughout the solid (uniform initial distribution of temperature). The solution of this simplest problem is presented in [6-44] in the form:

$$T^*(\tau) = \frac{x}{2} \sqrt{\frac{1}{a\tau_0}} \int_{\tau=x\sqrt{\tau_0/a}}^{\tau} \exp\left(-\frac{\tau}{2\tau_0}\right) \frac{I_1\left[\frac{1}{2\tau_0}\left(\tau^2 - \frac{x^2\tau_0}{a}\right)^{1/2}\right]}{\left(\tau^2 - \frac{x^2\tau_0}{a}\right)^{1/2}} d\tau$$

$$+ \exp\left[-\frac{x}{2} \sqrt{\frac{1}{a\tau_0}}\right] \quad (6-10-9)$$

for  $\tau \geq x\sqrt{\tau_0/a}$ ;

$$T^*(\tau) = 0$$

for  $\tau < x\sqrt{\tau_0/a}$ .

In this solution  $\tau_0$  is the relaxation period;  $a$  is the thermal diffusivity;  $I_1(z)$  is the modified Bessel function of the first kind.

From (6-10-9) it follows that at  $\tau = x\sqrt{\tau_0/a}$

$$T^* = e^{-\tau/2\tau_0} \quad (6-10-10)$$

Figure 6-12 is a plot of the temperature distribution versus distance  $x$ , which shows that at a certain distance  $\Delta$  the curve  $T^*(x)$  discontinues.

Hence, at  $x > \tau\sqrt{a/\tau_0}$  no temperature rise is observed (the heat flux front or discontinuity). In the classical heat-conduction theory no such discontinuity is found in the temperature distribution curve. This propagation front may be considered as the heat propagation depth. If this is denoted by  $\Delta$ , we get

$$\tau = \Delta\sqrt{\tau_0/a} \quad (6-10-11)$$

or

$$\Delta = \sqrt{\tau/\tau_0} \sqrt{a\tau} \quad (6-10-12)$$

The heat propagation (penetration) velocity

$$w_q = \frac{d\Delta}{d\tau} = \frac{d}{d\tau} (\tau \sqrt{a/\tau_0}) = \sqrt{a/\tau_0} \quad (6-10-13)$$

i.e. the propagation velocity depends on thermal diffusivity and  $\tau_0$ .

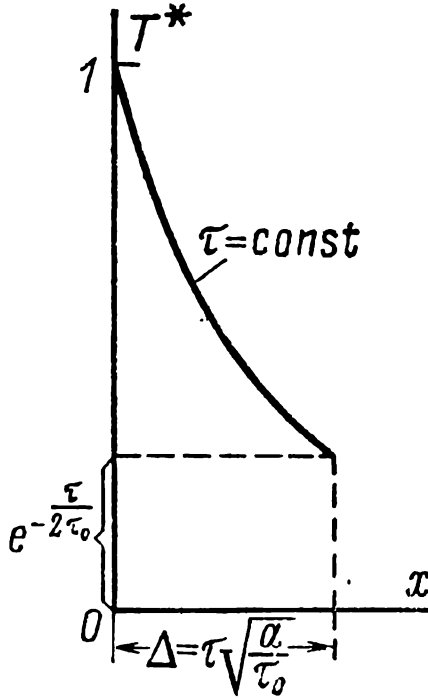


Fig. 6.12. Isochrone of a halfspace

For example, for solids (metals)  $\tau_0 \approx 10^{-11}$  s, hence for steel  $w_q = 1\,800$  m/s.

In the classical heat-conduction theory is found a concept about isotherm propagation velocity  $w_T$ . For the present case  $w_T = 2\sqrt{a/\tau}$ . Consequently, at  $\tau = 0$   $w_T = \infty$ , and at  $\tau \rightarrow \infty$   $w_T = 0$ , that is inconsistent with the concept of physical mechanism of heat propagation.

The heat flux density is found from the heat balance equation

$$q(\tau) = \frac{\sqrt{\lambda c \rho}}{\sqrt{\tau_0}} \exp\left(-\frac{\tau}{2\tau_0}\right) I_0\left(\frac{1}{2\tau_0} \sqrt{\tau^2 - \frac{x^2 \tau_0}{a}}\right) \quad (6-10-14)$$

at the solid surface ( $x = 0$ )

$$q_0(\tau) = \frac{\sqrt{\lambda c \rho}}{\sqrt{\tau_0}} \exp\left(-\frac{\tau}{2\tau_0}\right) I_0\left(\frac{\tau}{2\tau_0}\right) \quad (6-10-15)$$

At  $\tau = 0$  (initial moment)

$$q_0(0) = \frac{\sqrt{\lambda c \rho}}{\sqrt{\tau_0}} \quad (6-10-16)$$

The appropriate value from the classical heat-conduction theory

$$q_0(0) = \left(\frac{\sqrt{\lambda c \rho}}{\sqrt{\pi \tau}}\right)_{\tau=0} = \infty \quad (6-10-17)$$



i.e. the heat flux is infinite, this is also inconsistent with the physical mechanism of heat transfer.

Let us now compare solution (6-10-9) with the classical solution of a similar problem. For  $\tau \geq \tau_0$ , using the asymptotic expansion

$$I_1(z) \approx e^z \frac{1}{\sqrt{2\pi z}} \quad (6-10-18)$$

we get

$$T^* = \frac{x}{2\sqrt{\pi a}} \int_{x\sqrt{\tau_0/a}}^{\tau} e^{-\tau/2\tau_0} \frac{\exp\left(\frac{1}{2\tau_0} \sqrt{\tau^2 - \frac{x^2\tau_0}{a}}\right)}{\left(\tau^2 - \frac{x^2\tau_0}{a}\right)^{3/4}} d\tau + \exp\left(-\frac{x}{2} \sqrt{\frac{1}{a\tau_0}}\right) \quad (6-10-19)$$

If  $\frac{1}{4} \frac{x^2}{a\tau} = u^2$  is assumed, then for small values of  $\tau_0$  ( $\tau^2 \gg x^2\tau_0/a$ ) the term  $x^2\tau_0/a$  may be omitted:

$$\begin{aligned} T &\approx \frac{x}{2\sqrt{\pi a}} \int_{x\sqrt{\tau_0/a}}^{\tau} e^{-u^2} \tau^{-3/2} d\tau = \frac{2}{\sqrt{\pi}} \left[ \int_0^{\infty} e^{-u^2} du - \int_0^u e^{-u^2} du \right] \\ &= 1 - \frac{2}{\sqrt{\pi}} \int_0^u e^{-u^2} du = \operatorname{erfc} u \end{aligned} \quad (6-10-20)$$

that is a solution of the ordinary classical heat-conduction equation for given boundary conditions\*.

In a similar way it may be demonstrated that at small  $\tau_0$  the quantity  $q_0(\tau)$  is of the form

$$q_0(\tau) = \frac{\sqrt{\lambda c \rho}}{\sqrt{\tau_0}} \frac{1}{\sqrt{\pi \tau / \tau_0}} = \frac{\sqrt{\lambda c \rho}}{\sqrt{\pi \tau}} \quad (6-10-21)$$

this agrees with the heat-flux expression for the classical problem. Hence, the solutions, obtained from the hyperbolic heat-conduction equation are more general, and from these equations, as a particular case, the solutions of the classical heat-conduction theory are obtained.

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\* Here an approximate relation is used

$$\frac{1}{2\tau_0} \left[ -\tau + \left( \tau^2 - \frac{x^2\tau_0}{a} \right)^{1/2} \right] \approx -\frac{1}{4} \frac{x^2}{a\tau}$$

Let us introduce the following dimensionless quantities:

$$\theta = \frac{T}{T_0}; \xi = \frac{x}{\sqrt{a\tau_0}}, \quad \tau^* = \tau/\tau_0 \quad (6-10-22)$$

Then the solution for a temperature profile in a semi-infinite body may be written as:

$$\theta = \frac{\xi}{4} \int_{\xi}^{\tau^*} e^{-\frac{1}{2}\tau^*} \frac{I_1\left(\frac{1}{2}\tau^* \sqrt{1 - \xi^2/\tau^{*2}}\right)}{\frac{1}{2}\tau^* \sqrt{1 - \xi^2/\tau^{*2}}} d\tau^* + e^{-\frac{1}{2}\xi} \quad (6-10-23)$$

Accordingly for heat fluxes

$$q^*(\tau^*, \xi) = e^{-\frac{1}{2}\tau^*} I_0\left(\frac{1}{2}\tau^* \sqrt{1 - \xi^2/\tau^{*2}}\right) \quad (6-10-24)$$

$$q_0^*(\tau^*) = e^{-\frac{1}{2}\tau^*} I_0\left(\frac{1}{2}\tau^*\right) \quad (6-10-25)$$

where

$$q^*(\tau^*) = \frac{q(\tau)\sqrt{\tau_0}}{T_0\sqrt{\lambda c \rho}} \quad (6-10-26)$$

Formulas (6-10-24) and (6-10-25) were used for calculations. Figure 6-13 is a plot of  $q_0^*(\tau^*)$  versus the dimensionless time  $\tau^*$ . From this figure

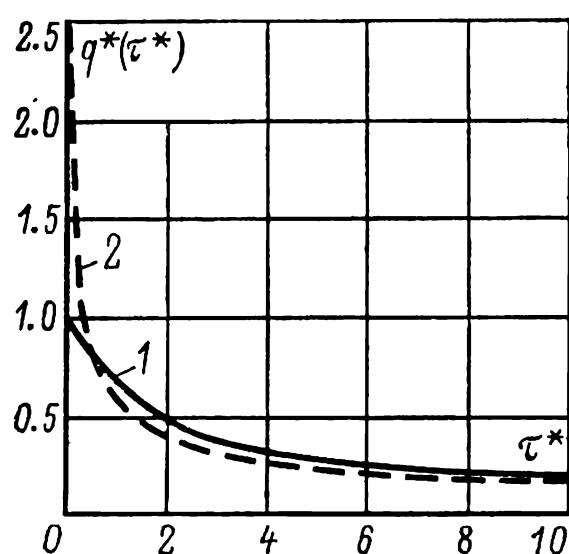


Fig. 6-13. Plot of  $q_0^*$  versus  $\tau^*$   
1 — according to the equation with allowance made for finite heat-transfer rate; 2 — classical solution

it is clear that at small  $\tau^*$  ( $\tau^* < 0.5$ ) the flow curves calculated by formulas (6-10-25) and (6-10-21) sharply diverge. At large  $\tau^*$  ( $\tau^* > 8$ ) the curves  $q_0(\tau^*)$  practically coincide.

It is of interest to compare the curves  $\theta(\xi)$  for the solutions with regard to the finite velocity  $w_q \neq \infty$  (see solution (6-10-23)) and for  $w_q = \infty$  (classical solution), when

$$\theta(\xi) = \frac{2}{\sqrt{n}} \int_0^u e^{-u^2} du = \operatorname{erfc} \left( \frac{\xi}{2\sqrt{\tau^*}} \right) \quad (6-10-27)$$

Solution (6-10-23) for small  $\tau^*$  may be simplified. Let us assume that

$$I_1(z) = \frac{1}{2} z \quad \text{at } 0 < z < 1. \text{ Then we get}$$

$$\theta = \frac{1}{4} \xi (e^{-\frac{1}{2}\xi} - e^{-\frac{1}{2}\tau^*}) + e^{-\frac{1}{2}\xi} \quad (6-10-28)$$

that is valid when

$$\frac{1}{2} \tau^* \sqrt{1 - \xi^2/\tau^{*2}} \leq 1 \quad \text{or} \quad \tau^{*2} \leq 4 + \xi^2 \quad (6-10-29)$$

In Fig. 6-14 is presented a family of isochrones for  $\tau^* = 0.25; 1.0$  and  $2.0$ .

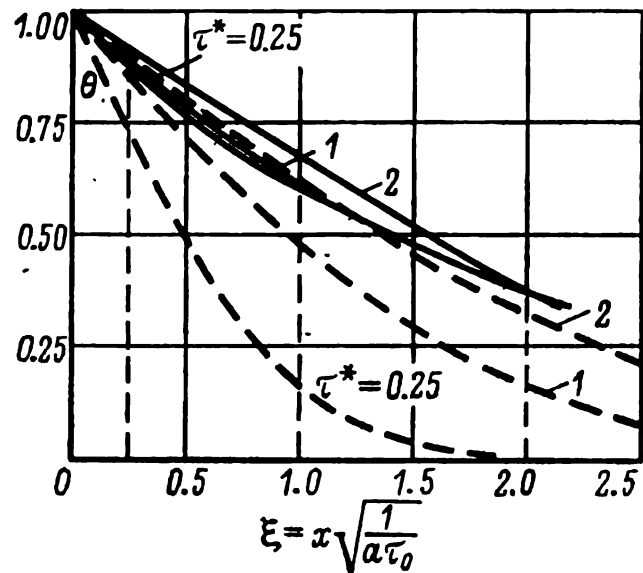


Fig. 6 14. Plot of dimensionless temperature  $\theta^*$  versus generalized coordinate of isochrone  $\xi$  for various dimensionless times  $\tau^*$  [6-44]

--- classical solution; — at  $w_q \neq \infty$ ; 1 —  $\tau^* = 1$ ; 2 —  $\tau^* = 2$

For  $\tau^* = 2$  both the curves  $\theta(\xi)$  give approximately one and the same value (deviation is 13%) up to  $\xi = \tau^* = 2$  (heat flow front or discontinuity). For  $\tau^* = 1$  the heat flow front passes through point  $\xi = 1$ . At this point the difference between  $\theta(\xi)$  curves is considerable, and amounts to about 20%. For small  $\tau^*$  the difference between the isochrones  $\theta(\xi)$  is large and stops at curve  $\theta = e^{-\frac{1}{2}\xi}$  at the heat wave front.

The slope of curves  $\theta(\xi)$  characterizes the rate of temperature change in the given direction. For classical isochrones this quantity tends to infinity at  $\tau^* \rightarrow 0$ . For isochrones which take account of the finite heat-transfer rate the slope is always finite and does not exceed 0.5.

Let us consider isochrones for large times ( $\tau^* > 2$ ). The graphical integration method was applied to solution (6-10-23) to calculate  $\theta(\xi)$  for  $\tau^*$  and to compare with classical isochrones  $\theta(\xi)$ . This comparison is presented in Fig. 6-15, which shows that isochrones  $\theta(\xi)$  practically coincide right up to the heat flow front.

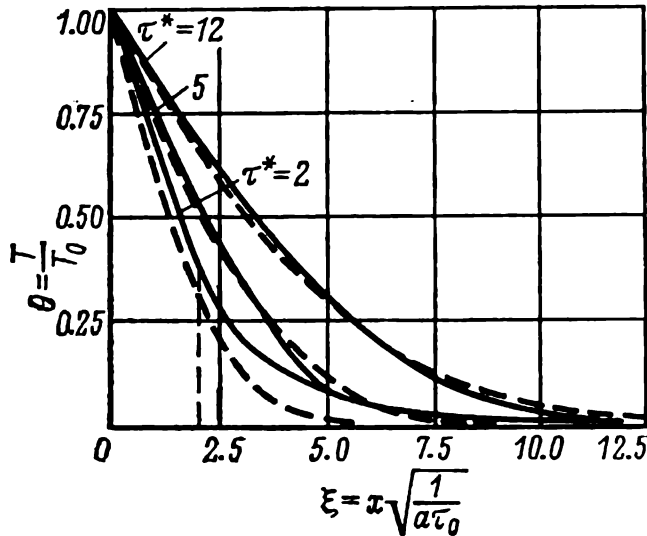


Fig. 6-15. Isochrones  $\theta(\xi)$  for a semi-infinite solid at various  $\tau^*$  [6-44]  
 ----- classical solution; — solution at  $w_q \neq \infty$

A more general problem was solved in [6-45] for a semi-infinite solid. The following boundary conditions were adopted:

$$T(x, 0) = f(x); \quad T(0, \tau) = \chi(\tau);$$

$$\frac{\partial T(\infty, \tau)}{\partial x} = 0; \quad \frac{\partial T(x, 0)}{\partial \tau} = F(x)$$

The solution is of the form

$$T(x, \tau) = \exp\left[-\frac{x}{2\sqrt{\tau_0 a}}\right] \chi\left(\tau - x\sqrt{\frac{\tau_0}{a}}\right) + \frac{x}{2\sqrt{\tau_0 a}} \int_{x\sqrt{\frac{\tau_0}{a}}}^{\tau} \chi(\tau - \tau_1) \\ \times \frac{\exp\left[-\frac{\tau_1}{2\tau_0}\right]}{\sqrt{\tau_1^2 - \frac{\tau_0}{a}x^2}} I_1\left(\frac{\sqrt{\tau_1^2 - \frac{\tau_0}{a}x^2}}{2\tau_0}\right) d\tau_1 - \frac{1}{2\sqrt{\tau_0 a}} \exp\left(-\frac{\tau}{2\tau_0}\right) \\ \times \int_0^{\tau\sqrt{\frac{a}{\tau_0}} - x} \left\{ \left[ \tau_0 F(\xi) + \frac{1}{2} f(\xi) \right] \left[ I_0\left(\frac{\sqrt{\tau^2 - \frac{\tau_0}{a}(x - \xi)^2}}{2\tau_0}\right) \right. \right.$$

$$\begin{aligned}
& - I_0 \left( \frac{\sqrt{\tau^2 - \frac{\tau_0}{a} (x + \xi)^2}}{2\tau_0} \right) \\
& + \frac{1}{2} \tau f(\xi) \left[ \frac{I_1 \left( \frac{1}{2\tau_0} \sqrt{\tau^2 - \frac{\tau_0}{a} (x - \xi)^2} \right)}{\sqrt{\tau^2 - \frac{\tau_0}{a} (x - \xi)^2}} - \frac{I_1 \left( \frac{1}{2\tau_0} \sqrt{\tau^2 - \frac{\tau_0}{a} (x + \xi)^2} \right)}{\sqrt{\tau^2 - \frac{\tau_0}{a} (x + \xi)^2}} \right] d\xi
\end{aligned}
\tag{6-10-29 a}$$

In [6-45], [6-47] a number of problems have been solved for classical configurations (infinite plate, sphere, cylinder) at boundary conditions of the first and second kinds, as well as at the fourth-kind boundary conditions (complicated configurations). These solutions may be successfully used for investigation of the mass-transfer mechanism in porous materials.

#### (b) Heat and Mass Transfer in Capillary-Porous Bodies

The system of differential heat- and mass-transfer equations with regard to finite rate of mass transfer ( $w_m \neq \infty$ ,  $w_q = \infty$ ), with the assumption that transfer properties of materials are constant in generalized variables, is of the form [6-45]:

$$\frac{\partial T}{\partial Fo} = \nabla^2 T - Ko^* \frac{\partial \theta}{\partial Fo} \tag{6-10-30}$$

$$\frac{\partial \theta}{\partial Fo} + K \frac{\partial^2 \theta}{\partial Fo^2} = Lu (\nabla^2 \theta - Pn \nabla^2 T) \tag{6-10-31}$$

where  $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\Gamma}{x} \frac{\partial}{\partial x}$ ;  $\Gamma = 0, 1, 2$ ;  $K = \frac{a\tau_{rm}}{R^2} = \frac{aa_m}{R^2 w_m^2}$ . Here  $a$  and  $a_m$  are coefficients of heat and mass diffusion, respectively;  $x$  is the dimensionless coordinate.

Equation (6-10-30) is a parabolic type equation. This is due to the fact that the heat propagation velocity  $w_q$  is very large [relaxation time  $\tau_{rq}$  for metals is about  $10^{-11}$  s (for aluminium), for gases  $\tau_{rq} \approx 10^{-9}$  s].

According to the data reported in [6-48] the heat propagation velocity equals the sound propagation velocity, i.e. it is very large ( $w_q = 330$  m/s).

As regards the mass propagation velocity during diffusion in capillary-porous bodies, it is less than  $w_q$  by about  $10^6$ - $10^7$  times and should, therefore, be included in mass-transfer equations.

Equations (6-10-30), (6-10-31) were solved by the integral transform method for an asymmetric infinite plate ( $\Gamma = 0$ ), cylinder ( $\Gamma = 1$ ), sphere ( $\Gamma = 2$ ) and a semi-infinite solid:

at the first-kind boundary conditions [6-45]

$$T(0, Fo) = \varphi_1(Fo), \theta(0, Fo) = \varphi_2(Fo) \quad (6-10-32)$$

$$T(1, Fo) = \psi_1(Fo), \theta(1, Fo) = \psi_2(Fo) \quad (6-10-33)$$

at the generalized boundary conditions of the second kind

$$\left. \begin{aligned} -\frac{\partial T(0, Fo)}{\partial X} + Ki_q^{(1)}(Fo) - (1 - \varepsilon)Lu Ko Ki_m^{(1)}(Fo) &= 0 \\ -\frac{\partial \theta(0, Fo)}{\partial X} + Pn \frac{\partial T(0, Fo)}{\partial X} + Ki_m^{(1)}(Fo) &= 0 \end{aligned} \right\} \quad (6-10-34)$$

$$\left. \begin{aligned} -\frac{\partial T(1, Fo)}{\partial X} + Ki_q^{(2)}(Fo) - (1 - \varepsilon)Lu Ko Ki_m^{(2)}(Fo) &= 0 \\ -\frac{\partial \theta(1, Fo)}{\partial X} + Pn \frac{\partial T(1, Fo)}{\partial X} + Ki_m^{(2)}(Fo) &= 0 \end{aligned} \right\} \quad (6-10-35)$$

and at arbitrary initial conditions

$$T(X, 0) = F_1(X), \quad \theta(X, 0) = F_2(X)$$

$$\frac{\partial \theta(X, 0)}{\partial Fo} = F_3(X) \quad (6-10-36)$$

Here, finite integral transformation was performed with respect to variable  $X$ :

$$u_{nF}^{(S)} = \int_0^1 u_F^{(S)}(X, Fo) \rho_F^{(S)}(X) K_F^{(S)}(\mu_n, X) dX \quad (6-10-37)$$

with kernel  $K_F^{(S)}(\mu_n, X)$  and weight  $\rho_F^{(S)}(X)$  ( $S = 1$ ) is taken for (6-10-32), (6-10-33),  $S = 2$  (6-10-34), (6-10-35). Thereafter was applied the Laplace integral transformation with respect to variable  $Fo$ .

Inversion with respect to  $X$  was performed by formula

$$u_F^{(S)}(X, Fo) = \omega_F^{(S)}(\mu_0, Fo) + \sum_{n=1}^{\infty} u_{nF}^{(S)}(\mu_n, Fo) H_F^{(S)}(\mu_n) K_F^{(S)}(\mu_n X) \quad (6-10-38)$$

where

$$\begin{aligned}
 H_R^{(S)}(\mu_n) &= \left[ \int_0^1 \rho_R^{(S)}(X) (K_R^{(S)}(\mu_n, X))^2 dX \right]^{-1} \\
 \omega_{\theta T}^{(2)}(\mu_0, Fo) &\equiv T_{00}^{(2)}(0, Fo) = \int_0^{Fo} [Ki_q^{(2)}(Fo') - Ki_q^{(1)}(Fo')] dFo' \\
 &+ Ko^* Lu \int_0^{Fo} [Ki_m^{(2)}(Fo') - Ki_m^{(1)}(Fo')] \left( \exp \left[ -\frac{Fo - Fo'}{K} \right] - 1 \right) dFo' \\
 &+ \int_0^1 F_1(X) dX + K Ko^* \left[ \exp \left( -\frac{Fo}{K} \right) - 1 \right] \int_0^1 F_3(X) dX \quad (6-10-39)
 \end{aligned}$$

$$\begin{aligned}
 \omega_{\theta\theta}^{(2)}(\mu_0, Fo) &\equiv \theta_{00}^{(2)}(0, Fo) = Lu \int_0^{Fo} [Ki_m^{(2)}(Fo') - Ki_m^{(1)}(Fo')] \\
 &\times \left[ 1 - \exp \left( -\frac{Fo - Fo'}{K} \right) \right] dFo' + \int_0^1 F_2(X) dX \\
 &+ K \left[ 1 - \exp \left( -\frac{Fo}{K} \right) \right] \int_0^1 F_3(X) dX \quad (6-10-40)
 \end{aligned}$$

$$\begin{aligned}
 \omega_{1T}^{(2)}(\mu_0, Fo) &\equiv 2T_{01}^{(2)}(0, Fo) = 2 \left\{ \int_0^{Fo} Ki_q^{(2)}(Fo') dFo' \right. \\
 &+ Ko^* Lu \int_0^{Fo} Ki_m^{(2)}(Fo') \left[ \exp \left( -\frac{Fo - Fo'}{K} \right) - 1 \right] dFo' \\
 &\left. + \int_0^1 XF_1(X) dX + K Ko^* \left[ \exp \left( -\frac{Fo}{K} \right) - 1 \right] \int_0^1 F_3(X) X dX \right\} \quad (6-10-41)
 \end{aligned}$$

$$\begin{aligned}
\omega_{1\theta}^{(2)}(\mu_0, \text{Fo}) &\equiv 2\theta_{01}^{(2)}(0, \text{Fo}) = 2 \left\{ \text{Lu} \int_0^{\text{Fo}} \text{Ki}_m^{(2)}(\text{Fo}') \right. \\
&\times \left[ 1 - \exp \left( -\frac{\text{Fo} - \text{Fo}'}{K} \right) \right] d\text{Fo}' + \int_0^1 F_2(X) X dX \\
&\left. + K \left[ 1 - \exp \left( -\frac{\text{Fo}}{K} \right) \right] \int_0^1 F_3(X) X dX \right\} \quad (6-10-42)
\end{aligned}$$

$$\begin{aligned}
\omega_{2T}^{(2)}(\mu_0, \text{Fo}) &\equiv 3T_{02}^{(2)}(0, \text{Fo}) \equiv 3 \left\{ \int_0^{\text{Fo}} \text{Ki}_q^{(2)}(\text{Fo}') d\text{Fo}' \right. \\
&+ \text{Ko}^* \text{Lu} \int_0^{\text{Fo}} \text{Ki}_m^{(2)}(\text{Fo}') \left[ \exp \left( -\frac{\text{Fo} - \text{Fo}'}{K} \right) - 1 \right] d\text{Fo}' + \int_0^1 F_1(X) X^2 dX \\
&\left. + K \text{Ko}^* \left[ \exp \left( -\frac{\text{Fo}}{K} \right) - 1 \right] \int_0^1 F_3(X) X^2 dX \right\} \quad (6-10-43)
\end{aligned}$$

$$\begin{aligned}
\omega_{2\theta}^{(2)}(\mu_0, \text{Fo}) &= 3\theta_{02}^{(2)}(0, \text{Fo}) = 3 \left\{ \text{Lu} \int_0^{\text{Fo}} \text{Ki}_m^{(2)}(\text{Fo}') \right. \\
&\times \left[ 1 - \exp \left( -\frac{\text{Fo} - \text{Fo}'}{K} \right) \right] d\text{Fo}' + \int_0^1 F_2(X) X^2 dX \\
&\left. + K \left[ 1 - \exp \left( -\frac{\text{Fo}}{K} \right) \right] \int_0^1 F_3(X) X^2 dX \right\} \quad (6-10-44)
\end{aligned}$$

where the subscript  $T$  refers to  $T^{(2)}(X, \text{Fo})$  and the subscript  $\theta$  to  $\theta^{(2)}(X, \text{Fo})$ .

In Table 6-7 are given the values of  $K_r^{(S)}$ ,  $\rho_r^{(S)}$ ,  $H_r^{(S)}(\mu_n)$ , as well as the characteristic equations for boundary conditions of the first and second kinds.

Solutions of equations (6-10-30), (6-10-31), for classical configurations at the boundary conditions of the first kind (6-10-32), (6-10-33)



Table 6-7

Kernels of Integral Transforms and Characteristic Equations

$\Gamma$	Kernels of integral transforms		$\rho_I^{(S)}(X)$		$H_I^{(S)}(\mu_n)$		Characteristic equations		$\omega_I^{(S)}(\mu_0, \text{Fo})$	
	$K_I^{(1)}(\mu_n, X)$	$K_I^{(2)}(\mu_n, X)$	$\rho_I^{(1)}(X)$	$\rho_I^{(2)}(X)$	$H_I^{(1)}(\mu_n)$	$H_I^{(2)}(\mu_n)$	of the 1-st kind	of the 2-nd kind	$\omega_I^{(1)}$	$\omega_I^{(2)}$
0	$\sin \mu_n X$	$\cos \mu_n X$	1	1	2	2	$\sin \mu = 0;$ $\mu_n = n\pi,$ $n = 1, 0, 2, \dots$	$\sin \mu = 0;$ $\mu_n = n\pi,$ $n = 0, 1, 2, \dots$	0	$u_{00}(0, \text{Fo})$
1	$J_0(\mu_n X)$	$J_0(\mu_n X)$	$X$	$X$	$\frac{2}{J_1^2(\mu_n)}$	$\frac{2}{J_0^2(\mu_n)}$	$J_0(\mu) = 0$	$J_1(\mu) = 0$	0	$2u_{01}(0, \text{Fo})$
2	$\frac{\sin \mu_n X}{X}$	$\frac{\sin \mu_n X}{X}$	$X^2$	$X^2$	2	$\frac{2\mu_n}{\sin^2 \mu_n}$	$\sin \mu = 0;$ $\mu_n = n\pi,$ $n = 0, 1, 2, \dots$	$\tan \mu = \mu$	0	$3u_{02}(0, \text{Fo})$

and of the second kind (6-10-34), (6-10-35) with regard to (6-10-37), (6-10-38) through (6-10-44) and Table 6-7 are written as follows:

$$\begin{aligned}
 T_F^{(S)}(X, \text{Fo}) &= \omega_{FT}^{(S)}(\mu_0, \text{Fo}) + \sum_{n=1}^{\infty} H_F^{(S)}(\mu_n) K_F^{(S)}(\mu_n, X) \\
 &\times \sum_{i=1}^3 [3Kp_i^2 + 2p_i(1 + \mu_n^2 K) + \mu_n^2(1 + \text{Lu} + \text{Lu Pn Ko}^*)]^{-1} \\
 &\times \left\{ (Kp_i^2 + p_i + \text{Lu} \mu_n^2) \int_0^{\text{Fo}} \Phi_{1F}^{(S)}(\text{Fo}') \exp[p_i(\text{Fo} - \text{Fo}')] d\text{Fo}' \right. \\
 &\left. - p_i \text{Ko}^* \int_0^{\text{Fo}} \Phi_{2F}^{(S)}(\text{Fo}') \exp[p_i(\text{Fo} - \text{Fo}')] d\text{Fo}' + N_{Fi} \exp[p_i \text{Fo}] \right\} \quad (6-10-45)
 \end{aligned}$$

$$\begin{aligned}
 \theta_F^{(S)}(X, \text{Fo}) &= \omega_{F0}^{(S)}(\mu_0, \text{Fo}) + \sum_{n=1}^{\infty} H_F^{(S)}(\mu_n) K_F^{(S)}(\mu_n, X) \\
 &\times \sum_{i=1}^3 [3Kp_i^2 + 2p_i(1 + \mu_n^2 K) + \mu_n^2(1 + \text{Lu} + \text{Lu Pn Ko}^*)]^{-1} \\
 &\times \left\{ \mu_n^2 \text{Lu Pn} \int_0^{\text{Fo}} \Phi_{1F}^{(S)}(\text{Fo}') \exp[p_i(\text{Fo} - \text{Fo}')] d\text{Fo}' + (p_i + \mu_n^2) \right. \\
 &\left. \times \int_0^{\text{Fo}} \Phi_{2F}^{(S)}(\text{Fo}') \exp[p_i(\text{Fo} - \text{Fo}')] d\text{Fo}' + M_{Fi} \exp[p_i \text{Fo}] \right\} \quad (6-10-46)
 \end{aligned}$$

where  $p_i$  are the non-multiple roots of the characteristic equation

$$\begin{aligned}
 \Delta(p) &\equiv Kp^3 + (1 + \mu_n^2 K)p^2 + \mu_n^2 p \\
 &\times (1 + \text{Lu} + \text{Lu Pn Ko}^*) + \mu_n^4 \text{Lu} = 0 \quad (6-10-47)
 \end{aligned}$$

$$\begin{aligned}
 N_{Fi} &= T_{nF}^0 (Kp_i^2 + p_i + \mu_n^2 \text{Lu}) + \theta_{nF}^0 \\
 &\times \text{Ko}^* \mu_n^2 \text{Lu} - p_i \text{Ko}^* K \theta_{0nF} \quad (6-10-48)
 \end{aligned}$$

$$\begin{aligned}
 M_{Fi} &= T_{nF}^0 \mu_n^2 \text{Lu Pn} + \theta_{nF}^0 [Kp_i^2 + p_i(1 + \mu_n^2 K) \\
 &+ \mu_n^2(1 + \text{Lu Pn Ko}^*)] + \theta_{0nF} K(p_i + \mu_n^2) \quad (6-10-49)
 \end{aligned}$$

$$\Phi_{10}^{(1)}(\text{Fo}) = \mu_n [\varphi_1(\text{Fo}) + (-1)^{n+1} \psi_1(\text{Fo})]$$

$$\Phi_{20}^{(1)}(\text{Fo}) = \mu_n \text{Lu} [\varphi_2(\text{Fo}) + (-1)^{n+1} \psi_2(\text{Fo})] - \text{Lu Pn} \Phi_{10}^{(1)}(\text{Fo})$$

$$\Phi_{11}^{(1)}(\text{Fo}) = \mu_n \psi_1(\text{Fo}) J_1(\mu_n)$$

$$\Phi_{21}^{(1)}(\text{Fo}) = \mu_n \psi_2(\text{Fo}) \text{Lu} J_1(\mu_n) - \text{Lu Pn} \Phi_{11}^{(1)}(\text{Fo})$$

$$\begin{aligned}
\Phi_{12}^{(1)}(\text{Fo}) &= (-1)^{n+1} \mu_n \psi_1(\text{Fo}) \\
\Phi_{22}^{(1)}(\text{Fo}) &= (-1)^{n+1} \mu_n \text{Lu} \psi_2(\text{Fo}) - \text{Lu Pn} \Phi_{12}^{(1)}(\text{Fo}) \\
\Phi_{10}^{(2)}(\text{Fo}) &= (-1)^n [\text{Ki}_q^{(2)}(\text{Fo}) - (1 - \varepsilon) \text{Lu Ko Ki}_m^{(2)}(\text{Fo})] \\
&\quad - [\text{Ki}_q^{(1)}(\text{Fo}) - (1 - \varepsilon) \text{Lu Ko Ki}_m^{(1)}(\text{Fo})] \\
\Phi_{20}^{(2)}(\text{Fo}) &= \text{Lu} [(-1)^n \text{Ki}_m^{(2)}(\text{Fo}) - \text{Ki}_m^{(1)}(\text{Fo})] \\
\Phi_{11}^{(2)}(\text{Fo}) &= [\text{Ki}_q^{(2)}(\text{Fo}) - (1 - \varepsilon) \text{Lu Ko Ki}_m^{(2)}(\text{Fo})] J_0(\mu_n) \\
\Phi_{21}^{(2)}(\text{Fo}) &= \text{Ki}_m^{(2)}(\text{Fo}) \text{Lu} J_0(\mu_n) \\
\Phi_{12}^{(2)}(\text{Fo}) &= [\text{Ki}_q^{(2)}(\text{Fo}) - (1 - \varepsilon) \text{Lu Ko Ki}_m^{(2)}(\text{Fo})] \cos \mu_n \\
\Phi_{22}^{(2)}(\text{Fo}) &= \text{Ki}_m^{(2)}(\text{Fo}) \text{Lu} \cos \mu_n
\end{aligned}$$

$T_{n\Gamma}^0$ ,  $\theta_{n\Gamma}^0$ ,  $\theta_{0n\Gamma}$  are the transforms of initial transfer potentials obtained by formula (6-10-37). In [6-45] are presented the solutions for the case of multiple roots of (6-10-47).

Appropriate solutions of the system of parabolic type equations may be obtained from (6-10-45), (6-10-46) (here, the condition  $\partial\theta(X, 0)/\partial\text{Fo} = F_3(X)$  is eliminated) at  $K \rightarrow 0$  in the form

$$\begin{aligned}
\tilde{T}_r^{(S)}(X, \text{Fo}) &\equiv \tilde{\omega}_{rT}^{(S)}(\mu_0, \text{Fo}) + \sum_{n=1}^{\infty} H_r^{(S)}(\mu_n) K_r^{(S)}(\mu_n, X) \\
&\times \sum_{j=1}^2 [2\tilde{p}_j + \mu_n^2(1 + \text{Lu} + \text{Lu Pn Ko}^*)]^{-1} \\
&\times \left\{ (\tilde{p}_j + \text{Lu} \mu_n^2) \int_0^{\text{Fo}} \Phi_{1r}^{(S)}(\text{Fo}') \exp [\tilde{p}_j(\text{Fo} - \text{Fo}')] d\text{Fo}' - \tilde{p}_j \text{Ko}^* \right. \\
&\times \left. \int_0^{\text{Fo}} \Phi_{2r}^{(S)}(\text{Fo}') \exp [\tilde{p}_j(\text{Fo} - \text{Fo}')] d\text{Fo}' + \tilde{N}_{rj} \exp [\tilde{p}_j \text{Fo}] \right\} \quad (6-10-50)
\end{aligned}$$

$$\begin{aligned}
\tilde{\theta}_r^{(S)} &\equiv \tilde{\omega}_{r\theta}^{(S)}(\mu_0, \text{Fo}) + \sum_{n=1}^{\infty} H_r^{(S)}(\mu_n) K_r^{(S)}(\mu_n, X) \\
&\times \sum_{j=1}^2 [2\tilde{p}_j + \mu_n^2(1 + \text{Lu} + \text{Lu Pn Ko}^*)]^{-1} \\
&\times \left\{ \mu_n^2 \text{Lu Pn} \int_0^{\text{Fo}} \Phi_{1r}^{(S)}(\text{Fo}') \exp [\tilde{p}_j(\text{Fo} - \text{Fo}')] d\text{Fo}' + (\tilde{p}_j + \mu_n^2) \right. \\
&\times \left. \int_0^{\text{Fo}} \Phi_{2r}^{(S)}(\text{Fo}') \exp [\tilde{p}_j(\text{Fo} - \text{Fo}')] d\text{Fo}' + \tilde{M}_{rj} \exp [\tilde{p}_j \text{Fo}] \right\} \quad (6-10-51)
\end{aligned}$$

where  $\tilde{p}_j$  are the roots of the characteristic equation

$$\tilde{\Delta}(p) \equiv p^2 + \mu_n^2(1 + \text{Lu} + \text{Lu Pn Ko}^*)p + \mu_n^4 \text{Lu} = 0 \quad (6-10-52)$$

[expression  $\tilde{\omega}_r^{(S)}$ ,  $\tilde{N}_{rj}$ ,  $\tilde{M}_{rj}$  are obtained respectively from (6-10-39) (6-10-44) at  $K \rightarrow 0$ ,  $\theta_{0nr} = 0$ ]. Using the notations employed in [6-1] we may write the roots of characteristic equation (6-10-52) as

$$\tilde{p}_j = -\mu_n^2 \text{Lu } v_j^2$$

where

$$v_j^2 = 0.5 \left[ \left( 1 + \text{Ko}^* \text{Pn} + \frac{1}{\text{Lu}} \right) + (-1)^j \right. \\ \left. \times \sqrt{\left( 1 + \text{Ko}^* \text{Pn} + \frac{1}{\text{Lu}} \right)^2 - \frac{4}{\text{Lu}}} \right], \quad (j = 1, 2)$$

The roots of equations (6-10-47) and (6-10-52), calculated with the aid of a computer for wide ranges of criteria appearing in these equations, are presented in [6-1] and [6-47], respectively.

In the case of a semi-infinite solid, equations (6-10-30), (6-10-31) at  $\Gamma = 0$  are solved by the method of Fourier integral transformation with respect to  $X$ :

$$U(\mu, \text{Fo}) = \int_0^\infty U(X, \text{Fo}) K(\mu, X) dX \quad (6-10-53)$$

and by the Laplace integral transformation method with respect to  $\text{Fo}$ . Inversion with respect to  $X$  is carried out by the formula

$$U(X, \text{Fo}) = \frac{2}{\pi} \int_0^\infty U(\mu, \text{Fo}) K(\mu, X) d\mu \quad (6-10-54)$$

Here, to conditions (6-10-32) and (6-10-34) at zero with boundary conditions of the first and second kinds, respectively, the condition at the infinity is added:

$$\frac{\partial T(\infty, \text{Fo})}{\partial X} = \frac{\partial \theta(\infty, \text{Fo})}{\partial X} = 0 \quad (6-10-55)$$

The solutions of equations (6-10-30), (6-10-31) for a semi-infinite solid with regard to (6-10-53), (6-10-54) are written as:

$$T^{(S)}(X, \text{Fo}) = \frac{2}{\pi} \int_0^\infty T^{(S)}(\mu, \text{Fo}) K^{(S)}(\mu, X) d\mu = \frac{2}{\pi} \int_0^\infty K^{(S)}(\mu, X) d\mu \\ \times \sum_{i=1}^3 [3Kp_i^2 + 2p_i(1 + K\mu^2) + \mu^2(1 + \text{Lu} + \text{Lu Pn Ko}^*)]^{-1}$$

$$\times \left\{ (K^2 p_i + p_i + \text{Lu} \mu^2) \int_0^{\text{Fo}} \Phi_1^{(S)}(\text{Fo}') \exp [p_i (\text{Fo} - \text{Fo}')] d\text{Fo}' \right. \\ \left. - p_i \text{Ko}^* \int_0^{\text{Fo}} \Phi_2^{(S)}(\text{Fo}') \exp [p_i (\text{Fo} - \text{Fo}')] d\text{Fo}' + N_i \exp [p_i \text{Fo}] \right\} \quad (6-10-56)$$

$$\theta^{(S)}(X, \text{Fo}) = \frac{2}{\pi} \int_0^{\infty} \theta^{(S)}(\mu, \text{Fo}) K^{(S)}(\mu, X) d\mu = \frac{2}{\pi} \int_0^{\infty} K^{(S)}(\mu, X) d\mu \\ \times \sum_{i=1}^3 [3Kp_i^2 + 2p_i(1 + K\mu^2) + \mu^2(1 + \text{Lu} + \text{Lu Pn Ko}^*)]^{-1} \\ \times \left\{ \mu^2 \text{Lu Pn} \int_0^{\text{Fo}} \Phi_1^{(S)}(\text{Fo}') \exp [p_i (\text{Fo} - \text{Fo}')] d\text{Fo}' + (p_i + \mu^2) \right. \\ \left. + \int_0^{\text{Fo}} \Phi_2^{(S)}(\text{Fo}') \exp [p_i (\text{Fo} - \text{Fo}')] d\text{Fo}' + M_i \exp [p_i \text{Fo}] \right\} \quad (6-10-57)$$

where

$$\Phi_1^{(1)}(\text{Fo}) = \mu \varphi_1(\text{Fo}); \quad \Phi_2^{(1)}(\text{Fo}) = \text{Lu} \mu \varphi_2(\text{Fo}) - \text{Lu Pn} \Phi_1^{(1)}(\text{Fo})$$

$$\Phi_1^{(2)}(\text{Fo}) = -[\text{Ki}_q^{(1)}(\text{Fo}) - (1 - \varepsilon) \text{Lu Ko Ki}_m^{(1)}(\text{Fo})]$$

$$\Phi_2^{(2)}(\text{Fo}) = -\text{Lu Ki}_m^{(1)}(\text{Fo})$$

$N_i$  and  $M_i$  are found from (6-10-48), (6-10-49) by substituting  $\mu_n$  by  $\mu$ . The appropriate solution for a system of ordinary differential heat- and mass-transfer equations at  $K=0$  are of the form

$$\tilde{T}^{(S)}(X, \text{Fo}) = \frac{2}{\pi} \int_0^{\infty} K^{(S)}(\mu, X) d\mu \sum_{j=1}^2 [2\tilde{p}_j + \mu^2 \\ \times (1 + \text{Lu} + \text{Lu Pn Ko}^*)]^{-1} \left\{ (\tilde{p}_j + \text{Lu} \mu^2) \int_0^{\text{Fo}} \Phi_1^{(S)}(\text{Fo}') \exp \right. \\ \times [\tilde{p}_j (\text{Fo} - \text{Fo}')] d\text{Fo}' - \tilde{p}_j \tilde{\text{Ko}}^* \int_0^{\text{Fo}} \Phi_2^{(S)}(\text{Fo}') \exp [\tilde{p}_j (\text{Fo} - \text{Fo}')] \\ \left. \times d\text{Fo}' + \tilde{N}_j \exp [\tilde{p}_j \text{Fo}] \right\}$$

$$\begin{aligned}
\tilde{\theta}^{(s)}(X, \text{Fo}) = & \frac{2}{\pi} \int_0^{\infty} K^{(s)}(\mu, X) d\mu \sum_{j=1}^2 [2\tilde{p}_j + \mu^2 \\
& \times (1 + \text{Lu} + \text{Lu Pn Ko}^*)]^{-1} \left\{ \mu^2 \text{Lu Pn} \int_0^{\text{Fo}} \Phi_1^{(s)}(\text{Fo}') \exp [\tilde{p}_j(\text{Fo} - \text{Fo}')] \right. \\
& \times d\text{Fo}' + (\tilde{p}_j + \mu^2) \int_0^{\text{Fo}} \Phi_2^{(s)}(\text{Fo}') \exp [\tilde{p}_j(\text{Fo} - \text{Fo}')] d\text{Fo}' + \tilde{M}_j \exp [\tilde{p}_j \text{Fo}] \left. \right\}
\end{aligned}$$

where  $\tilde{M}_j, \tilde{N}_j$  are found from (6-10-48), (6-10-49) at  $K \rightarrow 0$  substituting  $\mu_n$  by  $\mu$  with consideration for  $\theta_{0n\Gamma} = 0$ .

The number  $\text{Ki}_m$  is determined depending on the physical significance of  $\tau_{rm}$  ( $w_m$  or  $w_q$ ). To account for the finite mass-transfer rate the number  $\text{Ki}_m$  in the mass-transfer boundary conditions is substituted by  $\tilde{\text{Ki}}_m = \text{Ki}_m + \tau_{rm} \text{Ki}_m$  and in the initial conditions (6-10-36) the function  $F_3(X)$  is assumed zero, i.e.  $F_3(X) = 0$ .

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